CHAPTER - III

RELATED FIXED POINT THEOREMS FOR SET-VALUED MAPPINGS ON METRIC SPACES

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SET-VALUED MAPPINGS ON METRIC SPACES

(3.1) In 1975, Kaulgud and Pai [1] introduced the concept of set-valued mappings in fixed point theory. In 1981, Fisher [4] proved fixed point theorems on set-valued mappings. Fisher ([3], [5]) proved important related fixed point theorems. In 2000, Fisher and Turkoglu [1] have proved related fixed point theorems for set-valued mappings. We have extended these results in this chapter.

In this Chapter, we have proved related fixed point theorem one each for set-valued and single-valued mappings on complete metric spaces in section 3.2. In section 3.3, we have proved related fixed point theorems for set-valued mappings on three complete metric spaces and in section 3.4, we have proved related fixed point theorems for two pairs of set-valued mappings on two complete metric spaces.

Let (X, d) be a complete metric space and let B(X) be the set of all nonempty bounded subsets of X. The function \( \delta(A, B) \) with A and B in B(X) is defined by

\[
\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}
\]

If A consists of a single point a, we write

The result of section 3.3 has been published in Ultra Scientist Vol. 14(2), 186-191 (2002) under title "A related fixed point theorem for set-valued mappings on three metric spaces".
\[ \delta(A, B) = \delta(a, B) \]

and if \( B \) also consists of a single point \( b \), we write
\[ \delta(A, B) = \delta(a, b) = d(a, b) \]

It follows easily from the definition that
\[ \delta(A, B) = \delta(B, A) \geq 0, \]
\[ \delta(A, B) \leq \delta(A, C) + \delta(C, B) \]
for all \( A, B \) and \( C \) in \( B(X) \).

Now let \( \{A_n : n = 1, 2, \ldots\} \) be a sequence of nonempty subsets of \( X \). We say that the sequence \( \{A_n\} \) converges to the subset \( A \) of \( X \) if:

(i) each point \( a \) in \( A \) is the limit of a convergent sequence \( \{a_n\} \), where \( a_n \) is in \( A_n \) for \( n = 1, 2, \ldots \),

(ii) for arbitrary \( \varepsilon > 0 \), there exists an integer \( N \) such that \( A_n \subset A_\varepsilon \) for \( n > N \), where \( A_\varepsilon \) denotes the set of all points \( x \) in \( X \) for which there exists a point \( a \) in \( A \), depending on \( x \), such that \( d(x, a) < \varepsilon \). \( A \) is then said to be the limit of sequence \( \{A_n\} \). It follows easily from the definition that if \( A \) is the limit of sequence \( \{A_n\} \) then \( A \) is closed.

The following lemma was proved in [4].

**Lemma 1.** If \( \{A_n\} \) and \( \{B_n\} \) are sequences of bounded subsets of a complete metric space \( (X, d) \) which converges to bounded subsets \( A \) and \( B \) respectively, then the sequence \( \{\delta(A_n, B_n)\} \) converges to \( \delta(A, B) \).

Let \( F \) be a mapping of \( X \) into \( B(X) \). We say that the mapping \( F \) is continuous at a point \( x \) in \( X \) if whenever \( \{x_n\} \) is a sequence of points in \( X \) converging to \( x \), the sequence \( \{Fx_n\} \) in \( B(X) \) converges to \( Fx \) in \( B(X) \).

We say that \( F \) is continuous mapping of \( X \) into \( B(X) \) if \( F \) is continuous at each point \( x \) in \( X \). We say that a point \( z \) in \( X \) is a fixed point of \( F \) if \( z \) is in
F. If \( A \) is in \( B(X) \) we define the set 
\[
F_A = \bigcup_{a \in A} F_a.
\]

Fisher and Turkoglu [1] proved the following related fixed point theorem for set-valued mappings.

**Theorem A.** Let \( (X, d_1) \) and \( (Y, d_2) \) be complete metric spaces, let \( F \) be a mapping of \( X \) into \( B(Y) \) and \( G \) be mapping of \( Y \) into \( B(X) \) satisfying the inequalities

\[
\delta_1(GFx, GFx') \leq c \max \{d_1(x, x'), \delta_1(x, GFx), d_1(x', GFx'), \delta_2(Fx, Fx')\},
\]

\[
\delta_2(FGy, FGy') \leq c \max \{d_2(y, y'), \delta_2(y, FGy), d_2(y', FGy'), \delta_1(Gy, Gy')\},
\]

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \), where \( 0 \leq c < 1 \). If \( F \) is continuous, then \( GF \) has a unique fixed point \( z \) in \( X \) and \( FG \) has a unique fixed point \( w \) in \( Y \).

(3.2) In this section, we have extended the result of Fisher and Turkoglu [1] and proved following one related fixed point theorem for set-valued mappings and one related fixed point theorem for single-valued mappings using different contractive conditions on complete metric spaces.

**Theorem 1.** Let \( (X, d_1) \) and \( (Y, d_2) \) be complete metric spaces, let \( F \) be mapping of \( X \) into \( B(Y) \) and \( G \) be mapping of \( Y \) into \( B(X) \) satisfying the inequalities

\[
\delta_1(GFx, GFx') \leq c \frac{f(x, x', y, y')}{h(x, x')},
\]

\[
\delta_2(FGy, FGy') \leq c \frac{g(x, x', y, y')}{k(y, y')},
\]

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for all $x, x'$ in $X$ and $y, y'$ in $Y$, such that $h(x, x') \neq 0$ and $k(y, y') \neq 0$, where

$$f(x, x', y, y') = \max \left\{ \left[ d_1(x, x') \right]^2, \delta_2(Fx, Fx') \delta_1(Gy, Gy') \right\},$$

$$g(x, x', y, y') = \max \left\{ \left[ d_2(y, y') \right]^2, \delta_1(Gy, Gy') \delta_2(Fx, Fx') \right\},$$

$$h(x, x') = \max \left\{ d_1(x, x'), \delta_1(GFx, GFx'), \delta_2(Fx, Fx') \right\},$$

$$k(y, y') = \max \left\{ d_2(y, y'), \delta_2(FGy, FGy'), \delta_1(Gy, Gy') \right\},$$

and $0 \leq c < 1$. If $F$ is continuous, then $GF$ has a fixed point $z$ in $X$ and $FG$ has a fixed point $w$ in $Y$. Further, $Fz = \{w\}$ and $Gw = \{z\}$.

