CHAPTER II

RELATED FIXED POINT THEOREMS
ON TWO METRIC SPACES

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METRIC SPACES

2.1. S. Banach in 1922, established his famous fixed point theorem (1.1.1), popularly known as Banach's contraction theorem. Since then in next six decades there are a large number of research papers appeared which concerned the fixed point theorems of mappings defined from some space X to itself under various contractive conditions.

First time in 1981, B. Fisher established the concept of related fixed point theorem by taking two different complete metric spaces $(X,d)$ and $(Y, ξ)$ and two mappings $T : (X,d) \rightarrow (Y, ξ)$, $S : (Y, ξ) \rightarrow (X,d)$ and he proved that - under certain contractive conditions the composite mapping $ST$ has a unique fixed point in X and $TS$ has a unique fixed point in Y.

The results of this chapter are likely to be published in the Kobe University Journal in co-authorship of Prof. B. Fisher. Preprint of the paper is enclosed in the appendix.
In 1981, B. Fisher [8] introduced the following related fixed point theorem on two metric spaces.

**Theorem A**: Let \((X, d)\) and \((Y, \varphi)\) be complete metric spaces. If \(T\) is a mapping of \(X\) into \(Y\) and \(S\) is a mapping of \(Y\) into \(X\) satisfying the inequalities:

\[\varphi(Tx, TSy) \leq c \max \{d(x, Sy), \varphi(y, Tx), \varphi(y, TSy)\},\]

\[(2.1.1)\]

\[d(Sy, STx) \leq c \max \{\varphi(y, Tx), d(x, Sy), d(x, STx)\},\]

\[(2.1.2)\]

for all \(x\) in \(X\) and \(y\) in \(Y\), where \(0 \leq c < 1\), then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

Fisher [8] also proved an analogous result for compact metric spaces.

**Theorem B**: Let \((X, d)\) and \((Y, \varphi)\) be compact metric spaces. If \(T\) is a continuous mapping of \(X\) into \(Y\) and \(S\) is a continuous mapping of \(Y\) into \(X\) satisfying the inequalities:

\[\varphi(Tx, TSy) < \max \{d(x, Sy), \varphi(y, Tx), \varphi(y, TSy)\}\]

\[(2.1.3)\]

for all \(x\) in \(X\) and \(y\) in \(Y\) with \(x \neq Sy\) and

\[d(Sy, STx) < \max \{\varphi(y, Tx), d(x, Sy), d(x, STx)\}\]

\[(2.1.4)\]
for all \( x \) in \( X \) and \( y \) in \( Y \) with \( y \neq Tx \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further, \( Tz = w \) and \( Sw = z \).

Brain Fisher [9], in 1982, has proved another fixed point theorem on two metric spaces.

**Theorem C:** Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces. If \( T \) is a continuous mapping of \( X \) into \( Y \) and \( S \) is a mapping of \( Y \) into \( X \) satisfying the following inequalities:

\[
(2.1.5) \quad d(STx, STx') \leq c \max \left\{ d(x, x'), d(x, STx), d(x', STx'), \rho(Tx, Tx') \right\}
\]

\[
(2.1.6) \quad \rho(TSy, TSy') \leq c \max \left\{ \rho(y, y'), \rho(y, TSy), \rho(y', TSy'), d(Sy, Sy') \right\}
\]

for all \( x, x' \) in \( X \) and \( y, y' \) in \( Y \), where \( 0 \leq c < 1 \), then \( ST \) has a unique fixed point \( z \) in \( X \) and \( TS \) has a unique fixed point \( w \) in \( Y \). Further \( Tz = w \) and \( Sw = z \).

2.2: In this section, we prove new related fixed point theorems on two metric spaces which generalize the result of Fisher [9], Theorem C.
First, we prove that:

**Theorem 1**: Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces. Let \(T\) be a mapping of \(X\) into \(Y\) and \(S\) be a mapping of \(Y\) into \(X\) satisfying the inequalities:

\[
(2.2.1) \quad d(Sy, Sy') \leq c \max \left\{ d(x, Sy') \rho(Tx', Ty), d(x', Sy) \rho(Tx, Ty'), d(x, x') d(Sy, Sy') \right\}
\]

\[
(2.2.2) \quad \rho(Tx, Tx') \leq c \max \left\{ d(x, Sy') \rho(Tx', Ty), d(x', Sy) \rho(Tx, Ty'), \rho(y, y') \rho(Tx, Tx'), \rho(Tx, Ty) \rho(Tx', Ty') \right\}
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\), where \(0 \leq c < 1\). If either \(T\) or \(S\) is continuous then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\).

