CHAPTER VI

FIXED POINT THEOREMS ON QUASI-METRIC SPACES AND FOR DENSIFYING Mappings ON METRIC SPACES

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FIXED POINT THEOREMS ON QUASI-METRIC SPACES AND FOR DENSIFYING MAPPINGS ON METRIC SPACES

(6.1) In this Chapter, we have generalized the result of Chikkala and Baisnab [1] on quasi metric spaces and the result of Sharma and Shrivastava [1] for densifying mappings.

Quasi-metric spaces have been studied by Wilson [1], Kelly [1] and Lane [1] particularly, in Bi-topological spaces.

Later on, Hicks [1], Chikkala and Baisnab [1], Namdeo and Tiwari [2] and others proved some fixed point theorems on quasi-metric spaces.

Definition 1. A quasi-metric \( p \), for a non-empty set \( X \) is a non-negative real-valued function defined on \( X \times X \) satisfying

(i) \( p(x, y) = 0 \) iff \( x = y \) for all \( x, y \in X \),

(ii) \( p(x, z) \leq p(x, y) + p(y, z) \) for all \( x, y, z \in X \).

For a quasi-metric \( p \) on \( X \) there exists a quasi-metric \( q \) on \( X \) called the conjugate of \( p \) defined by \( q(x, y) = p(y, x) \) for all \( x, y \) in \( X \).

The pair \((X, p)\) is called quasi-metric space.

Definition 2. A sequence \( \{x_n\} \) of a quasi-metric space \((X, p)\) is said to be \( p \)-Cauchy iff for \( \varepsilon > 0 \), there exist a positive integer \( k \) such that

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p \left( x_m, x_n \right) < \varepsilon \text{ for } m > n \geq k \text{ and it } p\text{-converges at a point } x \text{ in } X, \text{ if } p \left( x, x_n \right) \to 0 \text{ as } n \to \infty \text{ and in this case } \{ x_n \} \text{ is called } p\text{-convergent in } X.

**Definition 3.** A quasi-metric space \((X, p)\) is said to be complete if every \(p\)-Cauchy sequence is \(p\)-convergent in \(X\).

**Definition 4.** A self-mapping \(T\) on a quasi-metric space \(X\) is said to be orbitally continuous at \(x_o \in X\) if for some \(x \in X\), 
\[ p(Tx, Tx_n) \to 0 \]
implies 
\[ p(Tx, Tx_n) \to 0, \]
where \(\{x_n\}\) is a sub-sequence of sequence \(\{x_n\}\) defined by \(x_{n+1} = Tx_n\) for \(n = 0, 1, 2, \ldots \)

In 1991, Chikkala, R. and Bainsab, A. P. [1] proved the following fixed point theorem on complete quasi-metric spaces.

**Theorem A.** Let \((X, p)\) be a complete quasi-metric space. Let \(T : X \to X\) satisfies the following conditions:

\[
(6.1.1) \quad p(Tx, Ty) \leq \alpha \left[ q(x, Tx) + q(y, Ty) \right] + \beta \ p(x, y),
\]
for all \(x, y \in X\), where \(\alpha, \beta \geq 0 ; 2\alpha + \beta < 1\),

\[
(6.1.2) \quad T \text{ is } p\text{-orbitally continuous at some point } x_o \text{ of } X.
\]

Then there is a unique fixed point of \(T\) in \(X\).


**Theorem B.** Let \((X, p)\) be a complete quasi-metric space. Let \(T : X \to X\) satisfies the condition (6.1.2) of Theorem A and

\[
(6.1.3) \quad p \left( Tx, Ty \right) \leq \alpha \left[ q(x, Tx) + q(y, Ty) \right] \\
+ \beta \ \frac{q(y, Ty) \left[ 1 + p(x, Ty) + q(y, Tx) \right]}{1 + q(x, Tx) + q(y, Ty)} \\
+ \gamma \ p \left( x, y \right),
\]
for all \(x, y \in X\), where \(\alpha, \beta, \gamma \geq 0 ; 2\alpha + \beta + \gamma < 1\). Then \(T\) has a unique fixed point in \(X\).
Here, we prove three fixed point theorems on complete quasi-metric spaces which generalizes the result of Chikkala and Bainsab [1]. Theorem 3 is a generalization of the result of Namdeo and Tiwari [2].

