COMMON FIXED POINT THEOREMS FOR
FOUR MAPPINGS SATISFYING A RATIONAL
INEQUALITY
CHAPTER V

COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS
SATISFYING A RATIONAL INEQUALITY

5.1 The following fixed point theorem was proved by Ahmad and Imdad[1].

THEOREM A. Let (S, T) and (T, J) be two weakly commuting pairs of mappings of a complete metric space (X, d) into itself such that

(5.1.1) \( T(X) \subseteq I(X), \ S(X) \subseteq J(X); \)

and for all \( x, y \in X \) either

\[
\frac{a\left(\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2\right)}{d(Sx, Ix) + d(Ty, Jy)} + bd(Ix, Jy)
\]

(5.1.2) \( d(Sx, Ty) \leq \frac{a\left(\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2\right)}{d(Sx, Ix) + d(Ty, Jy)} + bd(Ix, Jy) \)

If \( d(Sx, Ix) + d(Ty, Jy) \neq 0 \) where \( a, b > 0, a + b < 1, \)

or

(5.1.3) \( d(Sx, Ty) = 0 \) if \( d(Sx, Ix) + d(Ty, Jy) = 0. \)

If one of S, T, I or J is continuous, then S, T, I and J have a unique common fixed point \( z \). Further, \( z \) is the unique common fixed point of \( S \) and \( I \) and of \( T \) and \( J \).
Extending Theorem A, we now prove the following:

**Theorem 1.** Let \( \{S, T\} \) and \( \{T, J\} \) be two weakly commuting pairs of mappings of a complete metric space \((X, d)\) satisfying the conditions (5.1.1), (5.1.3) and the following:

\[
(5.1.4) \quad d(Sx, Ty) \leq \frac{\alpha [\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2]}{d(Sx, Ix) + d(Ty, Jy)}
\]

\[
+ \frac{\beta [1 + d(Jy, Ty)] d(Sx, Ix) + \gamma d(Ix, Jy)}{1 + d(Ix, Jy)}
\]

If \( d(Sx, Ix) + d(Ty, Jy) \neq 0 \) where \( \alpha, \beta, \gamma > 0, \alpha + \beta + \gamma < 1 \),
or,

\[
(5.1.5) \quad d(Sx, Ty) = 0 \quad \text{if} \quad d(Sx, Ix) + d(Ty, Jy) = 0
\]

If one of \( S, T, I \) or \( J \) is continuous then \( S, T, I, \) and \( J \) have a unique common fixed point \( z \). Further, \( z \) is the unique common fixed point of \( S \) and \( I \) and of \( T \) and \( J \).

**Proof** Let \( x_0 \) be an arbitrary point in \( X \). As \( S(X) \) is contained in \( J(X) \),
we can choose a point $x_1$ in $X$ such that $S_{x_1} = Jx_1$. Since $T(X)$ is also contained in $I(X)$, we can choose a point $x_2$ in $X$ such that $Tx_1 = Ix_2$. In this way, we can choose $x_{2n}, x_{2n+1}, x_{2n+2}$ such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Let us denote $U_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2})$.

We distinguish two cases:

**Case 1.** Suppose $U_{2n} + U_{2n+1} \neq 0$ for $n = 0, 1, 2 \ldots$,

then on using inequality (7.1.4), we get

$$U_{2n+1} \leq \frac{\alpha \{(U_{2n})^2 + (U_{2n+1})^2\}}{U_{2n} + U_{2n+1}} + \beta U_{2n+1} + \gamma U_{2n},$$

(5.1.6)