**Proof.** Let $x_1$ be an arbitrary point in $X$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $X$ and $Y$ respectively as follows. Choose a point $y_1$ in $Fx_1$ and then a point $x_2$ in $Gy_1$. In general, having chosen $x_n$ in $X$ and $Y_n$ in $Y$, choose $x_{n+1}$ in $Gy_n$ and then $y_{n+1}$ in $Fx_{n+1}$ for $n=1, 2, \ldots$.

Suppose first of all that $h(x_n, x_{n+1}) = 0$ for some $n$. Then

$$\max \left\{ d_1(x_n, x_{n+1}), \delta_1(GFx_n, GFx_{n+1}), \delta_2(Fx_n, Fx_{n+1}) \right\} = 0$$

and so on putting $x_n = x_{n+1} = z$ and $y_n = y_{n+1} = w$, we see that

$$GFz = \{z\}, Fz = \{w\}, FGw = \{w\}, Gw = \{z\} \quad (1)$$

Similarly, $k(y_n, y_{n+1}) = 0$ for some $n$ implies equations (1) hold.

We will now suppose that $h(x_n, x_{n+1}) \neq 0$ and $k(y_n, y_{n+1}) \neq 0$ for all $n$. Then using inequality (3.2.1), we have

$$d_1(x_{n+1}, x_{n+2}) \leq \delta_1(GFx_n, GFx_{n+1}),$$

$$\leq c \frac{f(x_n, x_{n+1}, y_n, y_{n+1})}{h(x_n, x_{n+1})} \quad (2)$$

If
\[ h(x_n, x_{n+1}) = \max\{d_1(x_n, x_{n+1}), \delta_1(GFx_n, GFx_{n+1}), \delta_2(Fx_n, Fx_{n+1})\} \]
\[ = d_1(x_n, x_{n+1}) \]

then inequality (2) gives
\[ d_1(x_n, x_{n+1}) \delta_1(GFx_n, GFx_{n+1}) \]
\[ \leq c \max \{[d_1(x_n, x_{n+1})]^2, \delta_2(Fx_n, Fx_{n+1}) \delta_1(Gy_n, Gy_{n+1}), \delta_2(y_n, Fx_{n+1}) \delta_2(y_{n+1}, Fx_n), \delta_1(x_n, Gy_n) \delta_1(x_{n+1}, Gy_{n+1})\} \]
\[ \leq c \max \{d_1(x_n, x_{n+1}) \delta_1(GFx_{n-1}, GFx_n), \delta_1(GFx_n, GFx_{n+1}) \delta_2(FGy_{n-1}, FGy_n)\}, \]

and so
\[ d_1(x_{n+1}, x_{n+2}) \leq \delta_1(GFx_n, GFx_{n+1}) \]
\[ \leq c \max \{\delta_1(GFx_{n-1}, GFx_n), \delta_2(FGy_{n-1}, FGy_n)\} \quad (3) \]

Next, if
\[ \max\{d_1(x_n, x_{n+1}), \delta_1(GFx_n, GFx_{n+1}), \delta_2(Fx_n, Fx_{n+1})\} \]
\[ = \delta_1(GFx_n, GFx_{n+1}) \]

then inequality (2) gives
\[ [\delta_1(GFx_n, GFx_{n+1})]^2 \leq c \max \{[d_1(x_n, x_{n+1})]^2, \delta_2(Fx_n, Fx_{n+1}) \delta_1(Gy_n, Gy_{n+1}), \delta_2(y_n, Fx_{n+1}) \delta_2(y_{n+1}, Fx_n), \delta_1(x_n, Gy_n) \delta_1(x_{n+1}, Gy_{n+1})\} \]
\[ \leq c \max \{\delta_1(GFx_n, GFx_{n+1}) \delta_1(GFx_{n-1}, GFx_n), \delta_1(GFx_n, GFx_{n+1}) \delta_2(FGy_{n-1}, FGy_n)\} \]

and again inequality (3) holds.

Finally, if
\[ \max\{d_1(x_n, x_{n+1}), \delta_1(GFx_n, GFx_{n+1}), \delta_2(Fx_n, Fx_{n+1})\} = \delta_2(Fx_n, Fx_{n+1}) \]

then inequality (2) gives
\[
\delta_2(Fx_n, Fx_{n+1}) \leq c \max \{d_1(x_n, x_{n+1})^2, \delta_2(Fx_n, Fx_{n+1}) \delta_1(Gy_n, Gy_{n+1}), \\
\delta_2(y_n, Fx_{n+1}) \delta_2(y_{n+1}, Fx_n), \delta_1(x_n, Gy_n) \delta_1(x_{n+1}, Gy_{n+1})\}
\]

\[
\delta_2(Fx_n, Fx_{n+1}) \leq c \max \{\delta_2(FGy_n, FGy_{n+1}), \delta_1(GFx_n, GFx_{n+1})\}
\]

and so inequality (3) holds for all cases.

We can prove similarly that
\[
d_1(x_{n+1}, x_{n+2}) \leq \delta_1(GFx_n, GFx_{n+1})
\]

\[
\leq c \max \{\delta_2(FGy_n, FGy_{n+1}), \delta_1(GFx_n, GFx_{n+1})\}
\]

and it follows easily by induction from inequalities (3) and (4) that
\[
d_1(x_{n+1}, x_{n+2}) \leq \delta_1(GFx_n, GFx_{n+1})
\]

\[
\leq c^{n-1} \max \{\delta_1(GFx_1, GFx_2), \delta_2(FGy_1, FGy_2)\}
\]

The sequence \(\{x_n\}\) is therefore a Cauchy sequence with a limit \(z\) in the complete metric space \(X\) and the sequence \(\{GFx_n\}\) is also a Cauchy sequence with a limit \(\{z_n\}\) in \(B(X)\). However, since \(x_{n+1}\) is in \(GFx_n\), it follows that \(z = z_0\). Thus
\[
\lim_{n \to \infty} x_n = z, \quad \lim_{n \to \infty} GFx_n = \{z\}
\]

Similarly, there exists a point \(w\) in \(Y\) such that
\[
\lim_{n \to \infty} y_n = w, \quad \lim_{n \to \infty} FGy_n = \{w\}
\]