**Theorem 1**: Let \((X, p)\) be a complete quasi-metric space. Let \(T\) is a self-mapping of \(X\) satisfying the condition (6.1.2) and

\[
(6.2.1) \quad p(Tx, Ty) \leq \alpha [q(x, Tx) + q(y, Ty)]
\]

\[
+ \beta \frac{p(x, Ty)q(y, Tx) + p(y, Ty)q(x, Tx)}{p(x, Ty) + p(y, Ty)}
\]

\[+ \gamma p(x, y),\]

for all \(x, y\) in \(X\), where \(\alpha, \beta, \gamma \geq 0; 2\alpha + \beta + \gamma < 1\).

Then \(T\) has a unique fixed point in \(X\).

**Proof.** Let \(\{x_n\}\) be a sequence in \(X\) defined by \(x_{n+1} = Tx_n\) for \(n = 0, 1, 2, \ldots\),

If \(x_n = x_{n+1}\) then immediately it follows that \(\{x_n\}\) is a \(p\)-Cauchy sequence. Let \(x_n \neq x_{n+1}\) for \(n = 0, 1, 2, \ldots\). Then, by using inequality (6.2.1), we have

\[
p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1})
\]

\[
\leq \alpha [q(x_n, Tx_n) + q(x_{n-1}, Tx_{n-1})]
\]

\[
+ \beta \frac{p(x_n, Tx_{n-1})q(x_{n-1}, Tx_n) + p(x_{n-1}, Tx_{n-1})q(x_n, Tx_n)}{p(x_n, Tx_{n-1}) + p(x_{n-1}, Tx_{n-1})}
\]

\[+ \gamma p(x_n, x_{n-1})\]

\[= \alpha [q(x_n, x_{n+1}) + q(x_{n-1}, x_n)]
\]

\[
+ \beta \frac{p(x_n, x_{n+1})q(x_{n+1}, x_n) + p(x_{n-1}, x_n)q(x_n, x_{n+1})}{p(x_n, x_n) + p(x_{n-1}, x_n)}
\]

\[+ \gamma p(x_n, x_{n-1})
\]
i.e. \( p(x_{n+1}, x_n) \leq h q(x_{n-1}, x_n) \) where \( h = \frac{\alpha + \gamma}{1 - \alpha - \beta} < 1 \)

Proceeding in similar manner, we get

\[
p(x_{n+1}, x_n) \leq h^n q(x_0, x_1)
\]

Now, for \( m > n \)

\[
p(x_m, x_n) \leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \ldots + p(x_{n+1}, x_n)
\]

\[
\leq (h^{m-1} + h^{m-2} + \ldots + h^n) q(x_0, x_1)
\]

\[
< \frac{h^n}{1-h} q(x_0, x_1)
\]

It follows from above that \( \{x_n\} \) is a \( p \)-Cauchy sequence in \( X \). Since \( X \) is complete therefore there exist a point \( z \) in \( X \) such that \( p \)-\( \lim_{n \to \infty} x_n = z \).

By the orbital continuity of \( T \), we get

\[
Tz = p \lim_{n \to \infty} Tx_n = p \lim_{n \to \infty} x_{n+1} = z,
\]

showing that \( z \) is a fixed point of \( T \).

To prove uniqueness of \( z \), let \( w \) be another fixed point of \( T \). Then by using condition (6.2.1), we have

\[
p(z, w) = p(Tz, Tw)
\]

\[
\leq \alpha [q(z, Tz) + q(w, Tw)]
\]

\[
+ \beta \frac{p(z, Tw) q(w, Tz) + p(w, Tw) q(z, Tz)}{p(z, Tw) + p(w, Tw)}
\]

\[
+ \gamma p(z, w)
\]

\[
= \alpha [q(z, z) + p(w, w)]
\]

\[
+ \beta \frac{p(z, w) q(w, z) + p(w, w) q(z, z)}{p(z, w) + p(w, w)}
\]

\[
+ \gamma p(z, w)
\]

i.e. \( p(z, w) \leq (\beta + \gamma) p(z, w) < p(z, w) \),

a contradiction. Thus, \( z = w \).

