CHAPTER III

FIXED POINT THEOREMS ON QUASI METRIC SPACES AND IN BI-METRIC SPACES

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3.1. Quasi-metric spaces have been studied by Kelley [1] and Lane [1] particularly in bi-topological spaces.

Before going to state our results we give the following definitions:

**DEFINITION 1:** A quasi-metric $p$ on a non-empty set $X$ is a non-negative real valued function on $X \times X$ satisfying:

(i) $p(x, y) = 0$ iff $x = y$; $(x, y \in X)$

(ii) $p(x, z) \leq p(x, y) + p(y, z)$; $(x, y, z \in X)$.

For a quasi-metric $p$ on $X$ there exists a quasi-metric $q$ on $X$ called the conjugate of $p$ given by $q(x, y) = p(y, x)$, for all $x, y \in X$.

**DEFINITION 2:** A sequence $\{x_n\}$ of $X$ is said to be $p$-Cauchy if and only if for $\varepsilon > 0$, there exists a positive integer $k$ such that $p(x_m, x_n) < \varepsilon$ for $m > n \geq k$, and it is called $p$-convergent at a point $x$ in $X$, if $p(x, x_n) \to 0$.
as \( n \to \infty \) and in this case \( \{x_n\} \) is called \( p \)-convergent in \( X \).

**Definition 3**: If every \( p \)-Cauchy sequence in \((X, p)\) is \( p \)-convergent in \( X \), then \((X, p)\) is said to be complete.

**Definition 4**: A self-mapping \( T \) on quasi-metric space \((X, p)\) is said to be orbital continuous at \( x_0 \in X \) if for some \( x \in X \), \( p(x, x_{n_1}) \to 0 \) implies \( p(Tx, Tx_{n_1}) \to 0 \),

where \( \{x_{n_1}\} \) is a subsequence of sequence \( \{x_n\} \) given by \( x_{n+1} = Tx_n \) for \( n = 0, 1, 2, \ldots \).

3.2. In this section, we shall prove some fixed point theorems on quasi-metric spaces.

Following theorem have been obtained by Chikkala and Baisnab [1].

**Theorem 1**: Let \((X, p)\) be a complete quasi-metric space.

Let \( T : X \to X \) satisfy the following conditions.

\[
(3.2.1) \quad p(Tx, Ty) \leq \alpha [q(x, Tx) + q(y, Ty)] + \beta p(x, y), \quad \text{where} \quad \alpha, \beta \geq 0; \quad 2\alpha + \beta < 1.
\]

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(3.2.2) $T$ is $p$-orbitally continuous at some point $x_0$ of $X$. Then there is a unique fixed point of $T$ in $X$.

Now, we prove the following theorems on complete quasi-metric spaces.

**THEOREM 2**: If in Theorem 1, condition (3.2.1) is replaced by

\[(3.2.3) \quad p(Tx,Ty) \leq \alpha \left[ q(x,Tx) + q(y,Ty) \right]
+ \beta \left[ p(x,Ty) + q(y,Tx) \right] + \gamma p(x,y),\]

where $\alpha, \beta, \gamma \geq 0$, $2\alpha + 2\beta + \gamma < 1$. Then $T$ has a unique fixed point in $X$.

**PROOF**: We define a sequence $\{x_n\}$ in $X$ given by

$x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$ Let $x_n \neq x_{n+1}$, then by using (3.2.3), we get

\[p(x_2, x_1) = p(Tx_1, Tx_0) \leq \alpha \left[ q(x_1, Tx_1) + q(x_0, Tx_0) \right]
+ \beta \left[ p(x_1, Tx_0) + q(x_0, Tx_1) \right]
+ \gamma p(x_1, x_0) \leq \alpha \left[ q(x_1, x_2) + q(x_0, x_1) \right]
+ \beta \left[ p(x_1, x_1) + q(x_0, x_2) \right] + \gamma p(x_1, x_0)\]
\[ \leq \alpha [ p(x_2, x_1) + q(x_0, x_1) ] \\
+ \beta [ q(x_0, x_1) + q(x_1, x_2) ] \\
+ \gamma q(x_0, x_1) \]

i.e. \( p(x_2, x_1) \leq \alpha [ p(x_2, x_1) + q(x_0, x_1) ] \\
+ \beta [ q(x_0, x_1) + p(x_2, x_1) ] \\
+ \gamma q(x_0, x_1) \)

which gives

\[ (1 - \alpha - \beta) p(x_2, x_1) \leq (\alpha + \beta + \gamma) q(x_0, x_1) \]

implies, \( p(x_2, x_1) \leq h q(x_0, x_1) \), where \( h = \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta} < 1 \).