So that

$$(1 - \alpha - \beta) U_{2n+1}^2 + (1 - \beta - \gamma) U_{2n} U_{2n+1} - (\alpha + \gamma) U_{2n}^2 \leq 0$$

the positive root $K$ of the quadratic equation

$$(1-\alpha-\beta) t^2 + 1(1-\beta-\gamma)t - (\alpha+\gamma) = 0$$

is

$$[((1-\beta-\gamma)^2 + 4(\alpha+\gamma)(1-\alpha-\beta))^{1/2} - (1-\beta-\gamma)] / (2-2\alpha-2\beta)$$

and since $\alpha + \beta + \gamma < 1$, it follows that $K < 1$, Thus

$$U_{2n+1} \leq KU_{2n}.$$ 

Similarly, if $U_{2n} + U_{2n-1} \neq 0$, $n = 1, 2, 3 \ldots$

then the inequality
\[ U_{2n} \leq \frac{\alpha \left( (U_{2n-1})^2 + (u_{2n}) \right)}{U_{2n-1} + U_{2n}} + \beta U_{2n} + \gamma U_{2n-1}. \]

like the earlier one, gives

\[ U_{2n} \leq K U_{2n-1}. \]

Thus, in general, we have shown that for \( k = 0, 1, 2, \ldots \),

\[ U_{k+1} \leq \frac{\alpha \left( (U_k)^2 + (U_{k+1})^2 \right)}{U_k + U_{k+1}} + \beta U_{k+1} + \gamma U_k. \]

Having this we see that \( U_{k+1} \leq K U_k \) which yields \( U_k \leq K^k U_0 \). Now it follows that the sequence

\[ (5.1.7) \quad \{ Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots \} \]

is a Cauchy sequence in the complete metric space \((X, d)\) and so has a limit point \( z \) in \( X \). Hence the sequences

\[ \{ Sx_{2n} \} = \{ Jx_{2n+1} \} \text{ and } \{ Tx_{2n-1} \} = \{ Ix_{2n} \} \]

Which are subsequences of \((5.1.7)\) also converge to the point \( z \).

Let us suppose that \( I \) is continuous so that the sequences \( \{ Ix_{2n} \} \)
and \( \{ ISx_{2n} \} \) converge to the point \( Iz \). Since \( S \) and \( I \) are weakly commuting, we have

\[ d(SIx_{2n}, ISx_{2n}) \leq d(Ix_{2n}, Sx_{2n}). \]
and so the sequence \( \{ S_{2n} \} \) also converges to the point \( I_z \).

We have

\[
\alpha \left[ \left\{ d\left( I_{2n}^{2}, S_{2n} \right) \right\}^2 + \left\{ d\left( T_{2n+1}, J_{2n+1} \right) \right\}^2 \right] \\
\leq \frac{d\left( S_{2n}, T_{2n+1} \right)}{d\left( I_{2n}^{2}, S_{2n} \right) + d\left( T_{2n+1}, J_{2n+1} \right)}
\]

\[
\beta \left[ 1 + d\left( J_{2n+1}, T_{2n+1} \right) \right] d\left( S_{2n}, I_{2n}^{2} \right) \\
+ \frac{1 + d\left( I_{2n}^{2}, J_{2n+1} \right)}{1 + d\left( I_{2n}^{2}, J_{2n+1} \right)}
\]

\[
+ \gamma \ d\left( I_{2n}^{2}, J_{2n+1} \right).
\]

Letting \( n \to \infty \), we have

\[
d\left( I_z, z \right) \leq \gamma \ d\left( I_z, z \right),
\]

a contradiction. It follows that \( I_z = z \). Further

\[
\alpha \left[ \left\{ d\left( I_z, S_z \right) \right\}^2 + \left\{ d\left( T_{2n+1}, J_{2n+1} \right) \right\}^2 \right] \\
\leq \frac{d\left( S_z, T_{2n+1} \right)}{d\left( I_z, S_z \right) + d\left( T_{2n+1}, J_{2n+1} \right)}
\]

\[
\beta \left[ 1 + d\left( J_{2n+1}, T_{2n+1} \right) \right] d\left( S_z, I_z \right) \\
+ \frac{1 + d\left( I_z, J_{2n+1} \right)}{1 + d\left( I_z, J_{2n+1} \right)}
\]
\[ + \gamma d (Iz, Jx_{2n+1}). \]

and letting \( n \to \infty \), we get

\[
d (Sz, z) \leq \alpha d (Sz, z) + \beta d (Sz, z) \leq (\alpha + \beta) d (Sz, z),\]

again a contradiction. Hence \( Sz = z \).

