CHAPTER - V

FIXED POINT THEOREMS ON

NORMED VECTOR SPACES

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(5.1) In this Chapter, we have studied the role of different iterative schemes in existence of fixed points in normed vector spaces. Mann [1] introduced a very useful iteration scheme called "Mann iteration scheme" and Cho, Fisher and Kang [1] generalized the scheme by introducing Mann-type iteration scheme. We have used these schemes in finding our results in this chapter.

Telci, Tas and Fisher [1] proved an important common fixed point theorem using Mann-iteration scheme on normed vector spaces. We have generalized this result in section 5.2 of this chapter.

Cho, Fisher and Kang [1] proved common fixed point theorems for four mappings on closed convex subset of Banach spaces. We have generalized these results in section 5.3 of this chapter.

Some preliminaries, which we need in order to prove our results are as follows.

Definition 1. A subset C of a real vector space X is said to be convex if

\[ \lambda x + (1 - \lambda) y \in C \]

for all \( x, y \) in \( C \) where \( 0 \leq \lambda < 1 \).

Definition 2. A normed vector space \( X \) is said to be uniformly convex if for all \( \varepsilon > 0 \), \( x, y \) in \( X \) such that \( ||x|| = ||y|| = 1 \) and \( ||x-y|| > 0 \) imply that

\[ ||1/2(x+y)|| \leq 1 - \delta \]
where \( \delta = \delta (\varepsilon) \) is independent of \( x \) and \( y \) and \( 0 < \delta < 1 \).

**Definition 3.** Let \( T \) be a selfmapping defined on a complete metric space \( X \). A sequence \( \{x_n\} \) associated with \( T \) is called iterative sequence if for an arbitrary point \( x_0 \) in \( X \), \( T^n : X \rightarrow X \) is defined by

\[
T^0 x_0 = x_0, \quad T x_0 = x_1, \quad T^2 x_0 = T x_1 = x_2, \quad \ldots, \quad T^n x_0 = x_n, \quad \ldots \text{ for } n \geq 1.
\]

Krasnoselskii [1] proved that if \( X \) be a Banach space and \( F \) be a mapping from a non-empty closed convex subset \( C \) of \( X \) into itself and if \( F \) is non-expansive for all \( x, y \) in \( C \), then for some \( x_0 \) in \( C \), the sequence \( \{F^n x_0\} \) does not converge necessarily to a fixed point of \( F \).

He proved that if \( X \) is uniformly convex and \( C \) is a compact subset of \( X \) and if the sequence \( \{F^n x_0\} \) for some \( x_0 \) in \( C \), where

\[
(S_1) \quad F_\lambda = (1-\lambda) I + \lambda F, \quad 0 < \lambda \leq 1,
\]

where \( I \) is the identity mapping and \( \lambda = 1/2 \), then the sequence may converge to a fixed point of \( F \). In [1], Schaefer extended the result for general \( \lambda \).

The scheme \((S_1)\) has been extended by Mann [1] by another iterative scheme called "Mann iteration scheme" associated with \( F \), where \( F \) be a mapping from a non-empty closed convex subset \( C \) of a Banach space \( X \) into itself, in the following way:

For some \( x_0 \) in \( X \), define a sequence \( \{x_n\} \) in \( X \) by

\[
(S_2) \quad x_{n+1} = (1-c_n) x_n + c_n F x_n
\]

for \( n = 0, 1, 2, \ldots \), where \( \{c_n\} \) is a sequence of real numbers such that

\[
(i) \quad 0 < c_n < 1 \quad (ii) \quad \sum_{n=0}^{\infty} c_n = +\infty.
\]
In 1971, Goebel and Zlotkiewicz [1] proved following important theorem.

**Theorem A.** If \( C \) is a closed and convex subset of a Banach space and if

(i) \( F^2 = I \), where \( I \) is the identity mapping,

(ii) \( \|Fx - Fy\| \leq k \|x - y\| \),

for all \( x, y \) in \( C \), where \( 0 \leq k < 2 \). Then \( F \) has at least one fixed point.

Pathak [1] proved the following theorem for two mappings.