Again,

\[ p(x_3, x_2) = p(Tx_2, Tx_1) \]

\[ \leq \alpha [ q(x_2, Tx_2) + q(x_1, Tx_1) ] \\
+ \beta [ p(x_2, Tx_1) + q(x_1, Tx_2) ] \\
+ \gamma p(x_2, x_1) \]

\[ \leq \alpha [ p(x_3, x_2) + q(x_1, x_2) ] \\
+ \beta [ q(x_1, x_2) + q(x_2, x_3) ] + \gamma q(x_1, x_2) \]

implies, \( p(x_3, x_2) \leq h q(x_1, x_2) = h p(x_2, x_1) \leq h^2 q(x_0, x_1) \).
On continuing in this way, we get

\[ p(x_{n+1}, x_n) \leq h^n q(x_0, x_1) \] (iii)

Now, for \( m > n \),

\[ p(x_m, x_n) \leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \ldots \]

\[ \ldots + p(x_{n+1}, x_n). \]

\[ \leq (h^{m-1} + h^{m-2} + \ldots + h^n) q(x_0, x_1) \]

\[ < \left( \frac{h^n}{1-h} \right) q(x_0, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

This shows \( \{ x_n \} \) is a \( p \)-Cauchy sequence in \( X \) and therefore there exists a point \( z \) in \( X \) such that \( p\lim x_n = z \). By the orbital continuity of \( T \) we get

\( Tz = p\lim Tx_n = p\lim x_{n+1} = z \), showing that \( z \) is a fixed point of \( T \).

To prove uniqueness, let \( w \) be another fixed point of \( T \). Then by using (3.2.3), we get

\[ p(z, w) = p(Tz, Tw) \]

\[ \leq \alpha [q(z, Tz) + q(w, Tw)] + \beta [p(z, Tw) + q(w, Tz)] \]

\[ + \gamma p(z, w) \]

\[ \leq (2\beta + \gamma) p(z, w) \leq p(z, w). \]
This contradiction proves that $T$ has a unique fixed point. This completes the proof of the theorem.

**REMARKS:**

1. If $\beta = 0$, we get Theorem 1 of Chikkala and Baisnab [1].

2. If $p$ is a metric $d$, then condition (3.2.2) of theorem 1 is not required and our theorem 2 reduces to Hardy and Rogers Theorem [1]. Again from (iii) we have

$$\sum d(T^{n+1}x, T^n x) < \infty$$

and hence by Theorem 2 of Bollensbacher and Hicks [1], there exists a lower semi-continuous function $\phi : X \to [0, \infty]$ which satisfies

$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

and therefore we get the Caristi fixed point theorem [1].

**THEOREM 3:** If in theorem 1, condition (3.2.1) is replaced by

(3.2.4) $p(Tx, Ty) \leq \alpha \max \left\{ p(x, y), \frac{1}{2} \left[p(Tx, x) + p(Ty, y)\right], \frac{1}{2} \left[p(Tx, y) + q(Ty, x)\right]\right\}$,

where $0 \leq \alpha < 1$, then $T$ has a unique fixed point.

**PROOF:** Sequence $\{x_n\}$ is defined in the same way as in Theorem 2. On applying (3.2.4) we get
\[ p(x_2, x_1) = p(Tx_1, Tx_0) \]

\[ \leq \alpha \max \left\{ p(x_1, x_0), \frac{1}{2} \left[ p(Tx_1, x_1) + p(Tx_0, x_0) \right], \right. \]

\[ \left. \frac{1}{2} \left[ p(Tx_1, x_0) + q(Tx_0, x_1) \right] \right\} \]

\[ \leq \alpha \max \left\{ p(x_1, x_0), \frac{1}{2} \left[ p(x_2, x_1) + p(x_1, x_0) \right], \right. \]

\[ \left. \frac{1}{2} \left[ p(x_2, x_0) + q(x_1, x_1) \right] \right\} \]

\[ \leq \alpha \max \left\{ p(x_1, x_0), \frac{1}{2} \left[ p(x_2, x_1) + p(x_1, x_0) \right], \right. \]

\[ \left. \frac{1}{2} p(x_2, x_0) \right\}, \]

we have the following possibilities:

\[ p(x_2, x_1) \leq \alpha p(x_1, x_0); \text{ or }, p(x_2, x_1) \leq \frac{\alpha}{2^{-\alpha}} p(x_1, x_0). \]