Further, \(Tz = w\) and \(Sw = z\).

**Proof**: For an arbitrary point \(x\) in \(X\). We define sequences \(\{x_n\}\) in \(X\) and \(\{y_n\}\) in \(Y\), by
\[(ST)^n x = x_n, \quad T(ST)^{n-1} x = y_n, \quad \text{for } n = 1, 2, 3, \ldots .\]

Applying inequality (2.2.1), we have

\[d(x_{n-1}, x_n) d(x_n, x_{n+1}) = d(Sy_{n-1}, Sy_n) d(STx_{n-1}, STx_n).\]

\[\leq c \max \left\{ d(x_{n-1}, Sy_n) \xi(Tx_n, TSy_{n-1}), \right.\]

\[d(x_n, Sy_{n-1}) \xi(Tx_{n-1}, TSy_n), \]

\[d(x_{n-1}, x_n) d(Sy_{n-1}, Sy_n), \]

\[d(Sy_{n-1}, STx_{n-1}) d(Sy_n, STx_n) \right\}\]

\[\leq c \max \left\{ d(x_{n-1}, x_n) \xi(y_n, y_{n+1}), \right.\]

\[d(x_{n-1}, x_n) \xi(y_n, y_{n+1}), \]

\[\left[ d(x_{n-1}, x_n) \right]^2, \]

\[d(x_{n-1}, x_n) d(x_n, x_{n+1}) \right\},\]

from which it follows that

\[d(x_n, x_{n+1}) \leq c \max \left\{ \xi(y_n, y_{n+1}), d(x_{n-1}, x_n) \right\}.\]

By applying inequality (2.2.2), we get

\[\left[ \xi(y_n, y_{n+1}) \right]^2 = \xi(Tx_{n-1}, Tx_n) \xi(TSy_{n-1}, TSy_n)\]

\[\leq c \max \left\{ d(x_{n-1}, Sy_n) \xi(Tx_n, TSy_{n-1}) \right\}.\]
\[ d(x_n, S_{y_{n-1}}) \leq \xi(T_{x_{n-1}}, TS_{y_n}), \]
\[ \xi(y_{n-1}, y_n) \leq \xi(T_{x_{n-1}}, T_{x_n}), \]
\[ \xi(T_{x_{n-1}}, TS_{y_{n-1}}) \leq \xi(T_{x_n}, TS_{y_n}) \]

\[ \leq c \max \left\{ d(x_{n-1}, x_n) \xi(y_n, y_{n+1}), \right. \]
\[ d(x_n, x_{n-1}) \xi(y_n, y_{n+1}), \]
\[ \xi(y_{n-1}, y_n) \xi(y_n, y_{n+1}), \]
\[ \left. \xi(y_n, y_n) \xi(y_{n+1}, y_{n+1}) \right\} \]

\[ \leq c \max \left\{ d(x_{n-1}, x_n) \xi(y_n, y_{n+1}), \right. \]
\[ d(x_{n-1}, x_n) \xi(y_n, y_{n+1}), \]
\[ \xi(y_{n-1}, y_n) \xi(y_n, y_{n+1}), \]
\[ o \right\}, \]

from which it follows that
\[ \xi(y_n, y_{n+1}) \leq c \max \left\{ d(x_{n-1}, x_n), \xi(y_{n-1}, y_n) \right\}. \]

It now follows easily by induction that
\[ d(x_n, x_{n+1}) \leq c^n \max \{ d(x, x_1), \xi(y_1, y_2) \} \]
\[ \xi(y_n, y_{n+1}) \leq c^{n-1} \max \{ d(x, x_1), \xi(y_1, y_2) \} , \]
for \( n = 1, 2, \ldots \). Since \( c < 1 \), it follows that \( \{ x_n \} \) and \( \{ y_n \} \) are Cauchy sequences with limit \( z \) in \( X \) and \( w \) in \( Y \).