Further, since \(x_{n+1}\) is in \(Gy_n\) we have
\[
\lim_{n \to \infty} FGy_n = \{w\} = \lim_{n \to \infty} Fx_n
\]

and using the continuity of \(F\), we see that
\[
\lim_{n \to \infty} Fx_n = Fz = \{w\}.
\]

If we now suppose that \(GFz \neq \{z\}\), then using inequality (3.2.1),
we have
\[ \max \{ d_1(x_n, z), \delta_1(GFx_n, GFz), \delta_2(Fx_n, Fz) \} \delta_1(GFx_n, GFz) \leq \]
\[ \leq c \max \{ [d_1(x_n, z)]^2, \delta_2(Fx_n, Fz) \delta_1(GFx_n, GFz), \]
\[ \delta_2(Fx_n, Fz) \delta_2(Fz, Fx_n), \delta_1(x_n, GFx_n) \delta_1(z, GFz) \} \]
Letting \( n \) tend to infinity and using equations (6) and (8), we get
\[ [\delta_1(z, GFz)]^2 \leq 0, \]
a contradiction. We must therefore have \( GFz = \{ z \} \), proving that \( z \) is a fixed point of \( GF \).

Further, using equation (8) we have \( FGw = FGFz = Fz = \{ w \} \),
proving that \( w \) is a fixed point of \( FG \).

This completes the proof of the Theorem 1.

Now we prove above theorem for single-valued mappings.

**Theorem 2.** Let \( (X, d_1) \) and \( (Y, d_2) \) be complete metric spaces, let \( S \) be mapping of \( X \) into \( Y \) and \( T \) be mapping of \( Y \) into \( X \) satisfying the inequalities
\begin{align*}
(3.2.3) \quad d_1(TSx, TSx') &\leq c \frac{f(x, x', y, y')}{h(x, x')} , \\
(3.2.4) \quad d_2(STy, STy') &\leq c \frac{g(x, x', y, y')}{k(y, y')} ,
\end{align*}
for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \) such that \( h(x, x') \neq 0 \) and \( k(y, y') \neq 0 \),
where
\begin{align*}
f(x, x', y, y') &= \max \{ [d_1(x, x')]^2, d_2(Sx, Sx') d_1(Ty, Ty'), \\
d_2(y, Sx') d_2(y', Sx), d_1(x, Ty) d_1(x', Ty') \}, \\
g(x, x', y, y') &= \max \{ [d_2(y, y')]^2, d_2(Ty, Ty') d_2(Sx, Sx'), \\
d_2(x, Ty') d_2(x', Ty), d_2(y, Sx) d_2(y', Sx') \},
\end{align*}

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\[ h(x, x') = \max \{d_1(x, x'), d_1(TSx, TSx'), d_2(Sx, Sx') \} \]

\[ k(y, y') = \max \{d_2(y, y'), d_2(STy, STy'), d_1(Ty, Ty') \} \]

and \(0 \leq c < 1\). If \(S\) is continuous, then \(TS\) has a unique fixed point \(z\) in \(X\) and \(ST\) has a unique fixed point \(w\) in \(Y\). Further, \(Sz = w\) and \(Tw = z\).

**Proof.** By letting the set valued mappings \(F\) and \(G\) be single-valued mappings \(S\) and \(T\) respectively. The existence of points \(z\) and \(w\) such that

\[ TSz = z, STw = w, Sz = w, Tw = z \]

follows immediately from Theorem 1.

All we have to do now to prove the uniqueness of \(z\) and \(w\). Suppose that \(TS\) has a second fixed point \(z'\). Then using inequality (3.2.3) with \(y = Sz\) and \(y' = Sz'\), we have

\[
\max \{d_1(z, z'), d_2(Sz, Sz') \} \quad d_1(z, z') \\
= \max \{d_1(z, z'), d_1(TSz, TSz'), \\
\quad d_2(Sz, Sz') \} \quad d_1(TSz, TSz') \\
\leq c \quad \max \{(d_1(z, z'))^2, d_2(Sz, Sz') \quad d_1(TSz, TSz'), \\
\quad [d_2(Sz, Sz')]^2, d_1(z, TSz) \quad d_1(z', TSz') \}
\]

\[
= c \quad \max \{(d_1(z, z'))^2, d_2(Sz, Sz') \quad d_1(z, z'), [d_2(Sz, Sz')]^2 \}
\]

It follows that either

\[ d_2(z, z') \leq c \quad d_1(z, z') < d_1(z, z'), \]

a contradiction, or,

\[ d_1(z, z') \leq c \quad d_2(Sz, Sz') \]

(9)

Inequality (9) must therefore hold.

Next, using inequality (3.2.4) with \(x = TSz = z\) and \(x' = TSz' = z'\), we have
\[
\max \{d_1(z, z'), \; d_2(Sz, Sz')\} = \max \{d_2(Sz, Sz'), \; d_2(STSz, STSz'), \; d_1(TSz, TSz')\} \\
\leq c \max \{[d_2(Sz, Sz')]^2, \; d_1(TSz, TSz') \; d_2(STSz, STSz'), \; [d_1(TSz, TSz')]^2, \; d_2(Sz, STSz) \; d_1(Sz', STSz')\} \\
= c \max \{[d_2(Sz, Sz')]^2, \; d_1(z, z') \; d_2(Sz, Sz'), \; [d_1(z, z')]^2\}
\]

It follows that either
\[
d_2(Sz, Sz') \leq c \; d_1(z, z') \tag{10}
\]
or,
\[
d_2(Sz, Sz') \leq c \; d_2(Sz, Sz') \tag{11}
\]

Now inequalities (9) and (10) imply that
\[
d_1(z, z') \leq c \; d_2(Sz, Sz') \leq c^2 \; d_1(z, z') < d_1(z, z'),
\]
giving a contradiction. Thus \(z = z'\). Inequality (11) must therefore hold which implies that
\[
d_2(Sz, Sz') = 0 \tag{12}
\]
Inequality (9) and (12) now imply that
\[
d_1(z, z') = 0
\]
again a contradiction. Our assumption that \(z \neq z'\) was therefore false and the uniqueness of \(z\) follows.

Similarly, \(w\) is the unique fixed point of \(ST\).

This completes the proof of Theorem 2.