Or,

\[ p(x_2, x_1) \leq \frac{\alpha}{2} p(x_2, x_0) \leq \frac{\alpha}{2} \left[ p(x_2, x_1) + p(x_1, x_0) \right] \]

implies,

\[ p(x_2, x_1) \leq \frac{\alpha}{2^{-\alpha}} p(x_1, x_0). \]

Thus, we have

\[ p(x_2, x_1) \leq r p(x_1, x_0) \]

\[ = r q(x_0, x_1), \text{ where } 0 \leq r < 1. \]

On proceeding in similar manner, we can get easily,

\[ p(x_{n+1}, x_n) \leq r^n q(x_0, x_1). \]
Now, for $m > n$

\[ p(x_m, x_n) \leq p(x_m, x_{m-1}) + \ldots + p(x_{n+1}, x_n) \]

\[ \leq (r^{m-1} + \ldots + r^n) \, q(x_0, x_1) \]

\[ < \left( \frac{r^n}{1-r} \right) \, q(x_0, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty . \]

This shows $\{x_n\}$ is a $p$-Cauchy sequence in $X$ and therefore there exists a point $z$ in $X$ such that $p$-$\lim x_n = z$.

By the orbital continuity of $T$ we get

$Tz = p$-$\lim Tx_n = p$-$\lim x_{n+1} = z$, showing that $z$ is a fixed point of $T$.

To prove uniqueness, let $w$ be another fixed point of $T$.

Then by applying (3.2.4), we get

\[ p(z, w) = p(Tz, Tw) \]

\[ \leq \alpha \, \max \left\{ p(z, w), \frac{1}{2} \left[ p(Tz, z) + p(Tw, w) \right] \right\} , \]

\[ \frac{1}{2} \left[ p(Tz, w) + q(Tw, z) \right] \}

implies, $p(z, w) \leq \alpha \, p(z, w)$, a contradiction. Hence, $z = w$ is unique fixed point of $T$. This completes the proof.

Now, we prove
THEOREM 4: If in Theorem 1, condition (3.2.1) is replaced by

\[(3.2.5)\quad [p(Tx,Ty)]^2 \leq \alpha \max \left\{ p(x,y) q(x,Tx), \right.\]
\[\left. \quad p(x,y) q(y,Ty), \quad p(x,Ty)q(y,Tx), \quad p(x,y) p(Tx,Ty), \quad q(x,Tx) q(y,Ty) \right\} \]

where \(0 \leq \alpha < 1\), then \(T\) has a unique fixed point.

PROOF: Sequence \(\{x_n\}\) is defined in the same way as in Theorem 2. On applying (3.2.5) we get

\[[p(x_2,x_1)]^2 = [p(Tx_1, Tx_0)]^2 \]
\[\leq \alpha \max \left\{ p(x_1, x_0) q(x_1, x_2), \quad p(x_1, x_0)q(x_0,x_1), \quad p(x_1,x_1) q(x_0,x_2), \quad p(x_1,x_0)p(x_2,x_1), \right.\]
\[\left. \quad q(x_1, x_2) q(x_0, x_1) \right\} \]
\[\leq \alpha \max \left\{ q(x_0, x_1) p(x_2, x_1), \quad [q(x_0, x_1)]^2, \quad o, \quad q(x_0, x_1) p(x_2, x_1), \quad p(x_2, x_1) q(x_0, x_1) \right\} \]

Then, we have either;

\[[p(x_2, x_1)]^2 \leq \alpha q(x_0, x_1) p(x_2, x_1)\]

implies, \(p(x_2, x_1) \leq \alpha q(x_0, x_1)\).
or, \( [p(x_2, x_1)]^2 \leq \alpha [q(x_0, x_1)]^2 \)

implies \( p(x_2, x_1) \leq \alpha^{\frac{1}{2}} q(x_0, x_1) \),
since \( 0 \leq \alpha < 1 \), therefore \( 0 \leq \alpha^{\frac{1}{2}} < 1 \).

Thus \( p(x_2, x_1) \leq r q(x_0, x_1) \), where \( 0 \leq r < 1 \).