Hence \( z \) is the unique fixed point of \( T \) in \( X \).
This completes the proof of Theorem 1.

**Remark.** If $\beta = 0$, we get Theorem A, of Chikkala and Baisnab [1].

**Theorem 2.** Let $(X, p)$ be a complete quasi-metric space. Let $T$ is a self-mapping of $X$ satisfying condition (6.1.2) and

(6.2.2) \[ p(Tx, Ty) \leq \alpha \max \left\{ p(x, y), \frac{1}{2} \left[ q(x, Tx) + q(y, Ty) \right] \right\}, \]

\[
\frac{p(x, Ty) q(y, Tx) + p(y, Ty) q(x, Tx)}{p(x, Ty) + p(y, Ty)}
\]

for all $x, y \in X$, where $0 \leq \alpha < 1$, then $T$ has a unique fixed point.

**Proof.** Suppose that sequence $\{x_n\}$ is defined in the same way as in the proof of Theorem 1.

Then, using inequality (6.2.2), we have

\[
p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1})
\]

\[
\leq \alpha \max \left\{ p(x_n, x_{n-1}), \frac{1}{2} \left[ q(x_n, Tx_n) + q(x_{n-1}, Tx_{n-1}) \right] \right\},
\]

\[
\frac{p(x_n, Tx_{n-1}) q(x_{n-1}, Tx_n) + p(x_{n-1}, Tx_{n-1}) q(x_n, Tx_n)}{p(x_n, Tx_{n-1}) + p(x_{n-1}, Tx_{n-1})}
\]

\[
= \alpha \max \left\{ p(x_n, x_{n-1}), \frac{1}{2} \left[ p(x_{n-1}, x_n) + p(x_n, x_{n-1}) \right] \right\},
\]

\[
\frac{p(x_n, x_{n-1}) q(x_{n-1}, x_n) + p(x_{n-1}, x_n) q(x_n, x_{n-1})}{p(x_n, x_{n-1}) + p(x_{n-1}, x_n)}
\]

i.e. \[ p(x_{n+1}, x_n) \leq \alpha \max \left\{ p(x_n, x_{n-1}), \frac{1}{2} \left[ p(x_{n-1}, x_n) + p(x_n, x_{n-1}) \right] \right\} \]

**Case I.** $p(x_{n+1}, x_n) \leq \alpha \ p(x_n, x_{n-1})$

**Case II.** $p(x_{n+1}, x_n) \leq \beta \ p(x_n, x_{n-1})$ where $\beta = \alpha/(2-\alpha)<1$

Proceeding in similar manner, we get

\[
p(x_{n+1}, x_n) \leq h^n(x_1, x_0)
\]

where $h = \alpha$ or $\beta$

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Now, for \( m > n \)

\[
p(x_m, x_n) \leq p(x_m, x_{m-1}) + \cdots + p(x_{n+1}, x_n) \\
\leq (h^{m-1} + \cdots + h^n) \ p(x_1, x_0) \\
< \frac{h^n}{1 - h} \ p(x_1, x_0)
\]

It follows from above that \( \{x_n\} \) is a \( p \)-Cauchy sequence in \( X \). Since \( X \) is complete therefore there exist a point \( z \) in \( X \) such that \( p \-\lim_{n \to \infty} x_n = z \).

By orbital continuity of \( T \), we get

\[
Tz = p \-\lim_{n \to \infty} T x_n = p \-\lim_{n \to \infty} x_{n+1} = z,
\]
showing that \( z \) is a fixed point of \( T \).

To prove uniqueness of \( z \), let \( w \) be another fixed point of \( T \). Then by using inequality (6.2.2), we have

\[
p(z, w) = p(Tz, Tw) \\
\leq \alpha \ \max \{p(z, w), \ \frac{1}{2} \ [q(z, Tz) + q(w, Tw)]\} \\
\]

\[
= \alpha \ p(z, w) < p(z, w),
\]
a contradiction. Thus, \( z = w \).