This means that \( z \) is in the range of \( S \) and since the range of \( J \) contains the range of \( S \), there exists a point \( z' \) such that \( Jz' = z \).

Thus,

\[
d (z, Tz') = d (Sz, Tz') = \alpha \left[ \{d (Sz, Iz)\}^2 + \{d(Tz', Jz')\}^2 \right] \leq \frac{d (Sz, Iz) + d (Tz', Jz')}{d (Sz, Iz) + d (Tz', Jz')} \]

\[
\beta [1 + d (Jz', Tz')]^2 d (Sz, Iz) + \frac{1 + d (Iz, Jz')}{1 + d (Iz, Jz')} + \gamma d (Iz, Jz') = \alpha d (z, Tz') < d (z, Tz'),
\]
Which implies that $Tz' = z$.

Since $T$ and $J$ weakly commute,
\[
d(Tz, Jz) = d(TJz', JTz') \\
\leq d(Jz', Tz') \\
= d(z, z) = 0
\]
giving there by $Tz = Jz$ and so,
\[
d(z, Tz) = d(Sz, Tz) \\
\leq \frac{\alpha \left[ \{d(Iz, Sz)\}^2 + \{d(Tz, Jz)\}^2 \right]}{d(Iz, Sz) + d(Tz, Jz)}
\]
\[
\beta \left[ 1 + d(Jz, Tz) \right] d(Sz, Iz) \\
+ \frac{1 + d(Iz, Jz)}{1 + d(Iz, Jz)}
\]
\[+ \gamma d(Iz, Jz) = 0\]

Which implies that $z = Tz = Jz$.

We therefore have proved that $z$ is a common fixed point of $S$, $T$, $I$, and $J$.

Now suppose that $S$ is continuous, so that the sequences $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to $Sz$. Since $S$ and $I$ weakly commute, it follows as above that the sequence $\{ISx_{2n}\}$ also converges to $Sz$. Thus
\[ d(S^2x_{2n}, T_{x_{2n+1}}) \leq \frac{\alpha\left[\left\{d\left(S^2x_{2n}, ISx_{2n}\right)\right\}^2 + \left\{d\left(T_{x_{2n+1}}, J_{x_{2n+1}}\right)\right\}^2\right]}{d(S^2x_{2n}, ISx_{2n}) + d(T_{x_{2n+1}}, J_{x_{2n+1}})} \]

\[ \beta\left[1 + d(J_{x_{2n+1}}, T_{x_{2n+1}})\right] d\left(S^2x_{2n}, ISx_{2n}\right) \]

\[ + \frac{1 + d(ISx_{2n}, J_{x_{2n+1}})}{1 + d(ISx_{2n}, J_{x_{2n+1}})} \]

\[ + \gamma d(ISx_{2n}, J_{x_{2n+1}}), \]

Letting \( n \to \infty \), we have

\[ d(Sz, z) \leq \gamma d(Sz, z) < d(Sz, z) \]

It follows that \( Sz = z \).