Applying inequality (2.2.1) we have

\[
d(Sw, x_n) d(STz, x_{n+1}) = d(Sw, Sy_n) d(STz, STx_n) \\
\leq c \max \left\{ \begin{array}{c}
    d(z, Sy_n) \xi(Tx_n, TSw), \\
    d(x_n, Sw) \xi(Tz, TSy_n), \\
    d(z, x_n) d(Sw, Sy_n), \\
    d(Sw, STz) d(Sy_n, STx_n) \end{array} \right\} \\
\leq c \max \left\{ \begin{array}{c}
    d(z, x_n) \xi(y_{n+1}, TSw), \\
    d(x_n, Sw) \xi(Tz, y_{n+1}), \\
    d(z, x_n) d(Sw, x_n), \\
    d(Sw, STz) d(x_n, x_{n+1}) \end{array} \right\}.
\]

Letting \( n \) tend to infinity, we have

\[
d(Sw, z) d(STz, z) \leq c \max \left\{ \begin{array}{c}
    d(z, z) \xi(w, TSw), \\
    d(z, Sw) \xi(Tz, w), d(z, z) d(Sw, z), \\
    d(Sw, STz) d(z, z) \end{array} \right\} \\
\leq c \max \left\{ o, d(z, Sw) \xi(Tz, w), o, o \right\}.
\]
i.e. \( d(Sw, z) \leq d(STz, z) \leq c \cdot d(z, Sw) \cdot \ell(Tz, w) \)

and so either

\[ Sw = z \]

or,

\[ d(STz, z) \leq c \cdot \ell(Tz, w) \]  \( (1) \)

By applying inequality (2.2.2), we get

\[
\ell(Tz, y_{n+1}) \cdot \ell(TSw, y_{n+1}) = \ell(Tz, Tx_n) \cdot \ell(TSw, TSy_n) \\
\leq c \cdot \max \left\{ d(z, Sy_n) \cdot \ell(Tx_n, TSw), \right. \\
\left. d(x_n, Sw) \cdot \ell(Tz, TSy_n), \right. \\
\left. \ell(w, y_n) \cdot \ell(Tz, Tx_n), \right. \\
\left. \ell(Tz, TSw) \cdot \ell(Tx_n, TSy_n) \right\} \\
\leq c \cdot \max \left\{ d(z, x_n) \cdot \ell(y_{n+1}, TSw), \right. \\
\left. d(x_n, Sw) \cdot \ell(Tz, y_{n+1}), \right. \\
\left. \ell(w, y_n) \cdot \ell(Tz, y_{n+1}), \right. \\
\left. \ell(Tz, TSw) \cdot \ell(y_{n+1}, y_{n+1}) \right\}. 
\]

Letting \( n \) tend to infinity, we get

\[ \ell(Tz, w) \cdot \ell(TSw, w) \leq c \cdot d(z, Sw) \cdot \ell(Tz, w) \]

and so either

\[ Tz = w \]
or
\[ \epsilon(T_{Sw}, w) \leq c \, d(z, Sw). \]  \hspace{1cm} (2)

If \( T \) is continuous, then

\[ w = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} Tx_n = Tz \]

and therefore inequality (1) implies \( z = STz = Sw \),

\( T_{Sw} = Tz = w \).

If \( S \) is continuous, then

\[ z = \lim_{n \to \infty} x_n = \lim_{n \to \infty} Sy_n = Sw, \]

and therefore inequality (2) implies \( w = T_{Sw} = Tz, \)

\( STz = Sw = z. \)