This shows that \( z \) is the unique fixed point of \( T \).

This completes the proof of Theorem 2.

**Theorem 3.** Let \( (X, p) \) be a complete quasi-metric space and \( T_i \) be selfmapping of \( X \) with fixed point \( u_i \) for \( i = 0, 1, 2, \ldots \) and let \( \{T_i\} \) converges uniformly to \( T_0 \) if

\[
(6.2.3) \quad p(T_0x, T_0y) \leq \alpha \ [p(x, T_0x) + q(y, T_0y)] \\
\quad + \beta \ \frac{q(y, T_0y) [1 + p(x, T_0x) + q(y, T_0x)]}{1 + p(x, T_0y) + q(y, T_0y)}
\]
\[
p(x, T_0x) q (y, T_0x) + p(x, T_0y) q (y, T_0y)
+ \gamma \frac{p(x, T_0x) + p (x, T_0y)}{p(x, T_0x) + p (x, T_0y)} p(x, y),
\]
for all \( x, y \in X \), where \( \alpha, \beta, \gamma \) and \( \delta \) are non-negative numbers and \( \alpha + \beta + \gamma + \delta < 1 ; \beta < 1/2 \).

Then, sequence \( \{u_i\} \) p-converges to \( u_0 \).

**Proof.** For a given \( \varepsilon > 0 \), by uniform convergence of sequence \( \{T_i\} \) to \( T_0 \) there exist a positive integer \( N \) such that

\[
p(T_0x, T_i x) \leq \frac{\varepsilon (1-\delta)}{(1+\alpha+\beta+\gamma)} \quad \text{for} \; i \geq N \tag{1}
\]

Using triangle inequality, we have

\[
p(u_0, u_i) = p(T_0u_0, T_i u_i)
\leq p(T_0u_0, T_i u_0) + p(T_0u_0, T_i u_i) \tag{2}
\]

Now, using inequality (6.2.3), we have

\[
p(T_0u_0, T_0u_i) \leq \alpha \left[ p(u_0, T_0u_0) + q(u_i, T_0u_0) \right]
+ \beta \frac{q(u_i, T_0u_i) [1 + p(u_0, T_0u_0) + q(u_i, T_0u_0)]}{1 + p(u_0, T_0u_i) + q(u_i, T_0u_i)}
+ \gamma \frac{p(u_0, T_0u_0) q(u_i, T_0u_0) + p(u_0, T_0u_i) q(u_i, T_0u_i)}{p(u_0, T_0u_0) + p(u_0, T_0u_i)}
+ \delta p(u_0, u_i)
= \alpha q(u_i, T_0u_i) + \beta \frac{q(u_i, T_0u_i)[1+q(u_i, T_0u_0)]}{1 + p(u_0, T_0u_i) + p(T_0u_i, u_i)}
+ \gamma p(T_0u_i, u_i) + \delta p(u_0, u_i)
= (\alpha + \beta + \gamma) p(T_0u_i, T_i u_i) + \delta p(T_0u_0, T_i u_i) \tag{3}
\]

From inequalities (2) and (3), we obtain
\[ p(u_0, u_i) \leq \frac{(1 + \alpha + \beta + \gamma)}{1 - \delta} p(T_0 u_i, T_i u_i) \]

Now, using inequality (1), we get

\[ p(u_0, u_i) \leq \frac{(1 + \alpha + \beta + \gamma)}{1 - \delta} \frac{(1 - \delta) \varepsilon}{(1 + \alpha + \beta + \gamma)} = \varepsilon \]

This proves that sequence \( \{u_i\} \) p-converges to \( u_0 \).

This completes the proof of Theorem 3.

**Remark.** If \( \gamma = 0 \), we get Theorem 3 of Namdeo and Tiwari [2].

(6.3) In this section of the Chapter, we study some fixed point theorems for densifying mappings on complete metric spaces.

In 1966, Kuratowski [1] has introduced and studied the important concept of “measure of non-compactness of bounded set”.

