CHAPTER III

Fixed Point Theorems in Complete 2-Metric Space
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FIXED POINT THEOREMS IN COMPLETE 2-METRIC SPACE.

3.1 Let \((M,d)\) be a metric space. A mapping \(T\) on \(M\) is orbitally continuous if \(\lim_{i} T^{n_i}x = z\) implies \(\lim_{i} T^{n_i}x = Tz\) for each \(x\) in \(M\). A space \(M\) is said to be \(T\)-orbitally complete if every Cauchy sequence of the form \(\{T^{n_i}x\}_{i=1}^{\infty}\), \(x\) in \(M\), converges in \(M\).

Ciric [144] has obtained a result regarding the existence of non-unique fixed points for orbitally continuous self mapping on a orbitally complete metric space:

**THEOREM C:** Let \((M,d)\) be orbitally complete metric space and \(T\) be a orbitally continuous self mapping of \(X\), satisfying:

\[(3.1.1) \quad \min \{d(Tx, Ty), d(x, Tx), d(y, Ty)\} \leq \min \{d(x, Ty), d(y, Tx)\} \leq q d(x, y)\]

[144] Ciric, L.B. (17)
for all \(x, y\) in \(M\) and \(q \in [0, 1]\).

Then for each \(x\) in \(M\) the sequence \(\{T^n x\}\) converges to a fixed point of \(T\).

Recently Dhage [145] obtain the following generalization of theorem C:

**Theorem D:** Let \(T : M \rightarrow M\) be an orbitally continuous self map of a metric space \(M\) and let \(M\) be \(T\)-orbitally complete.

If \(T\) satisfies the condition:

\[
(3.1.2) \quad \min \{d(Tx, Ty), d(x, Tx), d(y, Ty)\} \\
+ a \min \{d(x, Ty), d(y, Tx)\} \leq p \, d(x, y) + q \, d(x, Tx),
\]

for all \(x, y\) in \(M\) and \(a, p, q\) are real numbers such that \(0 < p + q < 1\), then for each \(x\) in \(M\), the sequence \(\{T^n x\}\) converges to a fixed point of \(T\).

Recently, Pachpatte [146] proved the following non-unique fixed point theorem for Cirić type mapping:

\[\text{[145]} \quad \text{Dhage, B.C.} \quad (21)\]
\[\text{[146]} \quad \text{Pachpatte, B.C.} \quad (87)\]
**THEOREM P**: Let $T : M \rightarrow M$ be an orbitally continuous mapping on $M$ and let $M$ be $T$-orbitally complete. If $T$ satisfies:

$$
(3.1.3) \quad \min\{([d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2)
- \min\{d(x, Tx) \ d(y, Ty), \ d(x, Ty) \ d(y, Tx)\}
\leq q \ d(x, Tx) \ d(y, Ty)
$$

for all $x, y \in M$ and $q \in (0, 1)$, then for each $x \in M$, then sequence $(T^n x)_{n=1}^{\infty}$ converges to a fixed point of $T$.

Many authors have extended and generalized Ciric theorem, Iseki [147], Pachpatte [148], Mishra [149], Lal and Das [150], Narayan, Thapliyal and Virendra [151].

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[147] Iseki, K. (44)
[148] Pachpatte, B.G. (88)
[149] Mishra, S.N. (80)
[150] Lal, S.N. and Das M. (70)
[151] Narayanan, K.A; Thapliyal, P.S. and Virendra. (85)
Singh and Iseki [152], Cho [153], [154], Dhage [155], T.Som [156], Pathak [157], Argyros [158].

In theorem of Ciric and Dhage fixed point is not unique.