**Theorem B.** Let \( C \) be a closed and convex subset of Banach space \( X \).

Let \( F : C \rightarrow C \) and \( G : C \rightarrow C \) satisfy the following conditions

(a) \( F \) and \( G \) commute,

(a) \( F^2 = I \) and \( G^2 = I \), where \( I \) denotes the identity mapping,

(a) \( \|Fx - Fy\|^2 \leq q \max \{ \|Gx - Fx\|, \|Gy - Fy\|, \|Gx - Fy\|, \|Gy - Fx\|, \}

\( \|Gx - Fx\|, \|Gy - Fx\|, \|Gx - Fy\|, \|Gy - Fy\| \}, \)

for all \( x, y \) in \( C \), where \( q \in (0, 1) \). Let \( x_1 \) in \( C \) be arbitrary, \( t \in (0, 1) \) and \( Gx_{n+1} = (1-t) Gx_n + t Fx_n \) for each integer \( n \geq 1 \). If the sequence \( \{Gx_n\} \) converges to a point \( u \) in \( C \), then \( u \) is the unique common fixed point of \( F \) and \( G \).

In 1993, Telci, Tas and Fisher [1] proved the following fixed point theorem using Mann-iteration scheme on normed vector space.

**Theorem C.** Let \( C \) be a non-empty, closed convex subset of a normed vector space \( X \) and let \( S, T \) be mappings of \( C \) into itself satisfying the inequality

\[ \|S^p x - Ty\| \leq k \max \{ \|S^p x - x\|, \|S^p x - y\|, \|Ty - y\|, \|Ty - x\|, \|x - y\| \} \]

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for all $x, y$ in $C$ where $0 < k < 1$ and $p$ is a positive integer. If $\{x_n\}$ is a sequence of Mann-iterates associated with $T$ which converges to a point $z$ in $C$, then $z$ is the unique fixed point of $S$ and $T$.

In ([2], [3]) Jungck defined concept of compatibility of two mappings.

**Definition 4.** Let $A$ and $S$ be two mappings from a normed vector space $(X, \|\cdot\|)$ into itself. The mappings $A$ and $S$ are said to be compatible if

$$
\lim_{n \to \infty} \|ASx_n - SAx_n\| = 0
$$

whenever $\{x_n\}$ is a sequence in $X$ such that

$$
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z \text{ for some } z \in X.
$$

Jungck also proved the following useful lemma.

**Lemma 1.** Let $A$ and $S$ be compatible mappings of a normed vector space $(X, \|\cdot\|)$ into itself. If $Az = Sz$ for some $z \in X$, then

$$
ASz = S^2z = SAz = A^2z.
$$


**Theorem D.** Let $C$ be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$ and $A, B, S$ and $T$ be mappings from $C$ into itself satisfying the following conditions:

(5.1.1) there exist constants $\alpha, \beta, \gamma, \delta \geq 0$ such that

$$
\|Sx - Ty\| \leq \alpha \|Ax - By\| + \beta \|Ax - Sx\|
$$

$$
+ \gamma \max\{\|By - Ty\|, \|Ax - Ty\|\} + \delta \|By - Sx\|
$$

for all $x, y$ in $C$, where $0 \leq \max\{\alpha + \gamma + \delta, \beta + \delta\} < 1$. 

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(5.1.2) for some \( x_0 \in C \), there exists a constant \( k \in [0, 1) \) such that
\[
\| x_{n+2} - x_{n+1} \| \leq k \| x_{n+1} - x_n \|
\]
for \( n = 0, 1, 2 \ldots \ldots \), where \( \{ x_n \} \) is a sequence in \( C \) defined by
\[
\begin{align*}
(A) x_{2n+1} &= \frac{1}{2} A x_{2n} + \frac{1}{2} S x_{2n} \\
(B) x_{2n+2} &= \frac{1}{2} B x_{2n+1} + \frac{1}{2} T x_{2n+1}
\end{align*}
\]
(5.1.3) the pairs \( \{ A, S \} \) and \( \{ B, T \} \) are compatible.

Then, the sequence \( \{ x_n \} \) defined by \( (S_3) \) converges to a point \( z \in C \).

Further, if \( A \) and \( B \) are continuous at the point \( z \), then \( Tz \) is a unique common fixed point of \( A, B, S \) and \( T \).