**Definition 5.** Measure of non-compactness of bounded set \( A \) of a metric space \( X \), denoted by \( \alpha (A) \), is the infimum of all \( \varepsilon > 0 \) such that \( A \) admits a finite covering by sets with diameter less than \( \varepsilon \).

Nussbaum [1] and Iseki [2] have studied and obtained many useful properties of the measure of non-compactness of bounded set \( A \). Some of these important properties are as follows:

(i) \( 0 \leq \alpha (A) \leq \delta (A) \) where \( \delta (A) \) is the diameter of \( A \),

(ii) If \( X \) is complete and \( \alpha (A) = 0 \), then \( A \) is compact,

(iii) \( \alpha (\overline{A}) = 0 \) iff \( \alpha (A) = 0 \) where \( \overline{A} \) is the closure of \( A \),

(iv) \( \alpha (A \cup B) = \max \{ \alpha (A), \alpha (B) \} \) for any two bounded subsets \( A \) and \( B \) of \( X \).

The concept of densifying mapping was introduced and studied by Furi and Vignoli [1].
**Definition 6.** A mapping $T$ defined on a metric space $X$ to itself is called densifying if for every bounded subset $A$ of $X$ with $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

Furi and Vignoli [1] have proved the following fixed point theorem.

**Theorem C.** Let $T$ be a continuous densifying mapping of a bounded complete metric space $(X, d)$ into itself. If for every $x, y$ in $X, x \neq y$,

$$d(Tx, Ty) < d(x, y).$$

Then $T$ has a unique fixed point.


Sharma and Shrivastava [1] proved the following fixed point theorem for densifying mappings on complete metric space.

**Theorem D.** Let $S$ and $T$ are two densifying self mappings of a bounded complete metric space $(X, d)$ satisfying

$$(6.3.1) \quad d(Sx, STy) < \alpha_1 d(x, Ty) + \alpha_2 [d(x, Sx) + d(Ty, STy)]$$

$$\quad + \alpha_3 [d(x, STy) + d(Ty, Sx)],$$

for all $x, y$ in $X$ with $x \neq Ty$,

$$(6.3.2) \quad ST = TS,$$

where $\alpha_1, \alpha_2$ and $\alpha_3$ are non-negative real constants and $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1,$

then $S$ and $T$ have a common fixed point of $X$, which is unique whenever $\alpha_1 + 2\alpha_3 = 1$.

**6.4** Now, we prove following two fixed point theorems for densifying mappings which generalize the result of Sharma and Shrivastava [1].
Theorem 4. Let $S$ and $T$ be continuous densifying mappings of a bounded complete metric space $(X, d)$ into itself satisfying the condition (6.3.2) and inequality

\[(6.4.1) \quad d(Sx, STy) < \alpha_1 d(x, Ty) + \alpha_2 [d(x, Sx) + d(Ty, STy)] + \alpha_3 [d(x, STy) + d(Ty, Sx)] + \alpha_4 \frac{d(x, STy) d(Ty, Sx) + d(Ty, STy) d(x, Sx)}{d(Ty, Sx) + d(Ty, STy)},\]

for all $x, y$ in $X$ with $x \neq Ty$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are non-negative real constants and $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = 1$, then $S$ and $T$ have a common fixed point in $X$ which is unique whenever $\alpha_1 + 2\alpha_3 + \alpha_4 = 1$.

Proof. Let $x_0$ be a point of $X$ and let $\{x_n\}$ be a sequence of $X$ defined by

$Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$, for $n = 0, 1, 2, \ldots \ldots$

Put $A = \{x_{2n+1} : n = 0, 1, 2, \ldots \}$, then $ST(A) = \{x_{2n+1} : n = 1, 2, \ldots \}$, and so $ST(A) \subset A$. By the continuity of $S$ and $T$, $ST(\overline{A}) \subset \overline{ST(A)} \subset \overline{A}$, and hence $\overline{A}$ is invariant under $ST$ and is bounded. To prove $\overline{A}$ is compact it is sufficient to prove $\alpha(A) = 0$. $\alpha(A)$ is finite since $A$ is bounded. Let $\alpha(A) > 0$,

Since $A = ST(A) \cup \{x_1\}$ and hence

$\alpha(A) = \max \{\alpha(x_1), \alpha(ST(A))\} = \alpha(ST(A)) < \alpha(A),$

this contradiction proves that $\alpha(A) = 0$ and hence $\overline{A}$ is compact.