Once again, there exists a point \( z' \) in \( X \) such that \( Jz' = z \).

\[ d(S^2x_{2n}, Tz') \leq \frac{\alpha\left[\left\{d\left(S^2x_{2n}, ISx_{2n}\right)\right\}^2 + \left\{d(Tz', Jz')\right\}^2\right]}{d(S^2x_{2n}, ISx_{2n}) + d(Tz', Jz')} \]

\[ \beta \left[1 + d(Jz', Tz')\right] d(S^2x_{2n}, ISx_{2n}) \]

\[ + \frac{1 + d(ISx_{2n}, Jz')}{1 + d(ISx_{2n}, Jz')} \]

\[ + \gamma d(ISx_{2n}, Jz'). \]
Letting $n \to \infty$, we have
\[ d(z, Tz') \leq \gamma d(z, Tz') \]
So that
\[ z = Tz'. \]

Since $T$ and $J$ weakly commute, it again follows as above that
\[ Tz = Jz. \]
Further,
\[ d(Sx_{2n}, Tz) \leq \frac{\alpha \left[ \{ d(Sx_{2n},Ix_{2n}) \}^2 + \{ d(Tz,Jz) \}^2 \right]}{d(Sx_{2n},Ix_{2n}) + d(Tz,Jz)} \]
\[ + \beta \left[ 1 + d(Jz,Tz) \right] d(Sx_{2n},Ix_{2n}) \]
\[ + \frac{1 + d(Ix_{2n},Jz)}{1 + d(Ix_{2n},Jz)} \]
\[ + \gamma d(Ix_{2n}, Jz). \]

Letting $n \to \infty$, we have
\[ d(z, Tz) \leq \gamma d(z, Tz) \]
and so
\[ z = Tz = Jz. \]

The point $z$ therefore is in the range of $T$ and since the range of $I$
contains the range of $T$, there exist a point $z''$ in $X$ such that $Tz'' = z$. Thus

$$d (Sz'', z) = d (Sz'', Tz)$$

$$\leq \alpha \left[ \{d (Sz'', Iz'')\}^2 + \{d (Tz, Jz)\}^2 \right]$$

$$d (Sz'', Iz'') + d (Tz, Jz)$$

$$= \beta [1 + d (Jz, Tz)] d (Sz'', Iz'')$$

$$+ \gamma d (Iz'', Jz).$$

$$= (\alpha + \beta) d (Sz'', z).$$

and so $Sz'' = z$.

Again, since $S$ and $I$ weakly commute, we have

$$d (Sz, Iz) = d (SIz'', ISz'') \leq d (Iz'', Sz'') = d (z, z) = 0$$

Thus $Sz = Iz = z$.

We thus have proved again that $z$ is a common fixed point of $S$, $T$, $I$ and $J$.

If the mappings $T$ or $J$ is continuous instead of $S$ or $I$ then the proof that $z$ is a common fixed point of $S$, $T$, $I$ and $J$ is similar.
Case 2. Suppose $U_{2n} + U_{2n+1} = 0$. Then, for some $n$, the inequality (5.1.6) gives

$$U_{2n} = d(Sx_{2n}, Tx_{2n+1}) = 0,$$
and

$$U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2}) = 0,$$
giving there by

$$Sx_{2n} = Jx_{2n+1} = Tx_{2n+1} = Sx_{2n+2} = \ldots = z.$$ 

Now we assert that there exists a point $w$ such that

$$Sw = lw = Tw = Jw = z$$
because if $Sw = lw \neq z$, then

$$0 < d(lw, z) = d(Sw, Tx_{2n+1})$$

$$\leq \frac{\alpha[\{d(Sw, lw)\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2]}{d(Sw, lw) + d(Tx_{2n+1}, Jx_{2n+1})}$$

$$+ \frac{\beta [1 + d(Jx_{2n+1}, Tx_{2n+1})] d(Sw, lw)}{1 + d(lw, Jx_{2n+1})} + \gamma d(lw, Jx_{2n+1})$$

$$\leq \gamma d(lw, z) < d(lw, z)$$

Which yields that $lw = z = Sw$. Similarly, one can argue that $Tw = Jw = z$.

Now suppose that $I$ or $S$ is continuous. Proceeding as above, it can be shown that $lw = z$ is a common fixed point of $S$, $T$, $I$ and $J$. 
Furthermore, if $J$ or $T$ is continuous, then the proof that $z$ is a common fixed point of $S$, $T$, $I$, and $J$ is similar.