Rest of the proof is similar to that of Theorem 2, and uniqueness of the fixed point follows easily.

Finally, we prove a theorem for the continuity of fixed points in quasi-metric space.

**THEOREM 5:** Let \((X, p)\) be a complete quasi-metric space and \(T_i\) be self-mappings of \(X\) with fixed points \(u_i\) for \(i = 0, 1, 2, \ldots\) and let \(\{T_i\}\) converge uniformly to \(T_0\). If

\[
(3.2.6) \quad p(T_0 x, T_0 y) \leq \alpha \left[ p(x, T_0 x) + q(y, T_0 y) \right]
+ \beta \left[ p(x, y) + q(y, T_0 x) \right]
\]

where \( \alpha \) is positive and \( 0 \leq \beta < \frac{1}{2} \). Then sequence \(\{u_i\}\)
p-converges to \(u_0\).
PROOF: For a given $\varepsilon > 0$, by uniform convergence of sequence $\{T_i\}$ to $T_0$, there exists a positive integer $N$ such that $p(T_0x, T_i x) < \varepsilon \left(\frac{1-2\beta}{1+\alpha}\right)$ for all $x \in X$ and $i \geq N$.

By applying (3.2.6) we get

\[ p(u_o, u_i) = p(T_0 u_o, T_i u_i) \]

\[ \leq p(T_0 u_o, T_0 u_i) + p(T_0 u_i, T_i u_i) \]

\[ \leq \alpha [p(u_o, T_o u_o) + q(u_i, T_0 u_i)] + \beta [p(u_o, u_i) + q(u_i, u_o)] + p(T_0 u_i, T_i u_i) \]

\[ = \alpha q(u_i, T_0 u_i) + \beta [p(u_o, u_i) + q(u_i, u_o)] + p(T_0 u_i, T_i u_i) \]

\[ \leq \alpha [q(u_i, T_i u_i) + q(T_i u_i, T_0 u_i)] + \beta [p(u_o, u_i) + q(u_i, u_o)] + p(T_0 u_i, T_i u_i) \]

\[ = \alpha p(T_0 u_i, T_i u_i) + \beta [p(u_o, u_i) + q(u_i, u_o)] + p(T_0 u_i, T_i u_i) \]

\[ + p(T_0 u_i, T_i u_i) \]

implies, $p(u_o, u_i) \leq \left(\frac{1+\alpha}{1-2\beta}\right) p(T_0 u_i, T_i u_i)$

\[ < \left(\frac{1+\alpha}{1-2\beta}\right) \left(\frac{1 - 2\beta}{1 + \alpha}\right) \varepsilon = \varepsilon. \]
Thus $p(u_0, u_i) < \varepsilon$ for $i \geq N$.

This shows that the sequence $\{u_i\}$ $p$-converges to $u_0$.

This completes the proof of the theorem.

3.3. In this part of the chapter, we study some non-unique fixed point theorems in Bi-metric space.

Generalizing the Banach contraction principle [1], Maia [1] has been considered a metric space $X$ with two metrics $d$ and $d_1$ and proved the following theorem.

**Theorem A**: Let $(X, d, d_1)$ be a Bi-metric space, such that $X$ satisfying the following conditions:

(3.3.1) $d_1(x, y) \leq d(x, y)$ for all $x, y$ in $X$,

(3.3.2) $X$ is complete with respect to $d_1$.

(3.3.3) $T$ is continuous with respect to $d_1$.

(3.3.4) $d(Tx, Ty) \leq \alpha d(x, y)$,

for every $x, y \in X$ and $\alpha \in [0, 1]$, then $T$ has a unique fixed point.

Maia's theorem was further generalized in several ways by Iseki, K. [2], [3], Rus, I.A.[1], Singh, S.P.[1],


Rhoades, B.E. [1], Mishra, S.N. [1], and others, for different type of mappings.

Here, we shall prove some fixed point theorems in a Bi-metric space which generalize the result of Ciric type map. Ciric, Lj.B. [3] has proved the following result.

**THEOREM B**. Let $T$ be a self-mapping on an orbitally complete metric space $M$, which satisfy the condition of the type.

$$\min \left\{ d(Tx, Ty), d(y, Ty), d(x, Tx) \right\} - \min \left\{ d(x, Ty), d(y, Tx) \right\} \leq qd(x, y),$$

for all $x, y$ in $M$ and $q \in (0, 1)$.