3.2 In a Paper Gahler [159] investigated the notion of 2-metric: a real valued function for a point triples on a set X, whose abstract properties, were suggested by the area function for a triangle determined by a triple in Euclidean space. The theory of 2-metric space has been extensively studied and developed by Gahler [160], [161],

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White [162], Iseki [163] and others,

We need some definitions:

**DEFINITION 1:** A 2-metric space in a space $X$, with a real valued function $d$, defined on $X \times X \times X$ satisfying:

(3.2.1) For two distinct points, $a, b$ there is a point $c$ such that

\[ d(a, b, c) \neq 0, \]

(3.2.2) $d(a, b, c) = 0$ if at least two of three points are equal,

(3.2.3) $d(a, b, c) = d(a, c, b) = d(b, c, a)$ in symmetric about the three variables.

(3.2.4) $d(a, b, c) < d(a, b, p) + d(a, p, c) + d(p, b, c)$.

**DEFINITION 2:** A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if,

(3.2.5) $\lim_{m, n} d(x_m, x_n, a) = 0$ for all $x$ in $X$.

[162] White, A.G. (141)

[163] Iseki, K. (47)
**Definition 3:** The sequence \( \{x_n\} \) is convergent to \( x \) in \( X \) and \( x \) is said to be the limit point of the sequence if,

\[
\lim_{n \to \infty} d(x_n, x, a) = 0 \text{ for some } x \text{ in } X.
\]

A 2-metric space in which every Cauchy sequence converges is called complete.

Iseki, Sharma and Sharma [164], Rao and Rao [165], Khan [166], [167], Singh [168] and many others have studied contractive mapping on 2-metric space.

Gahler has proved that the two metric \( d \) is continuous function of any one coordinate (\( x \) or \( y \) or \( z \)) but it may not be continuous for two coordinates, if it is then it is continuous in all three coordinates.

**Definition 4:** (Compare Sessa [169]) Two mappings \( P \) and \( T \) on 2-metric space \( X \) are weakly commuting iff,

\[
d(PTx, TPx, a) \leq d(Px, Tx, a)
\]

for all \( x, a \) in \( X \).

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[164] Iseki, K., Sharma, P.L. and Sharma, B.K. (42)
[165] Rao, I.H.N. and Rao, K.P.R. (96)
[166] Khan, M.S. (59)
[167] Khan, M.S. (60)
[168] Singh, S.L. (123)
[169] Sessa, S. (116)
**DEFINITION 5:** (Compare Singh and Ram [170], Jungck [171].)

Two self mappings $P$ and $T$ on 2-metric space are said to be asymptotically commuting iff,

$$
\lim_{n \to \infty} d(PT_{x_n}, TP_{x_n}) = 0, \\
\{x_n\} \text{ is a sequence in } X \text{ such that for any point } u \text{ in } X,
$$

$$
\lim_{n \to \infty} Px_n = \lim_{n \to \infty} Tx_n = u.
$$

Sessa [172] has shown that commuting and also weakly commuting mapping with (3.2.9) are asymptotically commuting.

### 3.3

The object of this chapter is to prove:

**THEOREM 1:** Suppose $(X,d)$ is a 2-metric space where $d$ is continuous. Let $P,Q,T$ be self mappings on $X$ and $a_i, a_i$ ($i = 1, 2, ..., 9$) are real numbers such that

$$
0 < a_i < 1 \text{ and } \sum_{i=1}^{9} a_i < 1,
$$

$$
a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1, \text{then for all } x,y,a \text{ in } X, \text{ and}
$$

[170] Singh, S.L. and Ram, B. (132)

[171] Jungck, G. (53)

(3.3.3) \[ \min \{d(Px,Qy,a), d(Tx,Px,a), d(Ty,Qy,a)\} \]

\[ + \min \{d(Tx,Qy,a), d(Ty,Px,a)\} \]

\[ \leq a_1 d(Tx,Px,a) + a_2 d(Ty,Qy,a) + a_3 d(Tx,Qy,a) \]

\[ + a_4 d(Ty,Px,a) + a_5 d(Tx,TPx,a) + a_6 d(Ty,TPx,a) \]

\[ + a_7 d(Px,TPx,a) + a_8 d(Qy,TPx,a) + a_9 d(Tx,Ty). \]