In order to prove the uniqueness of the common fixed point $z$, let $w$ be a second common fixed point of $S$ and $I$. Then

\[
    d(w, z) = d(Sw, Tz) \\
    \leq \frac{\alpha \left[ \{d(Sw, Iw)\}^2 + \{d(Tz, Jz)\}^2 \right]}{d(Sw, Iw) + d(Tz, Jz)} \\
    + \frac{\beta [1 + d(Jz, Tz)] d(Sw, Iw)}{1 + d(Iw, Jz)} \\
    + \gamma d(Iw, Jz) = 0,
\]

Which yields that $w = z$.

Similarly it can be proved that $z$ is a unique common fixed point of $T$ and $J$.

This completes the proof.

**REMARK:**

On taking $\beta = 0$, $\alpha = a$ and $\gamma = b$ in Theorem 1, we get Theorem A.
Our next result is as follows:

**THEOREM 2** Let \( \{ S, T \} \) and \( \{ T, J \} \) be two weakly commuting pairs of mappings of a complete metric space \((X, d)\) satisfying the conditions (5.1.1), (5.1.3) and (5.1.5) the following:

\[
\{ \{ d(Sx, Ix) \}^2 + \{ d(Ty, Jy) \}^2 \} \leq C \max \left\{ \begin{array}{c}
\frac{d(Sx, Ix) + d(Ty, Jy)}{1 + d(Ix, Jy)} \\
\frac{d(Ix, Jy)}{1 + d(Ix, Jy)}
\end{array} \right\}
\]

(5.1.8)

If \( d(Sx, Ix) + d(Ty, Ty) \neq 0 \) where \( 0 \leq C < 1 \),

or,

If one of \( S, T, I \) or \( J \) is continuous then \( S, T, I \) and \( J \) have a unique common fixed point \( z \). Further, \( z \) is the unique common fixed point of \( S \) and \( I \) and of \( T \) and \( J \).

**PROOF** Let \( x_0 \) be an arbitrary point in \( X \). As \( S(X) \) is contained in \( J(X) \), we can choose a point \( x_1 \) in \( X \) such that \( Sx_0 = Jx_1 \). Since \( T(X) \) is also contained in \( I(X) \), we can choose a point \( x_2 \) in \( X \) such that \( Tx_1 = Ix_2 \). In this way, we can
choose \( x_{2n}, x_{2n+1}, x_{2n+2} \) such that \( Sx_{2n} = Jx_{2n+1} \) and \( Tx_{2n+1} = Ix_{2n+2} \) for \( n = 0, 1, 2, \ldots \). Let us denote \( U_{2n} = d(Sx_{2n}, Tx_{2n+1}) \) and \( U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2}) \).

We distinguish two cases:

**Case 1.** Suppose \( U_{2n} + U_{2n+1} \neq 0 \) for \( n = 0, 1, 2, \ldots \), then on using inequality (5.1.8), we get

\[
(7.1.9) \quad U_{2n+1} \leq C \max \left\{ \frac{(U_{2n})^2 + (U_{2n+1})^2}{U_{2n} + U_{2n+1}}, \frac{U_{2n}}{U_{2n+1}} \right\}
\]

If \( \max \left\{ \frac{(U_{2n})^2 + (U_{2n+1})^2}{U_{2n} + U_{2n+1}}, \frac{U_{2n}}{U_{2n+1}} \right\} = \frac{(U_{2n})^2 + (U_{2n+1})^2}{U_{2n} + U_{2n+1}} \),

Then,

\[
U_{2n+1} \leq C \left[ \frac{(U_{2n})^2 + (U_{2n+1})^2}{U_{2n} + U_{2n+1}} \right]
\]

So that

\[
(1 - C) U_{2n+1}^2 + U_{2n} U_{2n+1} - C U_{2n}^2 \leq 0
\]