Now, we prove that:

**THEOREM 6**. Let $T$ be a self-mapping of a Bi-metric space $(X, d, d_1)$ satisfying the following conditions:

1. (3.3.5) $d_1(x, y) \leq d(x, y)$ for all $x, y$ in $X$.
2. (3.3.6) $X$ is orbitally complete with respect to $d_1$.
3. (3.3.7) $T$ is orbitally continuous with respect to $d_1$.
4. (3.3.8) There exists a real number $q \in (0, 1)$ such that if $T$ satisfies:
\[
\min \left\{ d(x, Tx) d(x, y), d(y, Ty) d(x, y), d(Tx, Ty) d(x, y), \right.
\]

\[
\frac{[d(x, Tx)]^2}{d(Tx, Ty)}, \quad \frac{d(x, Ty) d(x, y) d(Tx, Ty)}{d(Tx, Ty)}
\]

\[
\frac{[d(Tx, Ty)]^2}{d(y, Ty)}, \quad \frac{[d(Tx, Ty)]^2 d(x, Tx)}{d(y, Ty)}
\]

\[
\left\{ \frac{d(x, Ty) d(x, y) d(Tx, Ty)}{[d(x, Tx) + d(y, Ty)]} \right\}
\]

\[
- \min \left\{ d(x, Tx) d(x, Ty), d(y, Tx) d(y, Ty) \right\} \leq q \left[ d(x, y) \right]^2
\]

for all \(x, y \in X\). Then the sequence \( \{ T^n x \} \) converges to a fixed point of \(T\).

**Proof:** Let \(x = x_0\) be an arbitrary element of \(X\). We define a sequence \(\{ x_n \}\) where \(\{ x_n \} = \{ T^n x_0 \}\).

If for some \(n\), \(x_n = x_{n+1}\), then \(\{ x_n \}\) is a Cauchy sequence and the limit of \(\{ x_n \}\) is a fixed point of \(T\).

Let \(x_n \neq x_{n+1}\), for each \(n = 0, 1, 2, \ldots\), then applying (3.3.8), for \(x = x_{n-1}\), \(y = x_n\), we have

\[
\min \left\{ d(x_{n-1}, x_n) d(x_{n-1}, x_n), d(x_n, x_{n+1}) d(x_{n-1}, x_n), \right.\]

\[
\left. d(x_n, x_{n+1}) d(x_{n-1}, x_n) \right\}
\]
\begin{align*}
\min \{ d(x_{n-1}, x_n) & \quad d(x_{n-1}, x_n) \quad [d(x_{n-1}, x_n)]^2, \\
\frac{d(x_{n-1}, x_n) \quad d(x_n, x_{n+1}) \quad d(x_{n-1}, x_n) \quad d(x_n, x_{n+1})}{d(x_n, x_{n+1})} \quad d(x_n, x_{n+1}) \}
\end{align*}

\begin{align*}
\min \{ d(x_{n-1}, x_n) & \quad d(x_{n-1}, x_n) \quad d(x_n, x_{n+1}) \quad d(x_n, x_{n+1}) \}
\leq q [d(x_{n-1}, x_n)]^2
\end{align*}

i.e. \begin{align*}
\min \{ [d(x_{n-1}, x_n)]^2, & \quad d(x_{n-1}, x_n) \quad d(x_n, x_{n+1}), \\
\frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{d(x_n, x_{n+1})} \quad d(x_n, x_{n+1}) \}
\end{align*}

\begin{align*}
\min \{ d(x_{n-1}, x_n) & \quad d(x_{n-1}, x_n) \quad 0 \}
\leq q[d(x_{n-1}, x_n)]^2
\end{align*}

implies, \begin{align*}
\min \{ [d(x_{n-1}, x_n)]^2, & \quad d(x_{n-1}, x_n) \quad d(x_n, x_{n+1}) \}
\leq q[d(x_{n-1}, x_n)]^2
\end{align*}
Then, we have either,

\[ [d(x_{n-1}, x_n)]^2 \leq q[d(x_{n-1}, x_n)]^2, \text{ implies } q \geq 1, \text{ a contradiction.} \]

Or,

\[ d(x_n, x_{n+1}) d(x_{n-1}, x_n) \leq q[d(x_{n-1}, x_n)]^2 \]

implies,

\[ d(x_n, x_{n+1}) \leq q \cdot d(x_{n-1}, x_n). \]

On proceeding similar manner as above, we get

\[ d(x_n, x_{n+1}) \leq q \cdot d(x_{n-1}, x_n) \leq q^2 \cdot d(x_{n-2}, x_{n-1}) \leq \ldots \leq q^n d(x_0, x_1). \]

Now, by triangle inequality, we have

\[ d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+p-1}, x_{n+p}) \]

\[ \leq q^n (1 + q + q^2 + \ldots + q^{p-1}) d(x_0, x_1) \]

\[ \leq \frac{q^n}{1-q} d(x_0, x_1). \]

where, \( p \) is a positive integer.