(3.3) In this section, we have generalized the result of Fisher and Turkoglu [1] and proved related fixed point theorems for set-valued mappings on three complete metric spaces.
Theorem 3. Let \((X, d_x), (Y, d_y), \) and \((Z, d_z)\) be complete metric spaces, let \(F\) be mapping of \(X\) into \(B(Y)\), \(G\) be mapping of \(Y\) into \(B(Z)\) and let \(P\) be mapping of \(Z\) into \(B(X)\) satisfying the following inequalities:

\[
\delta_1(PGFx, PGFy') \leq c \max \{d_1(x, x'), \delta_1(x, PGFx), \\
\delta_1(x', PGFx'), \delta_2(Fx, Fy'), \delta_3(GFx, GFy')\},
\]

\[
\delta_2(FPGy, FPGy') \leq c \max \{d_2(y, y'), \delta_2(y, FPGy), \\
\delta_2(y', FPGy'), \delta_3(Gy, Gy'), \delta_1(PGy, PGy')\},
\]

\[
\delta_3(GFpz, GFpz') \leq c \max \{d_3(z, z'), \delta_3(z, GFpz), \\
\delta_3(z', GFpz'), \delta_2(Pz, Pz'), \delta_1(FPz, FPz')\},
\]

for all \(x, x'\) in \(X\), \(y, y'\) in \(Y\) and \(z, z'\) in \(Z\), where \(0 \leq c < 1\). If \(F\) and \(G\) are continuous, then \(PGF\) has a unique fixed point \(u\) in \(X\), \(FPG\) has a unique fixed point \(v\) in \(Y\) and \(GFP\) has a unique fixed point \(w\) in \(Z\).

Proof. Let \(x_1\) be an arbitrary point in \(X\). Define sequences \(\{x_n\}\) in \(X\), \(\{y_n\}\) in \(Y\) and \(\{z_n\}\) in \(Z\) as follows. Choose a point \(y_1\) in \(Fx_1\), \(z_1\) in \(Gy_1\) and then \(x_2\) in \(Pz_1\). In general, having chosen \(x_n\) in \(X\), \(y_n\) in \(Y\) and \(z_n\) in \(Z\), then choose \(x_{n+1}\) in \(Pz_n\), \(z_{n+1}\) in \(Gy_{n+1}\) and \(y_{n+1}\) in \(Fx_{n+1}\) for \(n = 1, 2, \ldots\). On using inequality (3.3.1), we have

\[
d_1(x_{n+1}, x_{n+2}) \leq \delta_1(PGFx_n, PGFx_{n+1})
\]

\[
\leq c \max \{d_1(x_n, x_{n-1}), \delta_1(x_n, PGFx_n), \\
\delta_1(x_{n+1}, PGFx_{n+1}), \delta_2(Fx_n, Fx_{n-1}), \delta_3(GFx_n, GFx_{n-1})\}
\]

\[
\leq c \max \{\delta_1(PGFx_{n-1}, PGFx_n), \delta_1(PGFx_n, PGFx_{n+1}), \\
\delta_2(Fx_n, Fx_{n+1}), \delta_3(GFx_n, GFx_{n+1})\}
\]

\[
= c \max \{\delta_1(PGFx_{n-1}, PGFx_n), \delta_2(Fx_n, Fx_{n+1}), \delta_3(GFx_n, GFx_{n+1})\} \quad (13)
\]
and similarly, using inequalities (3.3.2) and (3.3.3), we obtain
\[ d_2(y_{n+1}, y_{n+2}) \leq \delta_2(FPG_y, FPG_{y_{n+1}}) \]
\[ \leq c \max \{\delta_2(FPG_{y_{n-1}}, FPG_y), \]
\[ \delta_2(Gy, Gy_{n+1}), \delta_2(PGy, PGy_{n+1})\} \]  \hspace{1cm} (14)
\[ d_3(z_{n+1}, z_{n+2}) \leq \delta_3(GFPz, GFPz_{n+1}) \]
\[ \leq c \max \{\delta_3(GFPz_{n-1}, GFPz), \]
\[ \delta_3(Pz, Pz_{n+1}), \delta_2(FPz, FPz_{n+1})\} \]  \hspace{1cm} (15)
It follows that, for \( r = 1, 2, \ldots \ldots \)
\[ d_1(x_{n+1}, x_{n+r+1}) \leq \delta_1(PGFx, PGFx_{n+r}) \]
\[ \leq \delta_1(PGFx, PGFx_{n+r}) + \delta_1(PGFx_{n-1}, PGFx) + \ldots \ldots \]
\[ + \delta_1(PGFx_{n+r-1}, PGFx_{n+r}) \]
\[ \leq (c^n + c^{n+1} + \ldots + c^{n+r-1}) \delta_1(x_1, PGFx) \]
\[ < \varepsilon \]  \hspace{1cm} (16)
for \( n \) greater than some \( N \), since \( c < 1 \). The sequence \( \{x_n\} \) is a Cauchy sequence in the complete metric space \( X \) and so has a limit \( u \) in \( X \).
Similarly, the sequences \( \{y_n\} \) and \( \{z_n\} \) have limits \( v \) and \( w \) respectively.

Further
\[ \delta_1(u, PGFx_n) \leq d_1(u, x_{m+1}) + \delta_1(x_{m+1}, PGFx) \]
\[ \leq d_1(u, x_{m+1}) + \delta_1(PGFx_m, PGFx) \]
since \( x_{m+1} \) in \( PGFx_m \). Thus, on using inequality (16), we get
\[ \delta_1(u, PGFx_n) \leq d_1(u, x_{m+1}) + \varepsilon \]
for \( m, n \geq N \). Letting \( n \) tend to infinity, it follows that
\[ \delta_1(u, PGFx_n) < \varepsilon \]  \hspace{1cm} (17)
for \( n > N \) and so
\[
\lim_{n \to \infty} \text{PGFx}_n = \{u\} \tag{18}
\]
since \( \varepsilon \) is arbitrary. Similarly
\[
\lim_{n \to \infty} \text{FPGy}_n = \{v\} = \lim_{n \to \infty} \text{FPz}_n = \lim_{n \to \infty} \text{Fx}_{n+1} \tag{19}
\]
since \( z_n \) in \( \text{Gy}_n \) and \( x_{n+1} \) in \( \text{Pz}_n \) and
\[
\lim_{n \to \infty} \text{GFPz}_n = \{w\} = \lim_{n \to \infty} \text{GFx}_{n+1} = \lim_{n \to \infty} \text{Gy}_{n+1} \tag{20}
\]
since \( x_{n+1} \) in \( \text{Pz}_n \) and \( y_{n+1} \) in \( \text{Fx}_{n+1} \).