We now define a real-valued function $f$ by $f(x) = d(Tx, STx)$ for all $x \in \overline{A}$. $f$ being composite of continuous functions $d$, $T$ and $S$, is continuous on compact space $\overline{A}$ and hence there exists $z \in \overline{A}$ such that $f(z) = \inf \{f(x) : x \in \overline{A}\}$.

Let $Sz \neq z$, then by using inequality (6.4.1) and condition (6.3.2),
we have

\[ f(Sz) = d(TSz, STSz) = d(STz, STSz) \]

\[ < \alpha_1 d(Tz, TSz) + \alpha_2 [d(Tz, STz) + d(TSz, STSz)] \]

\[ + \alpha_3 [d(Tz, STSz) + d(TSz, STz)] \]

\[ + \alpha_4 \frac{d(Tz, STSz) d(TSz, STz) + d(TSz, STSz) d(Tz, TSz)}{d(TSz, STz) + d(TSz, STSz)} \]

\[ = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(Tz, STz) + (\alpha_2 + \alpha_3) d(TSz, STSz) \]

i.e. \( f(Sz) < \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3} d(Tz, STz) \)

\[ = h d(Tz, STz) \quad \text{where} \quad h = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3} = 1 \]

\[ = d(Tz, STz) = f(z), \]

this contradicts to the choice of \( f(z) \) which proves that \( Sz = z \). Therefore, \( STz = TSz = Tz \).

Now we shall prove that \( Tz = z \). Let \( Tz \neq z \), then using inequality (6.4.1), we have

\[ d(z, Tz) = d(Sz, STz) \]

\[ < \alpha_1 d(z, Tz) + \alpha_2 [d(z, Sz) + d(Tz, STz)] \]

\[ + \alpha_3 [d(z, STz) + d(Tz, Sz)] \]

\[ + \alpha_4 \frac{d(z, STz) d(Tz, Sz) + d(Tz, STz) d(z, Sz)}{d(Tz, Sz) + d(Tz, STz)} \]

\[ = \alpha_1 d(z, Tz) + \alpha_2 [d(z, z) + d(Tz, Tz)] \]

\[ + \alpha_3 [d(z, Tz) + d(Tz, z)] \]

\[ + \alpha_4 \frac{d(z, Tz) d(Tz, z) + d(Tz, Tz) d(z, z)}{d(Tz, z) + d(Tz, Tz)} \]

\[ = (\alpha_1 + 2\alpha_3 + \alpha_4) d(z, Tz) < d(z, Tz), \]
a contradiction which proves that $Tz = z$.

i.e. $z$ is the common fixed point of $S$ and $T$.

To prove uniqueness, let $w$ be another common fixed point of $S$ and $T$, then using inequality (6.4.1), we have

$$d(z, w) = d(Sz, STw)$$

$$< \alpha_1 d(z, Tw) + \alpha_2 [d(z, Sz) + d(Tw, STw)]$$

$$+ \alpha_3 [d(z, STw) + d(Tw, Sz)]$$

$$+ \alpha_4 \frac{d(z, STw) d(Tw, Sz) + d(Tw, STw) d(z, Sz)}{d(Tw, Sz) + d(Tw, STw)}$$

$$= (\alpha_1 + 2\alpha_3 + \alpha_4) d(z, w) < d(z, w),$$

a contradiction which proves that $z = w$. i.e. $z$ is the unique common fixed point of $S$ and $T$.

This completes the proof of Theorem 4.

**Remark**: If $\alpha_4 = 0$, we get Theorem D of Sharma and Shrivastava [1].

**Theorem 5.** Let $S$ and $T$ be two continuous densifying mappings of a bounded complete metric space $(X, d)$ into itself satisfying condition (6.3.2) and inequality (6.4.2)

$$d(Sx, STy) < \max \{d(x, Ty), \frac{1}{2} [d(x, Sx) + d(Ty, STy)],$$

$$\frac{1}{2} [d(x, STy) + d(Ty, Sx)],$$

$$\frac{d(x, STy) d(Ty, Sx) + d(Ty, STy) d(x, Sx)}{d(Ty, Sx) + d(Ty, STy)}\},$$

for all $x, y$ in $X$ with $x \neq Ty$, then $S$ and $T$ have a common fixed point in $X$ which is unique.