By using (3.3.5), we get

\[ d_1(x_n, x_{n+p}) \leq d(x_n, x_{n+p}) \leq \frac{q^n}{1-q} d(x_0, x_1), \]
since \( q \in (0,1) \). It follows that \( \{ x_n \} = \{ T^n x_0 \} \)

is a Cauchy sequence with respect to \( d_1 \). By (3.3.6) there exists \( u \in X \) such that \( \lim_{n \to \infty} T^n x = u \). By (3.3.7) we have \( Tu = \lim_{n \to \infty} T^{n+1} x = u \). This shows that \( u \) is a fixed point of \( T \). This completes the proof.

Our next result is due to Taskovic [1].

**Theorem 7**: Let \( T \) be a self-mapping of a Bi-metric space \((X,d,d_1)\) satisfying the conditions (3.3.5), (3.3.6), (3.3.7) of the theorem 6 and

\[
(3.3.9) \left\{ \begin{array}{c}
\alpha_1 d(x,Tx) d(x,y) + \alpha_2 d(x,y) d(y,Ty) \\
+ \alpha_3 d(Tx,Ty) d(x,y) + \alpha_4 [d(x,Tx)]^2 \\
+ \alpha_5 \frac{d(x,Tx) d(y,Ty) d(x,y)}{d(Tx,Ty)} + \alpha_6 \frac{[d(y,Ty)]^2 d(x,y)}{d(Tx,Ty)} \\
+ \alpha_7 \frac{[d(Tx,Ty)]^2 d(x,Tx)}{d(y,Ty)} + \alpha_8 \frac{d(x,Ty)d(x,y)d(Tx,Ty)}{[d(x,Tx)+d(y,Ty)]}
\end{array} \right. \]

\[ -\min \left\{ d(x,Tx) d(x,Ty), d(y,Tx) d(y,Ty) \right\} \leq \beta [d(x,y)]^2 \]

where \( \alpha_i \ (1 \leq i \leq 8) \) and \( \beta \) are real numbers such that \( \Sigma \alpha_i > \beta, \beta - \alpha_1 - \alpha_4 - \alpha_5 \geq 0 \), then for each \( x \in X \), the sequence \( \{ T^n x \} \) converges to a fixed point of \( T \).
PROOF: As in proof of theorem 6, by applying (3.3.9),
for \(x = x_{n-1}, y = x_n\) we get
\[
\{ \alpha_1 d(x_{n-1}, x_n) d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) d(x_n, x_{n+1}) \\
+ \alpha_3 d(x_n, x_{n+1}) d(x_{n-1}, x_n) + \alpha_4 [d(x_{n-1}, x_n)]^2 \\
+ \alpha_5 \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1}) d(x_{n-1}, x_n)}{d(x_n, x_{n+1})} \\
+ \alpha_6 \frac{[d(x_n, x_{n+1})]^2 d(x_{n-1}, x_n)}{d(x_n, x_{n+1})} + \alpha_7 \frac{[d(x_n, x_{n+1})]^2 d(x_{n-1}, x_n)}{d(x_n, x_{n+1})} \\
+ \alpha_8 \frac{d(x_{n-1}, x_{n+1}) d(x_{n-1}, x_n) d(x_n, x_{n+1})}{[d(x_{n-1}, x_n)+d(x_n, x_{n+1})]} \}
\]
\[- \min \{ d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1}) \} d(x_n, x_{n+1}) \leq \beta [d(x_{n-1}, x_n)]^2 \]

\[(\alpha_1 + \alpha_4 + \alpha_5) [d(x_{n-1}, x_n)]^2 \\
+ (\alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8) d(x_{n-1}, x_n) d(x_n, x_{n+1}) \\
- \min \{ d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1}) \} \leq \beta [d(x_{n-1}, x_n)]^2 \]
i.e. \((\alpha_1 + \alpha_4 + \alpha_5) [d(x_{n-1}, x_n)]^2 \\
+ (\alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8) d(x_{n-1}, x_n) d(x_n, x_{n+1}) \leq \beta [d(x_{n-1}, x_n)]^2 \]