Now, using continuity of \( F \) and \( G \), we have
\[
\text{Fu} = \lim_{n \to \infty} \text{Fx}_n = \{v\} \tag{21}
\]
\[
\text{Gv} = \lim_{n \to \infty} \text{Gy}_n = \{w\} \tag{22}
\]
Using inequality (3.3.1), we have
\[
\delta_1(\text{PGFx}_n, \text{PGFu}) \leq c \max \{d_1(x_n, u), \delta_1(x_n, \text{PGFx}_n), \delta_1(u, \text{PGFu}), \delta_2(\text{Fx}_n, \text{Fu}), \delta_3(\text{GFx}_n, \text{GFu})\}
\]
Letting \( n \) tend to infinity and using inequality (18), (20), (21) and (22), we get
\[
\delta_1(u, \text{PGFu}) \leq c \delta_1(u, \text{PGFu}) < \delta_1(u, \text{PGFu}),
\]
a contradiction. Thus \( \text{PGFu} = \{u\} \), and so \( u \) is a fixed point of \( \text{PGF} \).

Further, using (21) and (22), we get
\[
\text{FPGv} = \text{FPGFu} = \text{Fu} = \{v\}
\]
\[
\text{GFPw} = \text{GFPGv} = \text{Gv} = \{w\}
\]
and so, \( v \) is a fixed point of \( \text{FPG} \) and \( w \) is a fixed point of \( \text{GFP} \).

To prove uniqueness, suppose that \( \text{PGF} \) has another fixed point \( u' \). Then, using inequalities (3.3.1), (3.3.2) and (3.3.3), we have
\[
\delta_1(u', \text{PGFu'}) \leq \delta_1(\text{PGFu'}, \text{PGFu'})
\]
\[ \leq c \max \{d_i(u', u'), \delta_i(u', \text{PGFu'}), \]
\[ \delta_2(Fu', Fu'), \delta_3(GFu', GFu') \}\]
\[ = c \max \{\delta_2(Fu', Fu'), \delta_3(GFu', GFu')\} \]

**Case I:** \( \delta_1(u', \text{PGFu'}) \leq c \delta_2(Fu', Fu') \)
\[ \leq c \delta_2(Fu', \text{FPGFu'}) \]
\[ \leq c \delta_2(\text{FPGFu'}, \text{FPGFu'}) \]
\[ \leq c^2 \max \{\delta_2(Fu', Fu'), \delta_2(Fu', \text{FPGFu'}), \]
\[ \delta_3(GFu', GFu'), \delta_3(\text{PGFu'}, \text{PGFu'})\}\]
\[ = c^2 \max \{\delta_1(\text{PGFu'}, \text{PGFu'}), \delta_3(GFu', GFu')\} \]

and so, either
\[ \delta_1(u', \text{PGFu'}) \leq \delta_1(\text{PGFu'}, \text{PGFu'}) \]
\[ \leq c^2 \delta_1(\text{PGFu'}, \text{PGFu'}) < \delta_1(\text{PGFu'}, \text{PGFu'}), \]

a contradiction. Thus, PGFu' = \{u'\}.

or, if \( \delta_1(u', \text{PGFu'}) \leq c^2 \delta_2(GFu', GFu') \leq c^2 \delta_3(GFu', GFPFu') \)
\[ \leq c^2 \delta_3(GFPFu', GFPGFu') \]
\[ \leq c^3 \max \{\delta_3(GFu', GFu'), \delta_3(\text{PGFu'}, \text{PGFu'}), \]
\[ \delta_2(\text{FPGFu'}, \text{FPGFu'})\}\]
\[ \leq c^3 \max \{\delta_1(\text{PGFu'}, \text{PGFu'}), \delta_2(Fu', Fu')\} \]

and so,
\[ \delta_1(u', \text{PGFu'}) \leq \delta_1(\text{PGFu'}, \text{PGFu'}) \leq c^3 \delta_1(\text{PGFu'}, \text{PGFu'}) \]

and so, since \( c < 1 \), Fu' and GFu' are singletons and PGFu' = \{u'\}.

**Case II.** \( \delta_1(u', \text{PGFu'}) \leq \delta_1(\text{PGFu'}, \text{PGFu'}) \leq c^2 \delta_3(GFu', GFu') \)

Then, we get the same result.

Now, using inequality (3.3.1), we have
\[ d_i(u, u') = \delta_i(\text{PGFu}, \text{PGFu'}) \]

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\[ \leq c \max \{ d_1(u, u'), \delta_1(u, \text{PGFu}), \delta_1(u', \text{PGFu}'), \\
\delta_2(\text{Fu}, \text{Fu}'), \delta_3(\text{GFu}, \text{GFu}') \} \]
\[ = c \max \{ d_2(\text{Fu}, \text{Fu}'), d_3(\text{GFu}, \text{GFu}') \} \]

**Case I.** \[ d_1(u, u') \leq c \; d_2(\text{Fu}, \text{Fu}') \] (23)

So, using inequality (3.3.2), we have

\[ d_2(\text{Fu}, \text{Fu}') \leq \delta_2(\text{FPFGFu}, \text{FPFGFu}') \]
\[ \leq c \max \{ d_2(\text{Fu}, \text{Fu}'), \delta_2(\text{Fu}, \text{FPFGFu}), \\
\delta_2(\text{Fu}', \text{FPFGFu}'), \delta_3(\text{GFu}, \text{GFu}'), \delta_1(\text{PGFu}, \text{PGFu}') \} \]
\[ = c \max \{ d_3(\text{GFu}, \text{GFu}'), d_1(u, u') \} \]

and so, either

\[ d_2(\text{Fu}, \text{Fu}') \leq c \; d_1(u, u') \] (24)

From inequalities (23) and (24), we get

\[ d_1(u, u') \leq c^2 \; d_1(u, u') < d_1(u, u'), \]

a contradiction. Thus, \( u = u' \)

or \[ d_2(\text{Fu}, \text{Fu}') \leq c \; d_3(\text{GFu}, \text{GFu}') \]

So, using inequality (3.3.3), we have

\[ d_3(\text{GFu}, \text{GFu}') \leq \delta_3(\text{FPFGFu}, \text{FPFGFu}') \]
\[ = c \max \{ d_1(u, u'), d_2(\text{Fu}, \text{Fu}') \} \]

and so, if

\[ d_3(\text{GFu}, \text{GFu}') \leq c \; d_1(u, u') \] (25)

From inequalities (23), (24) and (25), we get

\[ d_1(u, u') \leq c^3 \; d_1(u, u') < d_1(u, u'), \]

a contradiction again. Thus, \( u = u' \).