Cho, Fisher and Kang [1] gave the name to scheme \( (S_3) \) as "Mann-type iteration scheme".

(5.2) In this section, we generalize the Theorem C of Telci, Tas and Fisher [1] and proved fixed point theorem for two mappings on normed vector space using Mann-iteration scheme.

**Theorem 1.** Let \( C \) be a non-empty, closed, convex subset of a normed vector space \( X \) and let \( S \) and \( T \) be mappings of \( C \) into itself satisfying the following inequality
\[
(5.2.1) \quad \| S^p x - T y \|^2 \leq k \max \{ \| S^p x - x \| \| T y - y \|, \| S^p x - y \| \| T y - x \|, \| S^p x - x \| \| T y - x \|, \| S^p x - y \| \| T y - y \|, \| x - y \|^2 \},
\]
for all \( x, y \) in \( C \), where \( 0 < k < 1 \) and \( p \) is a positive integer. If \( \{ x_n \} \) is a sequence of Mann-iterates associated with \( T \) which converges to a point \( z \) in \( C \). Then, \( z \) is the unique common fixed point of \( S \) and \( T \).

**Proof.** Suppose that \( \{ x_n \} \) is a sequence of Mann-iterates with \( T \)
converging to $z$ in $C$. Then, by definition

$$x_{n+1} = (1 - \lambda_n) x_n + \lambda_n T x_n$$

and so

$$T x_n = \lambda_n^{-1} (x_{n+1} - x_n) + x_n$$

Letting $n$ tend to infinity, it follows that $\{T x_n\}$ is a sequence in $C$ converging to $z$.

Now, using inequality (5.2.1), we have

$$||S^p z - T x_n||^2 \leq k \max \{||S^p z - z||, ||T x_n - x_n||, ||S^p z - x_n||, ||T x_n - z||, ||S^p z - z||, ||S^p z - x_n||, ||T x_n - x_n||, ||z - x_n||^2\}$$

Letting $n$ tend to infinity, we obtain

$$||S^p z - z||^2 \leq k \max \{||S^p z - z||, ||z - z||, ||S^p z - z||, ||z - z||, ||S^p z - z||, ||T z - z||, ||z - z||^2\}$$

$$= 0$$

It follows that $z$ is a fixed point of $S^p$.

To show that $z$ is also a fixed point of $T$. Using inequality (5.2.1) again, we have

$$||z - T z||^2 = ||S^p z - T z||^2$$

$$\leq k \max \{||S^p z - z||, ||T z - z||, ||S^p z - z||, ||T z - z||, ||S^p z - z||, ||T z - z||, ||z - z||^2\}$$

$$= 0$$

It follows that $z$ is a fixed point of $T$.

We now prove that $z$ is also a fixed point of $S$. Using inequality (5.2.1), we have

$$||S z - z||^2 = ||S^p (S z) - T z||^2$$

$$\leq k \max \{||S^p (S z) - S z||, ||T z - z||, ||S^p (S z) - z||, ||T z - S z||, ||z - z||^2\}$$
\[\|S^p(Sz) - Sz\| \leq k \max \{0, \|Sz - z\|^2, 0, 0, \|Sz - z\|^2\}\]

\[= k \max \{0, \|Sz - z\|^2, 0, 0, \|Sz - z\|^2\}\]

i.e. \(\|Sz - z\|^2 \leq k \|Sz - z\|^2 < \|Sz - z\|^2\),

a contradiction. Thus, \(Sz = z\). i.e. \(z\) is a fixed point of \(S\) as well.

To prove uniqueness, let \(z'\) is another fixed point of \(S\). Then, using inequality (5.2.1), we have

\[\|z' - z\|^2 = \|S^p z' - Tz\|^2\]

\[\leq k \max \{\|S^p z' - z\| \|Tz - z\|, \|S^p z' - z\| \|Tz - z\|, \|S^p z' - z\| \|Tz - z\|, \|z' - z\|^2\}\]

\[= k \max \{0, \|z' - z\|^2, 0, 0, \|z' - z\|^2\}\]

i.e. \(\|z' - z\|^2 \leq k \|z' - z\|^2 < \|z' - z\|^2\),

a contradiction. Thus, \(z = z'\).