The positive root \( K \) of the quadratic equation

\[
(1 - C) t^2 + t - C = 0
\]

is

\[
\left[ \{1 + 4 (1-C) C \}^{1/2} - (1 - C) \right] / (2 - 2C)
\]
and since $C < 1$, it follows that $K < 1$. Thus

$$U_{2n+1} \leq KU_{2n}.$$ 

If

$$\max \left\{ \frac{(U_{2n})^2 + (U_{2n+1})^2}{U_{2n} + U_{2n+1}}, U_{2n+1}, U_{2n} \right\} = U_{2n+1}$$

Then

$$U_{2n+1} \leq C U_{2n+1}$$

Which is impossible, Since $C < 1$.

If

$$\max \left\{ \frac{(U_{2n})^2 + (U_{2n+1})^2}{U_{2n} + U_{2n+1}}, U_{2n+1}, U_{2n} \right\} = U_{2n}$$

Then

$$U_{2n+1} \leq C U_{2n}$$

So, we have

$$U_{2n+1} \leq K U_{2n}.$$ 

Similarly if $U_{2n} + U_{2n-1} \neq 0$, n= 1, 2, .... then inequality

$$\frac{(U_{2n-1})^2 + (U_{2n})^2}{U_{2n-1} + U_{2n}}$$

like the earlier are, gives

$$U_{2n} \leq K U_{2n-1}$$
Thus, in general, we have shown that for \( k = 0, 1, 2, \ldots \)

\[
U_{k+1} \leq C \max \left\{ \frac{(U_k)^2 + (U_{k+1})^2}{U_k + U_{k+1}}, U_{k+1}, U_k \right\}
\]

Having this we see that \( U_{k+1} \leq K U_k \) which yields \( U_k \leq K^k U_0 \), now it follows that the sequence

\[
\text{(7.1.10)} \quad \{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}
\]

is a cauchy sequence in the complete metric space \((X, d)\) and so has a limit point \( z \) in \( X \). Hence the sequences

\[
\{Sx_{2n}\} = \{Ix_{2n+1}\} \text{ and } \{Tx_{2n-1}\} = \{Ix_{2n}\}
\]

Which are subsequences of (5.1.10) also converge to the point \( z \).

Let us suppose that \( I \) is continuous so that the sequence \( \{I^2x_{2n}\} \)

and \( \{ISx_{2n}\} \) converge to the point \( Iz \). Since \( S \) and \( I \) are weakly commuting, we have

\[
d(ISx_{2n}, ISx_{2n}) \leq d(Ix_{2n}, Sx_{2n})
\]

and so the sequence \( \{SIx_{2n}\} \) also converges to the point \( Iz \). We have
\[ d(\text{SI}_x, \text{TX}_{2n+1}) \leq C \max \left\{ \frac{\left( \{d(I^2x_{2n}^{}, SIx_{2n}^{})\}^2 + \{d(Tx_{2n+1}^{}, Jx_{2n+1}^{})\}^2 \right)}{d(I^2x_{2n}^{}, SIx_{2n}^{}) + d(Tx_{2n+1}^{}, Jx_{2n+1}^{})}, \right\} \]

Letting \( n \to \infty \), we have

\[ d(Iz, z) \leq C d(Iz, z), \]

a contradiction. It follows that \( Iz = z \). Further

\[ d(Sz, Tx_{2n+1}) \leq C \max \left\{ \frac{\left( \{d(Iz, Sz)\}^2 + \{d(Tx_{2n+1}^{}, Jx_{2n+1}^{})\}^2 \right)}{d(Iz, Sz) + d(Tx_{2n+1}^{}, Jx_{2n+1}^{})}, \right\} \]

and letting \( n \to \infty \), we get

\[ d(Sz, z) \leq C d(Sz, z) \]

