CHAPTER IV

FIXED POINT THEOREMS USING FOUR MAPPINGS
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4.1 In 1974, Ciric [173] established a non-unique fixed point theorem by proving:

**Theorem C:** Let $T$ be orbitally continuous self mapping on orbitally complete metric space $(X,d)$ satisfying:

$$\text{min} \{d(Tx,Ty), d(x,Tx), d(y,Ty)\}$$

$$-\text{min} \{d(x,Ty), d(y,Tx)\} \leq q \ d(x,y)$$

for all $x,y \in X$ and for some $q \in [0,1)$, then for each $x \in X$, the sequence $\{x^n x\}_{n=1}^{\infty}$ converges to a fixed point of $T$.

Taskovic [174] obtained a generalization of the result of Ciric. He replaced the condition (4.1.1) by

$$a_1 \ d(Tx,Ty) + a_2 \ d(x,Tx)+ a_3 \ d(y,Ty)$$

$$+ a_4 \ \text{min} \ {d(x,Ty), d(y,Tx)} \leq q \ d(x,y)$$

for all $x,y \in X$ where $a_i$ (i=1,2,3,4) and $q$ are real numbers with $a_1 + a_2 + a_3 > q$ and $q - a_2 \geq 0$.

[173] Ciric, L.B. (17)

In 1984, Bajaj [175] proved:

**THEOREM B:** Let $T : M \rightarrow M$ be an orbitally continuous mapping on $M$ and let $M$ be $T$-orbitally complete metric space. If $T$ satisfies:

\[
\min\{d(x,Tx) \cdot d(Tx,Ty), [d(x,y)]^2, d(x,Tx) \cdot d(y,Ty)\} \\
- \min\{d(x,Tx) \cdot d(x,y), d(x,Ty) \cdot d(y,Tx)\} \\
\leq q \cdot d(x,Tx) \cdot d(x,y)
\]

for all $x, y \in M$ and $q \in (0,1)$, then for each $x \in M$, the sequence $\{T^n x\}$ converges to a fixed point of $T$.

Recently, Pachpatte [176] proved:

**THEOREM P:** Let $T : M \rightarrow M$ be an $T$-orbitally continuous mapping on $M$ and let $M$ be $T$-orbitally complete. If $T$ satisfies:

\[
\min\{[d(Tx,Ty)]^2, d(x,y) \cdot d(Tx,Ty), [d(y,Ty)]^2\} \\
- \min\{d(x,Tx) \cdot d(y,Ty), d(x,Ty) \cdot d(y,Tx)\} \\
\leq q \cdot d(x,Tx) \cdot d(y,Ty)
\]

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[175] Bajaj, N
[176] Pachpatte, B.G.
for all $x, y \in M$ and $q \in [0,1)$, then for each $x \in M$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of $T$.

4.2 First we shall prove:

**Theorem 1**: Suppose $(X,d)$ is a metric space where $d$ is continuous. Let $P, Q, S$ and $T$ be self mappings on $X$ and $a_3$ and $a_4$ are real numbers such that

$$a_1 + a_2 + 2a_4 + a_5 < 1 \text{ and } a_1 + a_3 + a_5 < 1;$$

then for all $x, y \in M$, if

$$\min\{d(Sx, Ty), d(Px, Sx), d(Qy, Ty)\} + a_1 \min\{d(Px, Ty), d(Qy, Sx)\} \leq a_1 d(Px, Sx) + a_2 d(Qy, Ty) + a_3 d(Px, Ty) + a_4 d(Qy, Sx) + a_5 d(Px, Qy)$$

and any point $x_0$ in $X$, sequence $\{x_n\}$ and $\{y_n\}$ are such that

$$P_{x_2n} = Tx_{2n+1} = y_{2n+1}$$

$$Qx_{2n+1} = Sx_{2n+2} = y_{2n+2}, \quad n = 0, 1, 2, \ldots$$

(4.2.4) any sequence of $\{Tx_n\}$ and $\{Sx_n\}$ converges to any point $z$ of $X$;
(4.2.5) mappings P,Q,S,T are continuous at point z,

(4.2.6) the pairs \{(P,S)\} and \{(Q,T)\} are
z-asymptotically commuting; then z is the unique common
fixed point of P,Q,S and T.