(3.3.4) For any point \( x_0 \) in \( X \), \( \{x_n\} \) is such that

\[ Tx_{2n+1} \neq Px_{2n}, \quad Tx_{2n+2} = Qx_{2n+1} \]

\[ Tx_{2n+1} \neq Tx_{n+2}, \quad n = 0,1,2, \ldots \]

(3.3.5) any one sequence of \( \{Tx_n\} \) converges to any point \( z \) of \( X \);

(3.3.6) mapping \( P,Q,T \) are continuous at point \( z \);

(3.3.7) the pairs \( \{T,P\} \) and \( \{T,Q\} \) are \( z \)-asymptotically commuting,

then \( z \) is a coincident point of the mappings \( P,Q,T \) i.e. \( Pz = Qz = Tz \). Again if \( \frac{k}{a} \epsilon (0,1) \) then \( z \) is a unique fixed point of \( P,Q \) and \( T \).
\textbf{Proof:} We put } x = x_{2n} \text{ and } y = x_{2n+1} \text{ in (3.3.3)} \text{ and write,}

\[
\min \{d(Tx_{2n+1}, Tx_{2n+2}, a), d(Tx_{2n}, Tx_{2n+1}, a), d(Tx_{2n+1}, Tx_{2n+2}, a)\}
\]

\[
\quad + a \min \{d(Tx_{2n}, Tx_{2n+2}, a), d(Tx_{2n+1}, Tx_{2n+1}, a)\}
\]

\[
\leq a_1 d(Tx_{2n}, Tx_{2n+1}, a) + a_2 d(Tx_{2n+1}, Tx_{2n+2}, a)
\]

\[
\quad + a_3 d(Tx_{2n}, Tx_{2n+2}, a) + a_4 d(Tx_{2n+1}, Tx_{2n+1}, a)
\]

\[
\quad + a_5 d(Tx_{2n}, Tx_{2n+2}, a) + a_6 d(Tx_{2n+1}, Tx_{2n+2}, a)
\]

\[
\quad + a_7 d(Tx_{2n+1}, Tx_{2n+2}, a) + a_8 d(Tx_{2n+2}, Tx_{2n+2}, a)
\]

\[
\quad + a_9 d(Tx_{2n}, Tx_{2n+1}, a)
\]

\text{i.e.}

\[
\min \{d(Tx_{2n+1}, Tx_{2n+2}, a), d(Tx_{2n}, Tx_{2n+1}, a)\}
\]

\[
\leq (a_1 + a_3 + a_5 + a_9) d(Tx_{2n}, Tx_{2n+1}, a)
\]

\[
\quad + (a_2 + a_3 + a_5 + a_6 + a_7) d(Tx_{2n+1}, Tx_{2n+2}, a)
\]

\[
\text{or, } d(Tx_{2n+1}, Tx_{2n+2}, a) \leq \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_9} d(Tx_{2n}, Tx_{2n+1}, a)
\]

Similarly, we obtain by putting

\[
x = x_{2n+1}, \quad y = x_{2n+2}
\]

\[
d(Tx_{2n+2}, Tx_{2n+3}, a) \leq \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_9} d(Tx_{2n+1}, Tx_{2n+2}, a)
\]
Thus we have

\[(3.3.8) \quad d(Tx_{2n+1}, Tx_{2n+2}, a) \leq k \cdot d(Tx_{2n}, Tx_{2n+1}, a)\]

where \( k = \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_9} \in (0, 1), \)

\[\leq k^2 \cdot d(Tx_{2n-1}, Tx_{2n}, a)\]

Proceeding in this manner, we get in the usual way \( \{Tx_n\} \)

is a Cauchy sequence, thus from \((3.3.5)\) \( Tx_{2n} \rightarrow z, \)

\( Px_{2n} \rightarrow z \) and \( Qx_{2n+1} \rightarrow z \) and from \((3.3.6)\) \( PTx_{n_i} \rightarrow Pz. \)

and \( TPx_{n_i} \rightarrow Tz, \) where \( \{n_i\} \) is a subsequence of \( \{n\}. \)