**Case II.** \[ d_1(u, u') \leq d_3(\text{GFu}, \text{GFu}') \]
Then, we get the same result again.

This completes the proof of Theorem 3.

Now, we give a Corollary which is a related fixed point theorem for single-valued mappings on three complete metric spaces.

**Corollary 1.** Let \((X, d_1), (Y, d_2)\) and \((Z, d_3)\) be complete metric spaces. If \(T\) is a continuous mapping of \(X\) into \(Y\), \(S\) is a continuous mapping of \(Y\) into \(Z\) and \(R\) is a mapping of \(Z\) into \(X\) satisfying the following inequalities:

(i) \[ d_1(RSTx, RSTx') \leq c \max \{d_1(x, x'), d_1(x, RSTx), d_1(x', RSTx'), d_2(Tx, Tx'), d_3(STx, STx')\}, \]

(ii) \[ d_2(TRSy, TRSy') \leq c \max \{d_2(y, y'), d_2(y, TRSy), d_2(y', TRSy'), d_3(Sy, Sy'), d_1(RSy, RSy')\}, \]

(iii) \[ d_3(STRz, STRz') \leq c \max \{d_3(z, z'), d_3(z, STRz), d_3(z', STRz'), d_1(Rz, Rz'), d_2(TRz, TRz')\}, \]

for all \(x, x'\) in \(X\), \(y, y'\) in \(Y\) and \(z, z'\) in \(Z\), where \(0 \leq c < 1\), then \(RST\) has a unique fixed point \(u\) in \(X\), \(TRS\) has a unique fixed point \(v\) in \(Y\) and \(STR\) has a unique fixed point \(w\) in \(Z\). Further, \(Tu = v\), \(Sv = w\) and \(Rw = u\).

**Proof.** By letting the set-valued mappings \(F\), \(G\) and \(P\) be the single-valued mappings \(T\), \(S\) and \(R\) respectively in Theorem 3, we get the proof.

(3.4) In this section, we have extended the result of Fisher & Turkoglu [1] and proved related fixed point theorem for two pairs of set-valued mappings on two complete metric spaces.
Theorem 4. Let \((X, d_1)\) and \((Y, d_2)\) be complete metric spaces, let \(F\) and \(G\) be mappings of \(X\) into \(B\) \((Y)\) and \(P\) and \(Q\) be mappings of \(Y\) into \(B\) \((X)\) satisfying the inequalities

\[
\delta_1(PFx, QGx') \leq c \max \{d_1(x, x'), \delta_1(x, PFx), \\
\delta_1(x', QGx'), \delta_2(Fx, Gx')\},
\]

\[
\delta_2(GPy, FQy') \leq c \max \{d_2(y, y'), \delta_2(y, GPy), \\
\delta_2(y', FQy'), \delta_1(Py, Qy')\},
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\), where \(0 \leq c < 1\). If \(F\) and \(G\) are continuous, then \(PF\) and \(QG\) have a unique fixed point \(z\) in \(X\) and \(GP\) and \(FQ\) have a unique fixed point \(w\) in \(Y\).

**Proof.** Let \(x\) be an arbitrary point in \(X\). Define sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) and \(Y\) respectively as follows. Choose a point \(y_1\) in \(Fx_1\), a point \(x_2\) in \(Py_1\), a point \(y_2\) in \(Gx_2\) and then a point \(x_3\) in \(Qy_2\). In general, having chosen \(x_n\) in \(X\) and \(y_n\) in \(Y\) choose a point \(y_{2n-1}\) in \(Fx_{2n-1}\), a point \(x_{2n}\) in \(Py_{2n-1}\), a point \(y_{2n}\) in \(Gx_{2n}\), and then a point \(x_{2n+1}\) in \(Qy_{2n}\) for \(n = 1, 2, \ldots\).

Then using inequality (3.4.1), we have

\[
d_1(x_{2n+2}, x_{2n+1}) \leq \delta_1(PFx_{2n+1}, QGx_{2n})
\]

\[
\leq c \max \{d_1(x_{2n+1}, x_{2n}), \delta_1(x_{2n+1}, PFx_{2n+1}), \\
\delta_1(x_{2n}, QGx_{2n}), \delta_2(Fx_{2n+1}, Gx_{2n})\}
\]

\[
\leq c \max \{\delta_1(QGx_{2n}, PFx_{2n+1}), \delta_1(QGx_{2n}, PFx_{2n+1}), \\
\delta_1(PFx_{2n+1}, QGx_{2n}), \delta_2(Fx_{2n+1}, Gx_{2n})\}
\]

\[
\leq c \max \{\delta_1(PFx_{2n-1}, QGx_{2n}), \\
\delta_2(GPy_{2n-1}, FQy_{2n})\}
\]

since \(\delta_2(Fx_{2n+1}, Gx_{2n}) \leq \delta_2(GPy_{2n-1}, FQy_{2n}).
\]

(26)
Similarly, using inequality (3.4.1), we have
\[ d_1(x_{2n+2}, x_{2n+3}) \leq \delta_1(PFx_{2n+1}, QGx_{2n+2}) \]
\[ \leq c \max \{d_1(x_{2n+1}, x_{2n+2}), \delta_1(x_{2n+1}, PFx_{2n+1}), \]
\[ \delta_1(x_{2n+2}, QGx_{2n+2}), \delta_2(Fx_{2n+1}, Gx_{2n+2})\} \]
\[ \leq c \max \{\delta_1(QGx_{2n}, PFx_{2n+1}), \delta_1(QGx_{2n}, PFx_{2n+1}), \]
\[ \delta_1(PFx_{2n+1}, QGx_{2n+2}), \delta_2(Fx_{2n+1}, Gx_{2n+2})\} \]
\[ \leq c \max \{\delta_1(PFx_{2n+1}, QGx_{2n}), \]
\[ \delta_2(GPy_{2n+1}, FQy_{2n})\} \] (27)

since \[ \delta_2(Fx_{2n+1}, Gx_{2n+2}) \leq \delta_2(GPy_{2n+1}, FQy_{2n}). \]