To prove uniqueness, let \( ST \) has another fixed point \( z' \) and \( TS \) has \( w' \). Then applying inequality (2.2.1) we have

\[ d(z, Sw') \, d(z, z') = d(Sw, Sw') \, d(STz, STz') \]

\[ \leq c \, \max \left\{ d(z, Sw') \, \epsilon(Tz', T_{Sw}), \right. \]

\[ d(z', Sw) \, \epsilon(Tz, TSw'), \, d(z, z') \, d(Sw, Sw'), \]

\[ \left. d(Sw, STz) \, d(Sw', STz') \right\} \]

\[ \leq c \, \max \left\{ d(z, Sw') \, \epsilon(Tz', w), \right. \]

\[ d(z', z) \, \epsilon(w, w'), \, d(z, z') \, d(z, Sw'), \]

\[ \left. d(z', z) \, \epsilon(w, w'), \, d(z, z') \, d(z, Sw') \right\} \]
\[ d(z, z) d(W, z') \leq c \max \left\{ d(z, Sw') \xi(Tz', w), d(z', z) \xi(w, w'), \right\} d(z, z') d(z, Sw'), o \right\}. \]

i.e. \[ d(z, Sw') d(z, z') \leq c \max \left\{ d(z, Sw') \xi(Tz', w), \right\} d(z', z) \xi(w, w'), d(z, z') d(z, Sw'). \]

Then, we have either

\[ d(z, Sw') d(z, z') \leq c d(z, Sw') \xi(Tz', w) \]

implies \[ d(z, z') \leq c \xi(Tz', w) \tag{3} \]

Or \[ d(z, Sw') d(z, z') \leq c d(z, z') \xi(w, w') \]

implies \[ d(z, Sw') \leq c \xi(w, w') \tag{4} \]

Further, applying inequality (2.2.2) we have

\[ \xi(w, Tz') \xi(w, w') = \xi(Tz, Tz') \xi(TSw, TSw') \leq c \max \left\{ d(z, Sw') \xi(Tz', TSw'), \right. \]

\[ \left. d(z', Sw) \xi(Tz, TSw'), \xi(w', w) \xi(Tz, Tz'), \xi(Tz, TSw) \xi(Tz', TSw') \right\}. \]

\[ \leq c \max \left\{ d(z, Sw') \xi(Tz', w), \right. \]

\[ \left. d(z', Sw) \xi(Tz, TSw'), \xi(w', w) \xi(Tz, Tz'), \xi(Tz, TSw) \xi(Tz', TSw') \right\}. \]
\[ d(z', z) \leq \xi(w, w') \leq \xi(w, w') \leq \xi(w, Tz') \leq \xi(w, Sw') \]

which implies, either

\[ \xi(w, w') \leq c \ d(z, Sw') \quad (5) \]

or

\[ \xi(w, Tz') \leq c \ d(z', z) \quad (6) \]

From (3) and (6), we have

\[ d(z, z') \leq c \ \xi(Tz', w) \leq c^2 \ d(z', z), \]

which gives \( z' = z \), since \( c < 1 \). This shows \( ST \) has unique fixed point \( z \).

Now, \( TSw' = w' \) implies \( STSw' = Sw' \) and hence \( Sw' = z \). Thus, \( w = TSw = Tz = TSw' = w' \), which shows that \( TS \) has unique fixed point \( w \).

From inequalities (5) and (4), we have

\[ \xi(w, w') \leq c \ d(z, Sw') \leq c^2 \ \xi(w, w'), \]

which gives \( w = w' \), since \( c < 1 \). This shows \( TS \) has unique fixed point \( w \).

Now, \( STz' = z' \implies TSTz' = Tz' \) and hence \( Tz' = w \).

Thus, \( z = STz = Sw = STz' = z' \), which shows that \( ST \) has unique fixed point \( z \). This completes the proof of the theorem.
**COROLLARY 1**: Let $(X,d)$ be a complete metric space. If $S$ and $T$ are mappings of $X$ into itself satisfying

$$
(2.2.3) \quad d(Sx, Sy) \leq c \max \left\{ d(x, Sy) d(Ty, TSx), d(y, Sx) d(Tx, TSy), d(x, y) d(Sx, Sy), d(Sx, STx) d(Sy, STy) \right\}
$$

$$
(2.2.4) \quad d(Tx, Ty) d(TSx, TSy) \leq c \max \left\{ d(x, Sy) d(Ty, TSx), d(y, Sx) d(Tx, TSy), d(x, y) d(Tx, Ty), d(Tx, TSx) d(Ty, TSy) \right\}
$$

For all $x, y$ in $X$, $0 \leq c < 1$. If either $T$ or $S$ is continuous then $ST$ has a unique fixed point $z$ and $TS$ has a unique fixed point $w$ in $X$. Further $Tz = w$ and $Sw = z$.