Since \( P \) and \( T \) are asymptotically commuting, therefore for \( a \) in \( X, \)

\[\lim_{n_i} d(PTx_{n_i}, TPx_{n_i}, a) = 0,\]

and due to continuity of \( d \) for each \( a \) in \( X, \)

\[d(Pz, Tz, a) = 0.\]

Therefore, \( Pz = Tz. \) Similarly, \( Qz = Tz. \)

Put \( x = x_{2n} \) and \( y = z \) and taking limits, we get

\[d(z, Tz, a) \leq \frac{k}{a} \cdot d(z, Tz, a)\]
Thus \( Tz = z \), \( z \) is the common fixed point of \( P, Q, T \). It is easy to prove that \( z \) is unique. Thus completes the proof of theorem 1.

**Remark 1:** If we put \( a = -1 \) and \( a_i = 0 \) for \( i = 1, 2, \ldots, 8 \) in the hypothesis and \( P = Q, T = I \) (Identity mapping), then for metric space we get the above mentioned theorem of Ciric.

**Remark 2:** If we put \( a_i = 0 \) for \( i = 2, 3, \ldots, 8 \) in the hypothesis, and \( P = Q, T = I \), then we get for metric space the result of Dhage (mentioned above).

3.4 Now we shall start with \( A(X) \subset S(X) \cap T(X) \) and prove:

**Theorem 2:** Suppose \((X, d)\) is a 2-metric space where \( d \) is continuous. Let \( A, S, T \) be self mapping on \( X \) and \( a, a_i \) for \( i = (i = 1, 2, \ldots, 9) \) are real number such that

\[
\begin{align*}
(3.4.1) & \quad 0 \leq a_i < 1 \text{ and } \sum_{i=1}^{9} a_i < 1, \\
(3.4.2) & \quad a_1 + a_2 + 2a_3 + 2a_5 + a_6 + a_7 + a_9 < 1, \text{ then for all } x, y, z \text{ in } X \text{ and }
\end{align*}
\]
(3.4.3) \[ \min\{d(Ax, Ay, a), d(Sx, Ax, a), d(Ty, Ay, a)\} \]
+ \(a \min\{d(Sx, Ay, a), d(Ty, Ax, a)\}\)
\[\leq a_1 d(Sx, Ax, a) + a_2 d(Ty, Ay, a) + a_3 d(Sx, Ay, a)\]
+ \(a_4 d(Ty, Ax, a) + a_5 d(Sx, SAx, a) + a_6 d(Ty, SAx, a)\)
+ \(a_7 d(Ax, SAx, a) + a_8 d(Ay, SAx, a) + a_9 d(Sx, Ty)\).

(3.4.4) for any point \(x_0\) in \(X\), \(\{x_n\}\) is such that

\[S_{x_2n+1} = Ax_{2n}, \quad T_{x_2n+2} = Ax_{2n+1}\]

\[Ax_{n+1} \neq Ax_{n+2}, \quad n = 0, 1, 2, \ldots\]

(3.4.5) any one sequence of \(\{Ax_n\}\) converges to any point \(z\) in \(X\);  

(3.4.6) mappings \(A, S, T\) are continuous at point \(z\);  

(3.4.7) the pairs \(\{A, S\}\) and \(\{A, T\}\) are \(z\)-asymptotically commuting

then \(z\) is a coincident point of the mapping \(A, S, T\) i.e. \(Az = Sz = Tz\). Again if \(\frac{k}{a} \in (0, 1)\), then \(z\) is a unique fixed point of \(A, S\) and \(T\).