Using inequality (3.4.2), we have
\[ d_2(y_{2n+1}, y_{2n+2}) \leq \delta_2(FQy_{2n}, GPy_{2n+1}) \]
\[ \leq c \max \{d_2(y_{2n}, y_{2n+1}), \delta_2(y_{2n+1}, GPy_{2n+1}), \]
\[ \delta_2(y_{2n}, FQy_{2n}), \delta_1(Py_{2n+1}, Qy_{2n})\} \]
\[ \leq c \max \{\delta_2(GPy_{2n-1}, FQy_{2n}), \delta_2(FQy_{2n}, GPy_{2n+1}), \]
\[ \delta_2(GPy_{2n-1}, FQy_{2n}), \delta_1(Py_{2n+1}, Qy_{2n})\} \]
\[ \leq c \max \{\delta_2(GPy_{2n-1}, FQy_{2n}), \]
\[ \delta_1(PFx_{2n+1}, QGx_{2n})\} \] (28)

since \[ \delta_1(Py_{2n+1}, Qy_{2n}) \leq \delta_1(PFx_{2n+1}, QGx_{2n}). \]

Similarly, using inequality (3.4.2) again, we have
\[ d_2(y_{2n+2}, y_{2n+3}) \leq \delta_2(GPy_{2n-1}, FQy_{2n+2}) \]
\[ \leq c \max \{d_2(y_{2n+1}, y_{2n+2}), \delta_2(y_{2n+1}, GPy_{2n+1}), \]
\[ \delta_2(y_{2n+2}, FQy_{2n+2}), \delta_1(Py_{2n+1}, Qy_{2n+2})\} \]
\[ \leq c \max \{\delta_2(FQy_{2n}, GPy_{2n+1}), \delta_2(FQy_{2n}, GPy_{2n+1}), \]
\[ \delta_2(GPy_{2n+1}, FQy_{2n+2}), \delta_1(Py_{2n+1}, Qy_{2n+2})\} \]
\[ \leq c \max \{ \delta_2(GPy_{2n+1}, FQy_{2n}), \delta_1(PFx_{2n+1}, QGx_{2n+2}) \} \]

since \( \delta_1(Py_{2n+1}, Qy_{2n+2}) \leq \delta_1(PFx_{2n+1}, QGx_{2n+2}) \).

We will now prove that

\[ \delta_1(PFx_{2n+1}, QGx_{2n}) \leq c^n K, \] (30)

\[ \delta_1(PFx_{2n+1}, QGx_{2n+2}) \leq c^n K, \] (31)

\[ \delta_2(FQy_{2n}, GPy_{2n+1}) \leq c^n K, \] (32)

\[ \delta_2(GPy_{2n+1}, FQy_{2n+2}) \leq c^n K, \] (33)

where \( K = \max \{ \delta_1(PFx_1, QGx_2), \delta_1(PFx_3, QGx_2), \delta_2(GPy_1, FQy_2), \delta_2(GPy_3, FQy_2), \} \)

for \( n = 1, 2, \ldots \).

Inequalities (30) to (33) clearly hold when \( n = 1 \). Suppose inequalities (30) to (33) hold for some \( n \). Then it follows from inequality (26) that

\[ \delta_1(PFx_{2n+3}, QGx_{2n+2}) \leq c \max \{ \delta_1(PFx_{2n+1}, QGx_{2n+2}), \delta_2(GPy_{2n+1}, FQy_{2n+2}) \} \]

\[ \leq c^{n+1} K \]

on using inequalities (31) and (33). Inequality (30) now follows by induction.

Using inequality (28), we have

\[ \delta_2(FQy_{2n+2}, GPy_{2n+3}) \leq c \max \{ \delta_2(GPy_{2n+1}, FQy_{2n+2}), \delta_1(PFx_{2n+3}, QGx_{2n+2}) \} \]

\[ \leq c^{n+1} K \]

on using inequality (33) and our assumption on inequality (30). Inequality
(32) now follows by induction.

Using inequality (27), we have

\[ \delta_1(\PFx_{2n+3}, \QGx_{2n+4}) \leq c \max \{ \delta_1(\PFx_{2n+3}, \QGx_{2n+2}), \delta_2(\GPy_{2n+3}, \FQy_{2n+2}) \} \]

\[ \leq c^{n+1} K \]

on using our assumptions on inequalities (30) and (32). Inequality (31) follows by induction.

Finally, using inequality (29), we have

\[ \delta_2(\GPy_{2n+3}, \FQy_{2n+4}) \leq c \max \{ \delta_2(\GPy_{2n+3}, \FQy_{2n+2}), \delta_1(\PFx_{2n+3}, \QGx_{2n+4}) \} \]

\[ \leq c^{n+1} K \]

on using our assumptions on inequalities (31) and (32). Inequality (33) now follows by induction.

It follows that for \( r = 1, 2, \ldots \ldots \)

\[ d_1(x_{2n+1}, x_{2n+r+1}) \leq d_1(x_{2n+1}, x_{2n+2}) + d_1(x_{2n+2}, x_{2n+3}) + \ldots \]

\[ + d_1(x_{2n+r}, x_{2n+r+1}) \]

\[ \leq \delta_1(\QGx_{2n}, \PFx_{2n+1}) + \delta_1(\PFx_{2n+1}, \QGx_{2n+2}) + \ldots \]

\[ \leq (c^n + c^n + c^{n+1} + c^{n+1} + \ldots) K \]

\[ < \varepsilon \]

for \( n \) greater than some \( N \), since \( c < 1 \). The sequence \( \{x_n\} \) is therefore a Cauchy sequence in the complete metric space \( X \) and so has a limit \( z \) in \( X \). Similarly the sequence \( \{y_n\} \) is a Cauchy sequence in complete metric space \( Y \) and so has a limit \( w \) in \( Y \).
Further, with $m > n$, we have

\[
\delta_1(QGx_{2n}, PFx_{2n+1}) \leq \delta_1(QGx_{2n}, PFx_{2n+1}) + \\
\delta_1(PFx_{2n+1}, QGx_{2n+2}) + \ldots \ldots \\
+ \delta_1(QGx_{2m}, PFx_{2m+1}) \\
\leq (c^n + c^{n+1} + c^{n+1} + \ldots \ldots) K \\
< \varepsilon
\] (34)

for $n > N$.