Now, we prove the following theorem:

**THEOREM 2**: Let $(X,d)$ and $(Y, \rho)$ be complete metric spaces. If $T$ is a continuous mapping of $X$ into $Y$ and $S$ is a mapping of $Y$ into $X$ satisfying the inequalities:

$$
(2.2.5) \quad d(STx, STx') \leq \frac{c f(x, x', y, y')}{f_1(x, x', y, y')}
$$


\[(2.2.6) \quad \zeta(Ty, Ty') \leq \frac{c \cdot g(x, x', y, y')}{{g}_1(x, x', y, y')}\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\) for which

\[f_1(x, x', y, y') \neq 0 \neq g_1(x, x', y, y')\]

where

\[f(x, x', y, y') = \max \left\{ d(x, x') \cdot \zeta(Tx, Tx'), \right.\]
\[d(x, TSx) \cdot \zeta(Tx, Tx'), \]
\[d(x', STx) \cdot \zeta(Tx, Tx') \left. \right\}\]

\[g(x, x', y, y') = \max \left\{ \zeta(y, y') \cdot d(Sy, Sy'), \right.\]
\[\zeta(y', Ty') \cdot d(Sy, Sy'), \]
\[\zeta(y', Ty) \cdot d(Sy, Sy') \left. \right\}\]

\[f_1(x, x', y, y') = \max \left\{ \zeta(Tx, Tx'), d(x, STx), d(x', STx) \right\}\]

\[g_1(x, x', y, y') = \max \left\{ d(Sy, Sy'), \zeta(y, Ty'), \zeta(y', Tsy) \right\}\]

and \(0 \leq c < 1\), then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

**Proof:** Define the sequences \(\{x_n\}\) in \(X\) and \(\{y_n\}\) in \(Y\) as in the proof of theorem 1.
We will assume that \( x_n \neq x_{n+1} \) and \( y_n \neq y_{n+1} \) for all \( n \), otherwise, if \( x_n = x_{n+1} \) and \( y_n = y_{n+1} \) for some \( n \), we could put \( x_n = z \) and \( y_n = w \). By applying inequality (2.2.5) we get

\[
d(x_n, x_{n+1}) = d(STx_{n-1}, STx_n)
\]

\[
c \max \left\{ d(x_{n-1}, x_n), \varepsilon(Tx_{n-1}, Tx_n), d(x_{n-1}, STx_{n-1}), \varepsilon(Tx_{n-1}, Tx_n), d(x_n, STx_{n-1}), \varepsilon(Tx_{n-1}, Tx_n) \right\} \leq \max \left\{ \varepsilon(Tx_{n-1}, Tx_n), d(x_{n-1}, STx_{n-1}), d(x_n, STx_{n-1}) \right\}
\]

\[
c \max \left\{ d(x_{n-1}, x_n)\varepsilon(y_n, y_{n+1}), d(x_{n-1}, x_n) \right\} \leq \max \left\{ \varepsilon(y_n, y_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n) \right\}
\]

\[
c \max \left\{ d(x_{n-1}, x_n) \varepsilon(y_n, y_{n+1}), d(x_{n-1}, x_n) \right\} \leq \max \left\{ \varepsilon(y_n, y_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n) \right\}
\]
\[ d(x_n, x_{n+1}) \leq \frac{c \cdot d(x_{n-1}, x_n) \cdot \xi(y_n, y_{n+1})}{\max \{ \xi(y_n, y_{n+1}), d(x_{n-1}, x_n) \}} \]

From which, it follows that

\[ d(x_n, x_{n+1}) \leq c \cdot \xi(y_n, y_{n+1}) \quad \text{(i)} \]

or

\[ d(x_n, x_{n+1}) \leq c \cdot d(x_{n-1}, x_n) \quad \text{(ii)} \]