Now, let \(z'\) be another fixed point of \(T\). Then, using inequality (5.2.1), we have

\[\|z - z'^2\| = \|S^p z - Tz'\|^2\]

\[\leq k \max \{\|S^p z - z\| \|Tz' - z\|, \|S^p z - z\| \|Tz' - z\|, \|S^p z - z\| \|Tz' - z\|, \|z - z'^2\|\}\]

\[= k \max \{0, \|z - z'^2\|^2, 0, 0, \|z - z'^2\|^2\}\]

i.e. \(\|z - z'^2\|^2 \leq k \|z - z'^2\|^2 < \|z - z'^2\|^2\),

a contradiction. Thus, \(z = z'\). i.e. \(z\) is the unique common fixed point of \(S\) and \(T\).

This completes the proof of Theorem 1.

Corollary 1. Let \(C\) be a non-empty, closed, convex subset of a normed vector space \(X\) and let \(T\) be mapping of \(C\) into itself satisfying the following inequality
\[ \|T^p x - Ty\|^2 \leq k \max \{ \|T^p x - x\| \|Ty - y\|, \|T^p x - y\| \|Ty - x\|, \|T^p x - x\| \|Ty - x\|, \|T^p x - y\| \|Ty - y\|, \|x - y\|^2 \}, \]

for all \( x, y \) in \( C \), where \( 0 < k < 1 \) and \( p \) is a positive integer. If \( \{x_n\} \) is a sequence of Mann-iterates associated with \( T \) which converges to a point \( z \) in \( C \). Then, \( z \) is the unique fixed point of \( T \).

**Proof.** On putting \( S = T \) in the proof of Theorem 1, we get the proof of this Corollary.

(5.3) Cho, Fisher and Kang [1] proved a fixed point theorem (Theorem D, Section 5.1) for two pairs of compatible mappings on Banach spaces using Mann-type iteration scheme. We have extended their result as follows.

**Theorem 2.** Let \( C \) be a nonempty, closed, convex subset of a Banach space \( (X, \|\cdot\|) \) and \( A, B, S \) and \( T \) be mappings from \( C \) into itself satisfying conditions (5.1.2), (5.1.3) of Theorem D and

\[ (5.3.1) \quad \|Sx - Ty\| \leq \alpha \frac{f(x, y)}{g(x, y)}, \]

for all \( x, y \) in \( C \), where \( 0 \leq \alpha < 1 \) and

\[ f(x, y) = \max \{ \|Ax - By\|^2, \|Ax - Sx\| \|By - Ty\|, \|Ax - Ty\| \|By - Sx\|, \|Ax - Sx\| \|By - Sx\|, \|Ax - Ty\| \|By - Ty\| \}, \]

\[ g(x, y) = \max \{ \|Ax - By\|, \|By - Ty\|, \|By - Sx\| \} \]

Then the sequence \( \{x_n\} \) defined by \( (S_n) \) converges to a point \( z \) in \( C \). Further, if \( A \) and \( B \) are continuous at the point \( z \), then \( Tz \) is a unique common fixed point of \( A, B, S \) and \( T \).
**Proof.** From condition (5.1.2), it follows that

\[ \|x_{n+2} - x_{n+1}\| \leq k^{n+1} \|x_1 - x_0\| \quad \text{for } n = 0, 1, 2, \ldots \]

and so \( \{x_n\} \) is a Cauchy sequence in \( C \). Since \( X \) is a Banach space, \( C \) is complete, and hence \( \{x_n\} \) converges to some point \( z \) in \( C \).