again a contradiction. Hence \( Sz = z \).
This means that \( z \) is in the range of \( S \) and since the range of \( J \) contains the range of \( S \), there exists a point \( z' \) such that \( Jz' = z \). Thus

\[
d(z, Tz') = d(Sz, Tz') \leq C \max \left\{ \frac{\{d(Sz, Iz)\}^2 + \{d(Tz', Jz')\}^2}{d(Sz, Iz) + d(Tz', Jz')} \right\},
\]

\[
\leq C \max \left\{ \frac{[1 + d(Jz', Tz')]d(Sz, Iz)}{1 + d(Iz, Jz')} \right\},
\]

\[
d(Iz, Jz') \}
\]

\[
\leq C d(z, Tz')
\]

Which implies that \( Tz' = z \).

Since \( T \) and \( J \) weakly commute,

\[
d(Tz, Jz) = d(TJz', JTz') \leq d(Jz', Tz') = d(z, z) = 0
\]

giving there by \( Tz = Jz \) and so

\[
d(z, Tz) = d(Sz, Tz)
\]
\[
\begin{align*}
\frac{{\{ d(Iz, Sz) \}^2 + \{ d(Tz, Jz) \}^2}}{d(Iz, Sz) + d(Tz, Jz)} \\
\frac{\{ 1 + d(Jz, Tz) \} d(Sz, Iz)}{1 + d(Iz, Jz)} \\
\frac{\{ d(Iz, Jz) \}^2}{d(Iz, Jz)} \leq 0
\end{align*}
\]

Which implies that \( z = Tz = Jz \).

We therefore have proved that \( z \) is a common fixed point of \( S \), \( T \), \( I \) and \( J \).

Now suppose that \( S \) is continuous, so that the sequences \( \{ S^2x_{2n} \} \) and \( \{ SIx_{2n} \} \) converge to \( Sz \). Since \( S \) and \( I \) weakly commute, it follows as above that the sequence \( \{ ISx_{2n} \} \) also converges to \( Sz \). Thus

\[
\begin{align*}
d(S^2x_{2n}, Tx_{2n+1}) \leq C \max \left\{ \frac{\{ d(S^2x_{2n}, ISx_{2n}) \}^2 + \{ d(Tx_{2n+1}, Jx_{2n+1}) \}^2}{d(S^2x_{2n}, SIx_{2n}) + d(Tx_{2n+1}, Jx_{2n+1})}, \right. \\
\left. \frac{\{ 1 + d(Jx_{2n+1}, Tx_{2n+1}) \} d(S^2x_{2n}, ISx_{2n})}{1 + d(ISx_{2n}, Jx_{2n+1})} \right\}
\end{align*}
\]

Letting \( n \to \infty \) we have

\[
d(Sz, z) \leq C d(Sz, z),
\]
It follows that \( S_z = z \).

Once again, there exists a point \( z' \) in \( X \) such that \( Jz' = z \).

\[
\left\{ \left[ d(S^2x_{2n}, ISx_{2n}) \right]^2 + \left[ d(Tz', Jz') \right]^2 \right\}
\leq C \max \left\{ \frac{d(S^2x_{2n}, Tz')}{d(S^2x_{2n}, ISx_{2n}) + d(Tz', Jz')}, \frac{[1 + d(Jz', Tz')] d(S^2x_{2n}, ISx_{2n})}{1 + d(ISx_{2n}, Jz')} \right\}
\]

Letting \( n \to \infty \), we have

\[ d(z, Tz') \leq C d(z, Tz'), \]