**Proof:** In (4.2.2) we put \(x = x_{2n}\) and \(y = x_{2n+1}\),

we get

\[
\min \{d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Sx_{2n}), d(Sx_{2n+2}, Tx_{2n+1}) \}
\]

\[+ \min \{d(Tx_{2n+1}, Tx_{2n+1}), d(Sx_{2n+2}, Sx_{2n}) \}
\]

\[\leq a_1 \ d(Tx_{2n+1}, Sx_{2n}) + a_2 \ d(Sx_{2n+2}, Tx_{2n+1})
\]

\[+ a_3 \ d(Tx_{2n+1}, Tx_{2n+1}) + a_4 \ d(Sx_{2n+2}, Sx_{2n})
\]

\[+ a_5 \ d(Tx_{2n+1}, Sx_{2n+2})
\]

i.e.

\[
\min \{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}) \}
\]

\[\leq a_1 d(y_{2n}, y_{2n+1}) + a_2 d(y_{2n+1}, y_{2n+2}) + a_3 d(y_{2n+1}, y_{2n+1})
\]

\[+ a_4 d(y_{2n+2}, y_{2n}) + a_5 d(y_{2n+1}, y_{2n+2})
\]

or

\[
\min \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}) \}
\]

\[\leq (a_1 + a_4) \ d(y_{2n}, y_{2n+1})
\]

\[+ (a_2 + a_4 + a_5) \ d(y_{2n+1}, y_{2n+2})
\]
\[ d(y_{2n+1}, y_{2n+2}) \leq \frac{a_1 + a_4}{1 - a_2 - a_4 - a_5} d(y_{2n}, y_{2n+1}) \]

Similarly, if we put \( y = x_{2n+1}, \ x = x_{2n+2} \), we get

\[ d(y_{2n+2}, y_{2n+3}) \leq \frac{a_1 + a_4}{1 - a_2 - a_4 - a_5} d(y_{2n+1}, y_{2n+2}) \]

Thus \( \{y_n\} \), i.e.

\[ \{Tx_1, Sx_2, Tx_3, Sx_4, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots \} \]

is Cauchy sequence and with (4.2.4) as \( n \to \infty \)

\[ Px_{2n} = Tx_{2n+1} \to z, \ Qx_{2n+1} = Sx_{2n+2} \to z. \]

But \( PPx_{2n} \to Pz \). We know that \( \{P, S\} \) is \( z \)-asymptotically regular, Thus

\[ d(SPx_{2n}, PSx_{2n}) \to 0. \]

Now we put \( x = Px_{2n} \) and \( y = x_{2n+1} \) in the hypothesis, we get,

\[
\min\{d(SPx_{2n}, Tx_{2n+1}), d(PPx_{2n}, SPx_{2n}), d(Qx_{2n+1}, Tx_{2n+1})
\]

\[ + \min\{d(PPx_{2n}, Tx_{2n+1}), d(Qx_{2n+1}, Sx_{2n})\} \]
\[ \leq a_1 d(Px_{2n}, Sx_{2n}) + a_2 d(Qx_{2n+1}, Tx_{2n+1}) \\
+ a_3 d(Px_{2n}, Tx_{2n+1}) + a_4 d(Qx_{2n+1}, Sx_{2n}) \\
+ a_5 d(Px_{2n}, Qx_{2n+1}) . \]

Taking \( n \rightarrow \infty \), we get

\[ d(Pz, z) \leq a_1 d(Pz, z) + a_2 d(z, z) + a_3 d(Pz, z) \\
+ a_4 d(z, z) + a_5 d(Pz, z) \]

or,

\[ d(Pz, z) \leq (a_1 + a_3 + a_5) d(Pz, z) \]

since \( a_1 + a_3 + a_5 < 1 \).