From condition \((S_3)\), it follows that

\[ \frac{1}{2} Sx_{2n} = Ax_{2n+1} - \frac{1}{2} Ax_{2n} \]

and since \( A \) is continuous at \( z \), we have

\[ Az = \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_{2n} \]

Similarly, since \( B \) is continuous at \( z \), we have

\[ Bz = \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_{2n+1} \]

Using condition (5.3.1), we have

\[ \max \{ \|Ax_{2n} - Bz\|^2, \|Ax_{2n} - Sx_{2n}\| \|Bz - Tz\|, \]

\[ \|Ax_{2n} - Tz\| \|Bz - Sx_{2n}\|, \|Ax_{2n} - Sx_{2n}\| \|Bz - Sx_{2n}\|, \]

\[ \|Ax_{2n} - Tz\| \|Bz - Tz\| \}

\[ \|Sx_{2n} - Tz\| \leq \alpha \frac{\|Ax_{2n} - Tz\| \|Bz - Tz\|}{\max \{ \|Ax_{2n} - Bz\|, \|Bz - Tz\|, \|Bz - Sx_{2n}\| \}} \]

Letting \( n \) tends to infinity, we obtain

\[ \max \{ \|Az - Bz\|^2, 0, \|Az - Tz\| \|Bz - Az\|, 0, \]

\[ \|Az - Tz\| \|Bz - Tz\| \}

\[ \|Az - Tz\| \leq \alpha \frac{\|Az - Tz\| \|Bz - Tz\|}{\max \{ \|Az - Bz\|, \|Bz - Tz\|, \|Bz - Az\| \}} \quad (1) \]

Using condition (5.3.1), we have

\[ \max \{ \|Az - Bx_{2n+1}\|^2, \|Az - Sz\| \|Bx_{2n+1} - Tx_{2n+1}\|, \]

\[ \|Az - Tx_{2n+1}\| \|Bx_{2n+1} - Sz\|, \|Az - Sz\| \|Bx_{2n+1} - Sz\|, \]

\[ \|Az - Tx_{2n+1}\| \|Bx_{2n+1} - Tx_{2n+1}\| \}

\[ \|Sz - Tx_{2n+1}\| \leq \alpha \frac{\|Az - Tx_{2n+1}\| \|Bx_{2n+1} - Tx_{2n+1}\|}{\max \{ \|Az - Bx_{2n+1}\|, \|Bx_{2n+1} - Tx_{2n+1}\|, \|Bx_{2n+1} - Sz\| \}} \]
Letting \( n \) tend to infinity, we obtain

\[
\max \left\{ \|Az - Bz\|^2, 0, \|Az - Bz\| \|Bz - Sz\| \right\}
\]

\[
\|Sz - Bz\| \leq \frac{\|Az - Sz\| \|Bz - Sz\|}{\max \left\{ \|Az - Bz\|, 0, \|Bz - Sz\| \right\}}
\]

(2)

Again, using condition (5.3.1), we have

\[
\max \left\{ \|Ax_{2n} - Bx_{2n+1}\|^2, \|Ax_{2n} - Sx_{2n}\| \|Bx_{2n+1} - Tx_{2n+1}\|, \right\}
\]

\[
\|Ax_{2n} - Tx_{2n+1}\| \|Bx_{2n+1} - Sx_{2n}\|, \|Ax_{2n} - Sx_{2n}\| \|Bx_{2n+1} - Sx_{2n}\|, \right\}
\]

\[
\|Sx_{2n} - Tx_{2n+1}\| \leq \frac{\|Ax_{2n} - Sx_{2n}\| \|Bx_{2n+1} - Tx_{2n+1}\|}{\max \left\{ \|Ax_{2n} - Bx_{2n+1}\|, \|Bx_{2n+1} - Tx_{2n+1}\|, \|Bx_{2n+1} - Sx_{2n}\| \right\}}
\]

Letting \( n \) tend to infinity, we obtain

\[
\|Az - Bz\| \leq \frac{\max \left\{ \|Az - Bz\|^2, 0, \|Az - Bz\|\right\}}{\max \left\{ \|Az - Bz\|, 0, \|Az - Bz\| \right\}}
\]

i.e. \( \|Az - Bz\| \leq \alpha \|Az - Bz\| < \|Az - Bz\| \)

a contradiction. Thus, \( Az = Bz \).

Now, putting \( Az = Bz \) in inequalities (1) and (2), we obtain

\( Az = Tz \) and \( Sz = Bz \) respectively. Therefore, combining the above results, we obtain

\( Az = Bz = Sz = Tz \).

(3)

i.e. \( z \) is a coincidence point of \( A, B, S \) and \( T \).