Next, we have

\[
\delta_1(z, QGx_{2n}) \leq d_1(z, x_{2m+2}) + \delta_1(x_{2m+2}, QGx_{2n}) \\
\leq d_1(z, x_{2m+2}) + \delta_1(PFx_{2m+1} + QGx_{2n}),
\]

since $x_{2m+2}$ in $PFx_{2m+1}$. Thus, on using inequality (34), we have

\[
\delta_1(z, QGx_{2n}) \leq d_1(z, x_{2m+2}) + \varepsilon
\]

for $m > n > N$. Letting $n$ tend to infinity it follows that

\[
\delta_1(z, QGx_{2n}) \leq \varepsilon
\]

for $n > N$ and so

\[
\lim_{n \to \infty} QGx_{2n} = \{z\}, \quad (35)
\]

since $\varepsilon$ is arbitrary.

Similarly,

\[
\lim_{n \to \infty} PFx_{2n+1} = \{z\}, \quad (36)
\]

\[
\lim_{n \to \infty} GPy_{2n+1} = \{w\} = \lim_{n \to \infty} FQy_{2n} \quad (37)
\]

From the continuity of $F$ and $G$, we have

\[
\lim_{n \to \infty} Fx_{2n+1} = Fz = \{w\}, \quad (38)
\]

\[
\lim_{n \to \infty} Gx_{2n} = Gz = \{w\}. \quad (39)
\]

Using inequality (3.4.1), we now have
\[ \delta_1(PFz, QGx_{2n}) \leq c \max \{d_1(z, x_{2n}), \delta_1(z, PFz), \delta_1(x_{2n}, QGx_{2n}), \delta_2(Fz, Gx_{2n}) \} \]

Letting \( n \) tend to infinity and using equations (35) and (39), we have
\[ \delta_1(PFz, z) \leq c \delta_1(PFz, z) \]
since \( c < 1 \), we must have
\[ PFz = \{z\} = Pw, \tag{40} \]
on using equation (38), proving that \( z \) is a fixed point of \( PF \).

Using inequality (3.4.1) again, we now have
\[ \delta_1(x_{2n+2}, QGz) \leq \delta_1(PFx_{2n+1}, QGz) \]
\[ \leq c \max \{d_1(x_{2n+1}, z), \delta_1(x_{2n+1}, PFx_{2n+1}), \delta_1(z, QGz), \delta_2(Fx_{2n+1}, Gz) \} \]
Letting \( n \) tend to infinity and using equations (36), (38) and (39), we have
\[ \delta_1(z, QGz) \leq c \delta_1(z, QGz) \]
since \( c < 1 \), we must have
\[ QGz = \{z\} = Qw, \tag{41} \]
on using equation (39), proving that \( z \) is a fixed point of \( QG \).

It now follows from equations (38) and (41) that
\[ GPw = Gz = \{w\}. \]
w is therefore a fixed point of \( FQ \) and \( GP \).

To prove uniqueness. Suppose that \( PF \) and \( QG \) have a second common fixed point \( z' \). Then using inequalities (3.4.1) and (3.4.2), we have
\[ \max \{\delta_1(z', QGz'), \delta_1(z', PFz')\} \leq \delta_1(PFz', QGz') \]
\[ \leq c \max \{d_1(z', z'), \delta_1(z', PFz'), \delta_1(z', QGz'), \delta_2(Fz', Gz')\} \]
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\[ = c \delta_2 (F'z, G'z) \]
\[ \leq c \delta_2 (GPF'z', FQG'z') \]
\[ \leq c^2 \max \{ \delta_2 (F'z', G'z'), \delta_2 (Fz', GPF'z'), \delta_2 (Gz', FQG'z'), \delta_1 (PF'z', QG'z') \} \]
\[ \leq c^2 \delta_1 (PF'z', QG'z') \]

and it follows that
\[ \max \{ \delta_1 (z', QG'z'), \delta_1 (z', PF'z') \} = \delta_1 (PF'z', QG'z') = \delta_2 (F'z', G'z') = 0, \]
since \( c < 1 \). Thus \( Fz' \) and \( Gz' \) are singletons and
\[ PF'z' = QG'z' = \{ z' \}. \]

Using inequalities (3.4.1) and (3.4.2) again, we have
\[ d_1(z, z') = \delta_1 (PFz, QGz') \]
\[ \leq c \max \{ d_1(z, z'), \delta_1(z, PFz), \delta_1(z', QGz'), \delta_2(Fz, Gz') \} \]
\[ = c d_2(Fz, Gz') \]
\[ \leq c \delta_2 (GPFz, FQGz') \]
\[ \leq c^2 \max \{ d_2(Fz, Gz'), \delta_2(Fz, GPFz), \delta_2(Gz', FQGz'), \delta_1(PFz, QGz') \} \]
\[ = c^2 \max \{ d_2(Fz, Gz'), d_1(z, z') \} \]
\[ \leq c^2 d_1(z, z') \]
since \( c < 1 \), the uniqueness of \( z \) follows.

Similarly, \( w \) is the unique fixed point GP and FQ.

This completes the proof of Theorem 4.
If we let $F$ and $G$ be single-valued mappings of $X$ into $Y$ and let $P$ and $Q$ be single valued mappings of $Y$ into $X$, we obtain the following Corollary, which is a result of Fisher & Murthy [1].

**Corollary 2.** Let $(X, d)$ and $(Y, d')$ be complete metric spaces. If $F$ and $G$ are continuous mappings of $X$ into $Y$ and $P$ and $Q$ are mappings of $Y$ into $X$ satisfying the inequalities

$$d_1(PFx, QGx') \leq c \max \{d_1(x, x'), d_1(x, PFx),$$

$$d_1(x', QGx'), d_1(Fx, Gx')\},$$

$$d_2(GPy, FQy') \leq c \max \{d_2(y, y'), d_2(y, GPy),$$

$$d_2(y', FQy'), d_1(Py, Qy')\},$$

for all $x, x'$ in $X$ and $y, y'$ in $Y$, where $0 \leq c < 1$, then $PF$ and $QG$ have a unique fixed point $z$ in $X$ and $GP$ and $FQ$ have a unique fixed point $w$ in $Y$. Further, $Fz = Gz = w$ and $Pw = Qw = z$. 