Thus,

\[ z = Pz. \]

Similarly, \( Tz = Qz \) and \( z = Qz \).

Thus \( z \) is a common fixed point of \( P, Q, S, T \).

It is easy to prove that \( z \) is unique.

4.3 Now we shall prove:

**Theorem 2:** Suppose \((X, d)\) is a metric space where \( d \) is continuous. Let \( P, Q, S, T \) be self-mapping on \( X \) and \( a_i \) are real numbers such that:

\[ (4.3.1) \quad \text{then for all } x, y \in M, \text{ if} \]


\[(4.3.2) \quad \min[d(Px, Qy), (d(Px, Sx)d(Qy, Ty))^\frac{1}{2}] \]

\[\quad + a \min \{d(Px, Ty), d(Qy, Sx)\} \leq a_1 d(Sx, Ty),\]

with \((4.2.2)\) to \((4.2.6)\) then \(z\) is the unique common fixed point of \(P, Q, S\) and \(T\).

**PROOF:** In the hypothesis \((4.3.2)\) we put

\[x = x_{2n} \quad \text{and} \quad y = x_{2n+1}, \quad \text{and we write;}\]

\[\min[d(Tx_{2n+1}, Sx_{2n+2}), (d(Tx_{2n+1}, Sx_{2n})d(Sx_{2n+2}, Tx_{2n+1}))^\frac{1}{2}] \]

\[\quad + a \min[d(Tx_{2n+1}, Tx_{2n+1}), d(Sx_{2n+2}, Sx_{2n})] \]

\[\quad \leq a_1 d(Sx_{2n}, Tx_{2n+1})\]

or

\[\min[d(y_{2n+1}, y_{2n+2}), (d(y_{2n}, y_{2n+1})d(y_{2n+1}, y_{2n+2}))^\frac{1}{2}] \]

\[\quad \leq a_1 d(y_{2n}, y_{2n+1})\]

This means, either

\[d(y_{2n+1}, y_{2n+2}) \leq a_1 d(y_{2n}, y_{2n+1})\]

or

\[d(y_{2n+1}, y_{2n+2}) \leq a_1^2 d(y_{2n}, y_{2n+1})\]
In either case:

\[ d(y_{2n+1}, y_{2n+2}) \leq a_1 \, d(y_{2n}, y_{2n+1}), \text{as } 0 < a_1 < 1 \]

Similarly, by putting

\[ x = x_{2n+2}, \quad y = x_{2n+1} \text{ in (4.3.2), we get} \]

\[ d(y_{2n+3}, y_{2n+2}) \leq a_1 \, d(y_{2n+2}, y_{2n+1}). \]

Thus

\[ T = 1.3332 \]

\[ d(y_{2n+2}, y_{2n+1}) \leq a_1 \, d(y_{2n+1}, y_{2n}) \]

Therefore \( \{Tx_1, Sx_2, Tx_3, Sx_4, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots, \} \)

is a Cauchy sequence. We know that

\[ Px_{2n} = Tx_{2n+1} \rightarrow z, \quad Qx_{2n+1} = Sx_{2n+2} \rightarrow z. \]

But \( PPx_{2n} \rightarrow Pz \) and since \( \{P, S\} \) is \( z \) asymptotically

\( \Delta \text{regular then.} \)

\[ d(SPx_{2n}, PSx_{2n}) \rightarrow 0. \]

Thus

\[ Sz = Pz. \]

Now put \( x = Px_{2n} \) and \( y = x_{2n+1} \) in the hypothesis, then

\[
\min \{ d(P Px_{2n}, Qx_{2n+1}), d(P Px_{2n}, SPx_{2n}) \cdot d(Qx_{2n+1}, Tx_{2n+1}) \}^{\frac{1}{2}}
+ a_1 \min \{ d(P Px_{2n}, Tx_{2n+1}), d(Qx_{2n+1}, Sx_{2n}) \}
\leq a_1 \, d(SPx_{2n}, Tx_{2n+1})
\]
Taking limit, we get \( d(Pz,z) \leq a_1 d(Pz,z) \).