implies \(d(x_n, x_{n+1}) \leq \frac{\beta - \alpha_1 - \alpha_4 - \alpha_5}{\alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8} d(x_{n-1}, x_n)\)
i.e. \( d(x_n, x_{n+1}) \leq q \cdot d(x_{n-1}, x_n) \), where \( q = \frac{\beta - \alpha_1 - \alpha_4 - \alpha_5}{\alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8} \)

and hence \( d(x_n, x_{n+1}) \leq q \cdot d(x_{n-1}, x_n) \leq q^2 \cdot d(x_{n-2}, x_{n-1}) \)

\[ \ldots \leq q^n \cdot d(x_0, x_1). \]

The rest of the proof is similar to theorem 6.

Finally, we obtain a localized version of

Theorem 7 with the help of Achari, J. [2].

**Theorem 8**: Let \( T \) be a self-mapping of a bi-metric space \((X, d, d_1)\) satisfying the conditions (3.3.5), (3.3.6), (3.3.7) of theorem 6, (3.3.9) of theorem 7, and

(3.3.10) \( d(x_0, Tx_0) \leq (1-q) r \), where \( q = \frac{\beta - \alpha_1 - \alpha_4 - \alpha_5}{\alpha_2 + \alpha_3 + \alpha_6 + \alpha_7 + \alpha_8} \)

and \( r \) is the radius of the ball (sphere).

\[ B = B(x_0, r) = \{ x \in X : d(x, x_0) \leq r \}. \]

Then \( T \) has a fixed point.

**Proof**: The sequence \( \{x_n\} \) is defined as in Theorem 6.

Now, by (3.3.10), \( x_1 = Tx_0 \in B(x_0, r) \). By applying (3.3.9) of theorem 7, for \( x=x_0, y=x_1 \), we can easily get
\[(a_1 + a_4 + a_5) [d(x_0, x_1)]^2 + (a_2 + a_3 + a_6 + a_7 + a_8) d(x_0, x_1) d(x_1, x_2) \leq \beta [d(x_0, x_1)]^2\]

i.e. \((a_2 + a_3 + a_6 + a_7 + a_8) d(x_1, x_2) \leq (\beta - a_1 + a_4 - a_5) d(x_0, x_1)\).

implies \(d(x_1, x_2) \leq q d(x_0, x_1)\)

\[\leq q (1-q)r, \text{ by (3.3.10)}\]

Similarly, \(d(x_2, x_3) \leq q d(x_1, x_2)\)

\[\leq q^2 (1-q) r\]

and finally, \(d(x_n, x_{n+1}) \leq q^n (1-q) r\).

Now, \(d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq (1-q)r + q(1-q)r\)

i.e. \(d(x_0, x_2) \leq (1+q)(1-q)r\).

\[d(x_0, x_3) \leq d(x_0, x_2) + d(x_2, x_3)\]

\[\leq (1+q)(1-q)r + q^2 (1-q)r\]

\[= (1+q+q^2)(1-q)r\]

On proceeding in this manner, we get

\[d(x_0, x_n) \leq (1+q+q^2+...+q^{n-1})(1-q)r=(1-q^n)r \leq r.\]
This shows that the sequence \( \{Tx_n\} \), where \( Tx_n = x_{n+1} \), for \( n \geq o \) is contained in \( B \).

By applying the triangle inequality, we have
\[
d(x_n, x_{n+p}) \leq q^n (1+q+q^2+\ldots+q^{p-1}) (1-q)r \leq q^n \cdot r.
\]

By using (3.3.5) we get
\[
d_1(x_n, x_{n+p}) \leq d(x_n, x_{n+p}) \leq q^n \cdot r.
\]

shows that \( \{x_n\} = \{T^n x_0\} \) is a Cauchy sequence with respect to \( d_1 \). Since \( X \) is orbitally complete with respect to \( d_1 \), there exists \( u \in X \) such that \( \lim_{n \to \infty} T^n x = u \).

By using the orbital continuity of \( T \) with respect to \( d_1 \), we have
\[
Tu = \lim_{n \to \infty} T^{n+1} x = u ,
\]
showing that \( u \) is a fixed point of \( T \).

This completes the proof of the theorem.