Now, again using inequality (5.3.1), we have

\[
\max \left\{ \|ASz - Bz\|^2, \|ASz - S^2z\| \|Bz - Tz\|, \right\}
\]

\[
\|ASz - Tz\| \|Bz - S^2z\|, \|ASz - S^2z\| \|Bz - S^2z\|, \right\}
\]

\[
\|S^2z - Tz\| \leq \frac{\|ASz - Tz\| \|Bz - Tz\|}{\max \left\{ \|ASz - Bz\|, \|Bz - Tz\|, \|Bz - S^2z\| \right\}}
\]

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\[
\max \left\{ \|A^2z-Tz\|^2, 0, \|A^2z-Tz\| \|Tz-S^2z\| \right\} \\
= \alpha \frac{\|A^2z-S^2z\| \|Tz-S^2z\|}{\max \left\{ \|A^2z-Tz\|, 0, \|Tz-S^2z\| \right\}}
\]

Now, since \{A, S\} is compatible and \(Az = Sz\), then by Lemma 1, we have \(A^2z = S^2z\) and hence we get
\[
\|S^2z-Tz\| \leq \alpha \|S^2z-Tz\| < \|S^2z-Tz\|,
\]
a contradiction, hence \(S^2z = Tz\).

Using equations (3), (4) and Lemma 1, we obtain
\[
SAz = S^2z = A^2z = STz = ATz = Tz,
\]
and so, \(Tz\) is a common fixed point of \(A\) and \(S\).

Similarly, by using compatibility of \{B, T\} we can prove
\[
BTz = T^2z = TBz = B^2z = Tz,
\]
and so \(Tz\) is also a common fixed point of \(B\) and \(T\).

Thus, \(Tz\) is a common fixed point of \(A, B, S\) and \(T\).

To prove uniqueness, let \(Tz'\) be another common fixed point of \(A, B, S\) and \(T\).

Then, on using inequality (5.3.1), we have
\[
\|Tz-Tz'\| = \|STz-T^2z'\|
\]
\[
= \alpha \frac{\|ATz-BTz'\|^2, \|ATz-STz\| \|BTz'-T^2z'\|}{\max \left\{ \|ATz-BTz'\|, \|BTz'-T^2z'\| \|BTz'-STz\| \right\}}
\]
\[
\leq \alpha \frac{\|ATz-T^2z\| \|BTz'-STz\|, \|ATz-STz\| \|BTz'-STz\|}{\max \left\{ \|ATz-BTz'\|, \|BTz'-T^2z'\| \|BTz'-STz\| \right\}}
\]
\[
= \alpha \frac{\max \left\{ \|Tz-Tz\|^2, 0, \|Tz-Tz\|^2 \right\}}{\max \left\{ \|Tz-Tz\|, 0, \|Tz-Tz\| \right\}}
\]
\[
i.e. \|Tz-Tz'\| \leq \alpha \|Tz-Tz'\| < \|Tz-Tz'\|,
\]

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a contradiction. Thus, Tz = Tz'.

Hence, Tz is the unique fixed point of A, B, S and T.

This completes the proof of Theorem 2.

**Corollary 2.** Let C be non-empty, closed, convex subset of a Banach space \((X, \|\cdot\|)\) and A, S be mappings from C into itself satisfying the following conditions:

\[
\max \left\{ \|Ax - Ay\|^2, \|Ax - Sx\| \|Ay - Sy\|, \|Ax - Sy\| \|Ay - Sx\|, \|Ax - Sx\| \|Ay - Sx\|, \|Ax - Sy\| \|Ay - Sy\| \right\} \\
\leq \alpha \frac{\|Ax - Sy\| \|Ay - Sy\|}{\max \left\{ \|Ax - Ay\|, \|Ay - Sx\|, \|Ay - Sx\| \right\}}
\]  

(5.3.2)  

for all x, y in C, where \(0 \leq \alpha < 1\),

(5.3.3)  

for some \(x_0\) in C, there exists a constant \(k \in [0, 1)\) such that

\[
\|x_{n+2} - x_{n+1}\| \leq k \|x_{n+1} - x_n\| \quad \text{for} \quad n = 0, 1, 2, \ldots \ldots \quad \text{where} \quad \{x_n\} \quad \text{is a sequence in C defined by}
\]

(5.3.4)  

\[
(S_\alpha) \quad Ax_{n+1} = \frac{1}{2} Ax_n + \frac{1}{2} Sx_n
\]

the pair \(\{A, S\}\) is compatible.