So, \( z = Pz \).

Thus \( Tz = Qz \) and \( z = Qz \).

Therefore \( z \) is the fixed point of \( P, Q, S \) and \( T \). It is easy to prove that \( z \) is unique.

4.4 Now we shall prove:

**THEOREM 3:** Suppose \( (X,d) \) is a metric space where \( d \) is continuous. Let \( P, Q, S \) and \( T \) be self mapping on \( X \) and \( a, a_1 \) are real numbers such that

\[
\begin{align*}
(4.4.1) & \quad a_1 + a_3 > 0, \quad 0 \leq a_1 < 1, \text{then for all } x, y \in M, \text{ and } \\
a & \quad a < a_1 + a_2 + a_3 < 1, \text{ if }
\end{align*}
\]

\[
\begin{align*}
(4.4.2) & \quad a_1 d(Px,Sx) d(Qy,Ty) + a_2 [d(Sx,Ty)]^2 \\
& \quad + a_3 d(Px,Sx) d(Px,Qy) \\
& \quad + a_4 \min\{d(Px,Ty)d(Px,Qy), d(Qy,Ty)d(Qy,Sx)\} \\
& \quad \leq a d(Px,Sx) d(Sx,Ty)
\end{align*}
\]

and \((4.2.3)\) to \((4.2.6)\) of theorem 1 hold, then \( z \) is the unique common fixed point of \( P, Q, S \) and \( T \).

**PROOF:** In \((4.4.2)\) we put

\[
x = x_{2n} \quad \text{and} \quad y = x_{2n+1}, \quad \text{we get}
\]
\[
\begin{align*}
& a_1 d(T_{x_{2n+1}}, S_{x_{2n}}) d(S_{x_{2n+2}}, T_{x_{2n+1}}) + a_2 [d(S_{x_{2n}}, T_{x_{2n+1}})]^2 \\
+ & a_3 d(T_{x_{2n+1}}, S_{x_{2n}}) d(T_{x_{2n+1}}, S_{x_{2n+2}}) \\
+ & a_4 \min \{d(T_{x_{2n+1}}, T_{x_{2n+1}}), d(T_{x_{2n+1}}, S_{x_{2n+2}})\}, \\
& d(S_{x_{2n+2}}, T_{x_{2n+1}}) d(S_{x_{2n+2}}, S_{x_{2n}}) \\
\leq & a d(T_{x_{2n+1}}, S_{x_{2n}}) d(S_{x_{2n+2}}, T_{x_{2n+1}}) \\
\text{i.e. } & d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}) + a_2 [d(y_{2n}, y_{2n+1})]^2 \\
+ & a_3 d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}) \\
+ & a_4 \min \{d(y_{2n+1}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}, \\
& d(y_{2n+1}, y_{2n+2}) d(y_{2n+2}, y_{2n}) \\
\leq & a d(y_{2n}, y_{2n+1}) d(y_{2n}, y_{2n+1}) \\
\end{align*}
\]

Hence
\[
\begin{align*}
d(y_{2n+1}, y_{2n+2}) \leq \frac{a - a_2}{a_1 + a_3} d(y_{2n}, y_{2n+1}) \\
\end{align*}
\]

Similarly by putting \( x = x_{2n+2}, y = x_{n+1} \) in the hypothesis, we get
\[
\begin{align*}
d(y_{2n+3}, y_{2n+2}) \leq \frac{a - a_2}{a_1 + a_3} d(y_{2n+2}, y_{2n+1}) \\
\end{align*}
\]