**PROOF:** By putting \(x = x_{2n}\) and \(y = x_{2n+1}\) in the hypothesis, we write:
\[
\min\{d(Ax_{2n}, Ax_{2n+1}, a), d(Ax_{2n-1}, Ax_{2n}, a), d(Ax_{2n}, Ax_{2n+1}, a)\}
\]

\[+ a \min\{d(Ax_{2n-1}, Ax_{2n+1}, a), d(Ax_{2n}, Ax_{2n}, a)\}\]

\[\leq a_1 d(Ax_{2n-1}, Ax_{2n}, a) + a_2 d(Ax_{2n}, Ax_{2n+1}, a)\]

\[+ a_3 d(Ax_{2n-1}, Ax_{2n+1}, a) + a_4 d(Ax_{2n}, Ax_{2n}, a)\]

\[+ a_5 d(Ax_{2n-1}, Ax_{2n+1}, a) + a_6 d(Ax_{2n}, Ax_{2n+1}, a)\]

\[+ a_7 d(Ax_{2n}, Ax_{2n+1}, a) + a_8 d(Ax_{2n+1}, Ax_{2n+1}, a)\]

\[+ a_9 d(Ax_{2n-1}, Ax_{2n}, a)\]

i.e. \[
\min\{d(Ax_{2n}, Ax_{2n+1}, a), d(Ax_{2n-1}, Ax_{2n}, a)\}\]

\[\leq (a_1 + a_3 + a_5 + a_9) d(Ax_{2n-1}, Ax_{2n}, a)\]

\[+ (a_2 + a_3 + a_5 + a_6 + a_7) d(Ax_{2n}, Ax_{2n+1}, a)\]

\[\leq \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_9} d(Ax_{2n-1}, Ax_{2n}, a).\]

Similarly we obtain by putting \(x = x_{2n+1}\) and \(y = x_{2n+2}\)

\[d(Ax_{2n+1}, Ax_{2n+2}, a) \leq \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_9} d(Ax_{2n}, Ax_{2n+1}, a).\]
Thus we have

$$(3.4.8) \quad d(Ax_{2n}, Ax_{2n+1}, a) \leq k \ d(Ax_{2n-1}, Ax_{2n}, a)$$

where

$$k = \frac{a_1 + a_3 + a_5 + a_9}{1 - a_2 - a_3 - a_5 - a_6 - a_9} \in (0, 1)$$

$$\leq k^2 \ d(Ax_{2n-2}, Ax_{2n-1}, a)$$

Proceeding in this manner, we get in the usual way \( \{Ax_n\} \) is a Cauchy sequence thus from \((3.4.5)\) \( Ax_{2n} \rightarrow z, Sx_{2n+1} \rightarrow z, \)

\( Tx_{2n+2} \rightarrow z \) and from \((3.4.6)\) \( ATx_{n_i} \rightarrow Az \) and \( TAx_{n_i} \rightarrow Tz \)

where \( \{n_i\} \) is a subsequence of \( \{n\} \). Since \( A \) and \( T \) are asymptotically commuting, therefore for \( a \) in \( X, \)

$$\lim_{n_i} d(ATx_{n_i}, TAx_{n_i}, a) = 0, \quad \forall n_i \$$

and due to continuity of \( d \) for each \( a \) in \( X, \)

$$d(Az, Tz, a) = 0$$

Hence,

$$Az = Tz$$

and similarly \( Az = Sz \).

By putting \( x = x_{2n} \) and \( y = z \) and taking limit in the hypothesis we write

$$d(z, Az, a) \leq \frac{k}{a} \ d(z, Az, a).$$

Thus \( Az = z \), \( z \) is the common fixed point of \( A, S, T \). It is easy to prove that \( z \) is unique.
3.5 Now we shall prove the following theorem:

**THEOREM 3**: Suppose \((X,d)\) is a 2-metric space where \(d\) is continuous. Let \(P,Q,T\) be self mapping on \(X\) and \(a, a_1(i = 1, 2, \ldots, 5)\) are real numbers such that

\[0 \leq a_1 + a_3 + a_5 < 1,\]

then for all \(x, y, a\) in \(X\) and

\[
\begin{align*}
(3.5.1) \quad & \min\{d(Px, Qy, a) + d(Tx, Ty, a)d(Px, Qy, a), \min\{d(Tx, Ty, a)d(Px, Qy, a), d(Ty, Qy, a)\}^2 \} \\
& \quad + a_1 d(Tx, Px, a)d(Ty, Qy, a) + a_2 d(Tx, Ty, a)d(Ty, Px, a) \\
& \quad + a_3 d(Ty, TPy, a)d(Tx, Ty, a) + a_4 d(Tx, TPy, a)d(Qy, TPy, a) \\
& \quad + a_5 d(Px, TPy, a)d(Tx, Ty),
\end{align*}
\]

\[
(3.5.2) \quad \text{For any point } x_0 \text{ in } X, \{x_n\} \text{ is such that } \\
Tx_{2n+1} = Px_{2n}, Tx_{2n+2} = Qx_{2n+1}, \\
Tx_{n+1} \neq Tx_{n+2}, \text{ for } n = 0, 1, 2, \ldots.
\]

\[
(3.5.3) \quad \text{any one sequence of } \{Tx_n\} \text{ converges to any point } z \text{ of } X;
\]

\[
(3.5.4) \quad \text{mappings } P, Q, T \text{ are continuous at point } z;
\]

\[
(3.5.5) \quad \text{the pairs } \{T, P\} \text{ and } \{T, Q\} \text{ are } z\text{-asymptotically commuting},
\]
then $z$ is coincident point of the mapping $P,Q,T$ i.e. $Pz = Qz = Tz$. Again if $\frac{k}{a} \in (0,1)$ then $z$ is a unique fixed point of $P,Q$ and $T$.

**Proof:** We put $x = x_{2n}$ and $y = x_{2n+1}$ in (3.5.1) and write

$$\min\{d(Tx_{2n+1}, Tx_{2n+2}, a)^2, d(Tx_{2n}, Tx_{2n+1}, a) d(Tx_{2n+1}, Tx_{2n+2}, a), \{d(Tx_{2n+1}, Tx_{2n+2}, a)^2 + a \min\{d(Tx_{2n}, Tx_{2n+1}, a) d(Tx_{2n+1}, Tx_{2n+2}, a)\}, d(Tx_{2n}, Tx_{2n+2}, a) d(Tx_{2n+1}, Tx_{2n+1}, a)\} \leq a_1 d(Tx_{2n}, Tx_{2n+1}, a) d(Tx_{2n+1}, Tx_{2n+2}, a) + a_2 d(Tx_{2n}, Tx_{2n+2}, a) d(Tx_{2n+1}, Tx_{2n+1}, a) + a_3 d(Tx_{2n+1}, Tx_{2n+2}, a) d(Tx_{2n}, Tx_{2n+1}, a) + a_4 d(Tx_{2n}, Tx_{2n+2}, a) d(Tx_{2n+2}, Tx_{2n+2}, a) + a_5 d(Tx_{2n+1}, Tx_{2n+2}, a) d(Tx_{2n}, Tx_{2n+1}, a) \leq (a_1 + a_3 + a_5) d(Tx_{2n}, Tx_{2n+1}, a) d(Tx_{2n+1}, Tx_{2n+2}, a)$
Since

\[ d(T_{x2n}, T_{x2n+1}, a) d(T_{x2n+1}, T_{x2n+2}, a) \]

\[ \leq (a_1 + a_3 + a_5) d(T_{x2n}, T_{x2n+1}, a) d(T_{x2n+1}, T_{x2n+2}, a) \]

is impossible, as \( a_1 + a_3 + a_5 < 1 \). Thus we have

\[ [d(T_{x2n+1}, T_{x2n+2}, a)]^2 \]

\[ \leq (a_1 + a_3 + a_5) d(T_{x2n}, T_{x2n+1}, a) d(T_{x2n+1}, T_{x2n+2}, a) \]

i.e.

\[ (T_{x2n+1}, T_{x2n+2}, a) \leq (a_1 + a_3 + a_5) d(T_{x2n}, T_{x2n+1}, a) \]

The rest of the proof follows from theorem 1.

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