So that \( z = Tz' \),

Since \( T \) and \( J \) weakly commute, it again follows as above that

\[ Tz = Jz. \] Further,

\[
\left\{ \left[ d(Sx_{2n}, Ix_{2n}) \right]^2 + \left[ d(Tz, Jz) \right]^2 \right\}
\leq C \max \left\{ \frac{d(Sx_{2n}, Tz)}{d(Sx_{2n}, Ix_{2n}) + d(Tz, Jz)}, \frac{[1 + d(Jz, Tz)] d(Sx_{2n}, Ix_{2n})}{1 + d(Ix_{2n}, Jz)} \right\}
\]
Letting $n \to \infty$, we have

$$d(z, Tz) \leq C \ d(z, Tz)$$

and so that

$$z = Tz = Jz.$$ 

The point $z$ therefore is in the range of $T$ and since the range of $I$ contains the range of $T$, there exists a point $z''$ in $X$ such that $Tz'' = z$. Thus,

$$d(Sz'', z) = d(Sz'', Tz')$$

$$\leq C \max \left\{ \frac{[\{d(Sz'', Iz'')\}^2 + \{d(Tz, Jz)\}^2]}{d(Sz'', Iz'') + d(Tz, Jz)} \right\}$$

$$\leq C \ d(Sz'', z)$$

and so

$$Sz'' = z.$$ 

Again, Since $S$ and $I$ weakly commute, we have

$$d(Sz, Iz) = d(SIz'', ISz'') \leq d(Iz'', Sz'') = d(z, z) = 0$$

Thus

$$Sz = Iz = z.$$ 

We thus have proved again that $z$ is a common fixed point of $S$, $T$, $I$ and $J$.

If the mappings $T$ or $J$ is continuous instead of $S$ or $I$ then the proof that $z$ is a common fixed point of $S$, $T$, $I$ and $J$ is similar.
Case 2  Suppose \( U_{2n} + U_{2n+1} = 0 \). Then, for some \( n \), the inequality (5.1.6) gives

\[
U_{2n} = d (Sx_{2n}, Tx_{2n+1}) = 0, \text{ and }
\]

\[
U_{2n+1} = d (Tx_{2n+1}, Sx_{2n+2}) = 0, \text{ giving there by}
\]

\[
Sx_{2n} = Jx_{2n+1} = Tx_{2n+1}, Sx_{2n+2} = \ldots \ldots = z.
\]

Now we assert that there exists a point \( w \) such that

\[
Sw = Iw = Tw = Jw = z \quad \text{because if} \quad Sw = Iw \neq z, \text{ then} \]

\[
0 < d (Iw, z) = d (Sw, Tx_{2n+1})
\]

\[
\leq C \max \left\{ \frac{[d (Sw, Iw))^2 + d (Tx_{2n+1}, Jx_{2n+1})^2]}{d(Sw, Iw) + d(Tx_{2n+1}, Jx_{2n+1})} \right\}
\]

\[
\leq C \max \left\{ \frac{[1 + d(Jx_{2n+1}, Tx_{2n+1})]d(Sw, Iw)}{1 + d(Iw, Jx_{2n+1})} \right\}
\]

Which yields that \( Iw = z = Sw \). Similarly, one can range the \( Tw = Jw = z \).

Now suppose that \( I \) or \( S \) or continuous. Proceeding as above, it can be shown that \( Iw = z \) is a common fixed point \( S, T, I \) and \( J \).

Further more, if \( J \) or \( T \) is continuous, then the proof that \( z \) is a

common fixed point of \( S, T, I \) and \( J \) is similar.
In order to prove the uniqueness of the common fixed point \( z \), let \( w \) be a second common fixed point of \( S \) and \( I \). Then,

\[
d(w, z) = d(Sw, Tz)
\]

\[
\leq C \max \left\{ \frac{[d(Sw, Iw)]^2 + [d(Tz, Jz)]^2}{d(Sw, Iw) + d(Tz, Jz)}, \frac{[1 + d(Jz, Tz)] d(Sw, Iw)}{1 + d(Iw, Jz)}, d(Iw, Jz) \right\}
\]

\[
\leq 0
\]

Which yields that \( w = z \).

Similarly it can be proved that \( z \) is a unique common fixed point of \( T \) and \( J \).

This completes the proof.