So,
\[
\begin{align*}
d(y_{2n+2}, y_{2n+1}) \leq \frac{a - a_2}{a_1 + a_3} d(y_{2n+1}, y_{2n}) \\
\end{align*}
\]

Thus
\[
\{y_n\} \text{ i.e.} \\
\{T_{x_1}, S_{x_2}, T_{x_3}, S_{x_4}, \ldots, S_{x_{2n}}, T_{x_{2n+1}}, \ldots\} \\
\]

is a Cauchy sequence and with hypothesis, as \( n \to \infty \).
\[ P_{x_{2n}} = T_{x_{2n+1}} \rightarrow z, \quad Q_{x_{2n+1}} = S_{x_{2n+2}} \rightarrow z. \]

But \( PP_{x_{2n}} \rightarrow Pz \). We know that \( \{P,S\} \) is \( z \)-asymptotically regular, thus

\[ d(SP_{x_{2n}}, PS_{x_{2n}}) \rightarrow 0. \]

Now we put \( x = P_{x_{2n}} \) and \( y = x_{2n+1} \) in the hypothesis, we get

\[
\begin{align*}
& a_1 d(PP_{x_{2n}}, SP_{x_{2n}}) d(Q_{x_{2n+1}}, Tx_{2n+1}) + a_2 [d(SP_{x_{2n}}, Tx_{2n+1})]^2 \\
& + a_3 d(PP_{x_{2n}}, SP_{x_{2n}}) d(PP_{x_{2n}}, Q_{x_{2n+1}}) \\
& + a_4 \min\{d(PP_{x_{2n}}, Tx_{2n+1})d(PP_{x_{2n}}, Q_{x_{2n+1}}), \\
& \quad d(Q_{x_{2n+1}}, Tx_{2n+1})d(Q_{x_{2n+1}}, Sx_{2n})\} \\
& \leq a d(PP_{x_{2n}}, SP_{x_{2n}}) d(SP_{x_{2n}}, Tx_{2n+1})
\end{align*}
\]

Taking \( n \rightarrow \infty \), we get

\[
\begin{align*}
& a_1 d(Pz, z) d(z, z) + a_2 [d(z, z)]^2 + a_3 d(Pz, z) d(Pz, z) \\
& \leq a d(Pz, z) d(z, z)
\end{align*}
\]

Hence \( Pz = z \).

Similarly, \( Tz = Qz \) and \( z = Qz \).

Thus \( z \) is a common fixed point of \( P, Q, S \) and \( T \). It is easy to prove that \( z \) is unique.
4.5 Now we shall prove:

**THEOREM 4:** Suppose \((X,d)\) is a metric space where \(d\) is continuous. Let \(P,Q,S\) and \(T\) be self mappings on \(X\) and a and \(a_i\) \((i=1,2,3,4,)\) are real numbers such that:

\[(4.5.1)\quad a_2 + a_3 > 0 \text{ and } a_1 \neq a_2 + a_3\]

then for all \(x,y \in M\); if

\[(4.5.2)\quad a_1[d(Qy, Ty)]^2 + a_2(Px, Qy)d(Px, Sx) + a_3[d(Px, Qy)]^2 + a_4 \min d(Px, Sx)d(Qy, Ty), d(Px, Ty)d(Qy, Sx)\]

\[\leq a d(Px, Sx)d(Px, Qy)\]

and \((4.2.3)\) to \((4.2.6)\) of theorem 1 hold, then \(z\) is the unique common fixed point of \(P,Q,S\) and \(T\).