**Proof.** Suppose that sequence $\{x_n\}$ is defined as in the proof of Theorem 4.

Put $A = \{x_{2n+1} : n = 0, 1, 2, \ldots\}$, then $ST(A) = \{x_{2n+1} : n = 1, 2, \ldots\}$,
and so, $ST(A) \subseteq A$. By the continuity of $S$ and $T$, $ST(\overline{A}) \subseteq \overline{ST(A)} \subseteq \overline{A}$, and hence $\overline{A}$ is invariant under $ST$ and is bounded. To prove $\overline{A}$ is compact it is sufficient to prove $\alpha(A) = 0$. $\alpha(A)$ is finite since $A$ is bounded. Let $\alpha(A) > 0$,

Since $A = ST(A) \cup \{x_i\}$ and hence

$\alpha(A) = \max \{\alpha(x_i), \alpha(ST(A))\} = \alpha(ST(A)) < \alpha(A),$

a contradiction which proves that $\alpha(A) = 0$ and hence $\overline{A}$ is compact.

We now define a real-valued function $f$ by $f(x) = d(Tx, STx)$ for all $x \in \overline{A}$. $f$ being composite of continuous functions $d$, $T$ and $S$, is continuous on compact space $\overline{A}$ and hence there exists $z \in \overline{A}$ such that $f(z) = \inf \{f(x) : x \in \overline{A}\}$.

Let $Sz \neq z$, then by using inequality (6.4.2) and condition (6.3.2), we have

$f(Sz) = d(TSz, STSz) = d(STz, STSz)$

\[ < \max \{d(Tz, TSz), \frac{1}{2} [d(Tz, STz) + d(TSz, STSz)]\}, \]

\[ \frac{1}{2} [d(Tz, STSz) + d(TSz, STz)], \]

\[ \frac{d(Tz, STSz) d(TSz, STz) + d(TSz, STSz) d(Tz, STz)}{d(TSz, STz) + d(TSz, STSz)} \]

i.e. $f(Sz) < d(Tz, TSz) = f(z)$,

this contradicts the choice of $f(z)$, which proves that $Sz = z$. Therefore, $STz = TSz = Tz$.

Now, let $Tz \neq z$, then using inequality (6.4.2), we have

$d(z, Tz) = d(Sz, STz)$

\[ < \max \{d(z, Tz), \frac{1}{2} [d(z, Sz) + d(Tz, STz)], \]

\[ \frac{1}{2} [d(z, STz) + d(Tz, Sz)] \]
\[
\frac{d(z, STz) \cdot d(Tz, Sz) + d(Tz, STz) \cdot d(z, Sz)}{d(Tz, Sz) + d(Tz, STz)}
\]

\[
= \max \{d(z, Tz), 0, d(z, Tz), d(z, Tz)\}
\]

\[
= d(z, Tz),
\]

a contradiction. Thus, Tz = z.

Hence, z is the common fixed point of S and T.

To prove uniqueness of z, let w be another common fixed point of S and T, then using inequality (6.4.2), we have

\[
d(z, w) = d(Sz, STw)
\]

\[
< \max \{d(z, Tw), \frac{1}{2} [d(z, Sz) + d(Tw, STw)],
\]

\[
\frac{1}{2} [d(z, STw) + d(Tw, Sz)],
\]

\[
\frac{d(z, STw) \cdot d(Tw, Sz) + d(Tw, STw) \cdot d(z, Sz)}{d(Tw, Sz) + d(Tw, STw)}
\]

\[
= \max \{d(z, w), 0, d(z, w), d(z, w)\}
\]

\[
= d(z, w),
\]

a contradiction. Thus, z = w, which proves that z is the unique common fixed point of S and T.

This completes the proof the Theorem 5.