Then the sequence defined by \((S_\alpha)\) converges to a point \(z\) in C. Further, if A is continuous at the point \(z\), then Sz is a unique common fixed point of A and S.

**Proof.** If we take \(S = T\) and \(A = B\) in the proof of Theorem 2, we get the proof of Corollary 2.

**Corollary 3.** Let C be a nonempty, closed, convex subset of a Banach space \((X, \|\cdot\|)\) and S be a mappings from C into itself satisfying the following conditions:
\[
\max \{ \|x-y\|^2, \|x-Sx\|, \|y-Sy\|, \|x-Sy\|, \|y-Sx\| \}, \\
\frac{\|x-Sx\| \|y-Sx\|, \|x-Sy\| \|y-Sy\|}{\max \{ \|x-y\|, \|y-Sy\|, \|y-Sx\| \}} 
\]

(5.3.5) \[\|Sx-Sy\| \leq \alpha \] 

for all \( x, y \) in \( C \), where \( 0 \leq \alpha < 1 \),

(5.3.6) \text{for some } \( x_0 \) in \( C \), there exists a constant \( k \in [0,1) \) such that \( \|x_{n+2}-x_{n+1}\| \leq k \|x_{n+1}-x_n\| \) for \( n = 0, 1, 2, \ldots \) where \( \{x_n\} \) is a sequence in \( C \) defined by

\[ x_{n+1} = \frac{1}{2} x_n - \frac{1}{2} Sx_n \] (S5)

Then the sequence \( \{x_n\} \) defined by \( (S_5) \) converges to a point \( z \) in \( C \). In fact, the point \( z \) is a unique fixed point of \( S \).

**Proof.** If we take \( T = S \) and \( A = B = I \), where \( I \) is the identity mapping in \( C \), in the proof of Theorem 2, the proof of this Corollary follows easily.

**Theorem 3.** Let \( C \) be a nonempty, closed, convex subset of a Banach space \((X, \|\cdot\|)\) and \( A, B, S \) and \( T \) be mappings from \( C \) into itself satisfying the conditions (5.1.2), (5.3.1) and

(5.3.7) \[ A^2 = S^2 = I \text{ or } B^2 = T^2 = I \], where \( I \) is the identity mapping in \( C \).

Then the sequence defined by \( (S_3) \) converges to a point \( z \) in \( C \). Further, if \( A \) and \( B \) are continuous at the point \( z \), then \( z \) is a unique common fixed point of \( A, B, S \) and \( T \).

**Proof.** As in the proof of Theorem 2, we know that the sequence defined by \( (S_3) \) converges to a point \( z \) in \( C \). and then we get \( Az = Bz = Sz = Tz \).

Suppose that \( A^2 = S^2 = I \). Then, using inequality (4) we obtain

\[ \|z-Tz\| \leq \alpha \|z-Tz\| < \|z-Tz\|, \]

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a contradiction. Thus, Tz = z.

Hence, z is the common fixed point of A, B, S and T.

To prove uniqueness of z, let z' be another common fixed point of A, B, S and T. Then, using condition (5.3.1), we have
\[
\max \{ \|Az-Bz\|^{2}, \|Az-Sz\| \|Bz-Tz\|, \\
\|Az-Tz\| \|Bz-Sz\|, \|Az-Sz\| \|Bz'-Sz\|, \\
\|Sz-Tz\| \leq \alpha \frac{\|Az-Tz\| \|Bz'-Tz\|}{\max \{ \|Az-Bz\|, \|Bz'-Tz\|, \|Bz'-Sz\| \}} \leq \alpha \frac{\|z-z'\|^2, 0}{\max \{ \|z-z\|, 0, \|z'-z\| \}}
\]

i.e. \|z-z'\| \leq \alpha \|z-z'\| < \|z-z\|,

a contradiction, Thus, z = z'.

i.e. z is the unique common fixed point of A, B, S and T.

This completes the proof of Theorem 3.