**PROOF:** In \((4.5.2)\) we put

\[x = x_{2n} \text{ and } y = x_{2n+1}\; \text{ we get}\]

\[a_1[d(Sx_{2n+2}, Tx_{2n+1})]^2 + a_2 d(Tx_{2n+1}, Sx_{2n+2})d(Tx_{2n+1}, Sx_{2n}) + a_3[d(Tx_{2n+1}, Sx_{2n+2})]^2 + a_4 \min d(Tx_{2n+1}, Sx_{2n})d(Sx_{2n+2}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+1})d(Sx_{2n+2}, Sx_{2n})\]

\[\leq a d(Tx_{2n+1}, Sx_{2n}) d(Tx_{2n+1}, Sx_{2n+2})\]
\[ i.e. \quad a_1[d(y_{2n+1}, y_{2n+2})]^2 + a_2 d(y_{2n+1}, y_{2n+2}) d(y_{2n}, y_{2n+1}) + a_3 [d(y_{2n+1}, y_{2n+2})]^2 \]
\[ + a_4 \min(d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+1}) d(y_{2n+2}, y_{2n})) \] \[ \leq a d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}) \]

Hence
\[ a_1 d(y_{2n+1}, y_{2n+2}) + a_2 d(y_{2n}, y_{2n+1}) + a_3 d(y_{2n+1}, y_{2n+2}) \]
\[ \leq a d(y_{2n}, y_{2n+1}) \]

which implies
\[ d(y_{2n+1}, y_{2n+2}) \leq \frac{a - a_2}{a_1 + a_3} d(y_{2n}, y_{2n+1}) \]

Similarly by putting \( x = x_{2n+2}, \ y = x_{2n+1} \) in the hypothesis, we get
\[ d(y_{2n+3}, y_{2n+2}) \leq \frac{a - a_2}{a_1 + a_3} d(y_{2n+2}, y_{2n+1}). \]

Thus
\[ d(y_{2n+2}, y_{2n+1}) \leq \frac{a - a_2}{a_1 + a_3} d(y_{2n+1}, y_{2n}) \]
Thus \( \{y_n\} \) i.e.

\[
\{T_{x_1}, S_{x_2}, T_{x_3}, S_{x_4}, \ldots, S_{x_{2n}}, T_{x_{2n+1}}, \ldots\}
\]

is a Cauchy sequence and with hypothesis, as \( n \to \infty \)

\( P_{x_{2n}} = T_{x_{2n+1}} \to z, \ Q_{x_{2n+1}} = S_{x_{2n+2}} \to z. \)

But \( PP_{x_{2n}} \to Pz. \) We know that \( \{P, S\} \) is \( z \)-asymptotically regular, Thus

\[
d(\text{SP}_{x_{2n}}, \text{PS}_{x_{2n}}) \to 0.\]

Now we put \( x = P_{x_{2n}} \) and \( y = x_{2n+1} \) in the hypothesis, we get

\[
a_1[d(Q_{x_{2n+1}}, T_{x_{2n+1}})]^2 + a_2 d(PP_{x_{2n}}, Q_{x_{2n+1}}) d(PP_{x_{2n}}, SP_{x_{2n}})
+a_3[d(PP_{x_{2n}}, Q_{x_{2n+1}})]^2
+a_4 \min\{d(PP_{x_{2n}}, SP_{x_{2n}}), d(Q_{x_{2n+1}}, T_{x_{2n+1}}),
\]
\[
d(PP_{x_{2n}}, T_{x_{2n+1}}) d(Q_{x_{2n+1}}, S_{x_{2n}})\}
\]
\[
\leq a d(PP_{x_{2n}}, SP_{x_{2n}}) d(PP_{x_{2n}}, Q_{x_{2n+1}}).
\]

Taking \( n \to \infty \) we get

\[
a_2[d(Pz, z)]^2 + a_3[d(Pz, z)]^2 \leq a[d(Pz, z)]^2
\]
which gives contradiction, hence

\[ Pz = z. \]

Similarly, \( Tz = Qz \) and \( z = Qz. \)

Thus \( z \) is the common fixed point of \( P, Q, S \) and \( T \). It is easy to prove that \( z \) is unique.

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