By applying inequality (2.2.6) we get

\[ \xi(y_n, y_{n+1}) = \xi(TS_{y_{n-1}}, TS_{y_n}) \]

\[ \leq \frac{c \cdot \max \{ \xi(y_{n-1}, y_n), \xi(y_n, TS_{y_n}) \cdot d(Sy_{n-1}, Sy_n), \xi(y_n, TS_{y_{n-1}}) \cdot d(Sy_{n-1}, Sy_n) \}}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]

\[ \leq \frac{c \cdot \max \{ \xi(y_{n-1}, y_n) \cdot d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \cdot d(x_{n-1}, x_n), \xi(y_n, y_n) \cdot d(x_{n-1}, x_n) \}}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]

\[ \leq \frac{c \cdot \max \{ \xi(y_{n-1}, y_n) \cdot d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \cdot d(x_{n-1}, x_n) \}}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]

\[ \leq \frac{\xi(y_n, y_{n+1}) \cdot d(x_{n-1}, x_n)}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]

\[ \leq \frac{\xi(y_n, y_{n+1}) \cdot d(x_{n-1}, x_n)}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]

\[ \leq \frac{\xi(y_n, y_{n+1}) \cdot d(x_{n-1}, x_n)}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]

\[ \leq \frac{\xi(y_n, y_{n+1}) \cdot d(x_{n-1}, x_n)}{\max \{ d(x_{n-1}, x_n), \xi(y_n, y_{n+1}) \}} \]
From which it follows that

\[ d(y_n, y_{n+1}) \leq c d(y_{n-1}, y_n) \quad (iii) \]

or

\[ d(y_n, y_{n+1}) \leq c \, d(x_{n-1}, x_n) \quad (iv). \]

By using inequalities (i) and (iii), we get

\[
d(x_n, x_{n+1}) \leq c \, d(y_n, y_{n+1}) \leq c^2 \, d(y_{n-1}, y_n) \leq \ldots \leq c^n \, d(y_1, y_2).
\]

By using inequalities (i) and (iv) we get

\[
d(x_n, x_{n+1}) \leq c \, d(y_n, y_{n+1}) \leq c^2 d(x_{n-1}, x_n) \leq c^3 d(y_{n-1}, y_n) \leq \ldots \leq c^{2n-1} \, d(y_1, y_2) \leq c^{2n} \, d(x, x_1).
\]

From inequality (ii), we have

\[
d(x_n, x_{n+1}) \leq c d(x_{n-1}, x_n) \leq \ldots \leq c^n \, d(x, x_1).
\]

Since \( c < 1 \), from above it follows that in each case \( \{x_n\} \)

is a Cauchy sequence with limit \( z \) in \( X \).

Similarly, by using inequalities (iv) and (i), we have

\[ d(y_n, y_{n+1}) \leq c^{2n-1} \, d(x, x_1) \leq c^{2n} \, d(y, y_1). \]

From inequalities (iv) and (ii), we get

\[ d(y_n, y_{n+1}) \leq c^n \, d(x, x_1), \]
and from inequality (iii), we have

\[ \varrho(y_n, y_{n+1}) \leq c^n \varrho(y, y_1). \]

Since \( c < 1 \), from above it follows that in each case \( \{ y_n \} \)
is a Cauchy sequence with limit \( w \) in \( y \).

Using the continuity of \( T \) we have

\[ w = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} T x_n = T z \quad (v) \]

Now using the inequality (2.2.5) and the assumption that \( STz \neq z \), we get

\[ d(S T z, x_n) = d(S T z, S T x_{n-1}) \]

\[ \leq \frac{c \max \left\{ d(z, x_{n-1}) \varrho(T z, T x_{n-1}), d(z, S T z) \varrho(T z, T x_{n-1}), \\
\quad d(x_{n-1}, S T z) \varrho(T z, T x_{n-1}) \right\}}{\max \left\{ \varrho(T z, T x_{n-1}), d(z, S T z), d(x_{n-1}, S T z) \right\}} \]

\[ \leq \frac{c \max \left\{ d(z, x_{n-1}) \varrho(T z, y_n), d(z, S T z) \varrho(T z, y_n), \\
\quad d(x_{n-1}, S T z) \varrho(T z, y_n) \right\}}{\max \left\{ \varrho(T z, y_n), d(z, S T z), d(x_{n-1}, S T z) \right\}} \]

Letting \( n \) tend to infinity, we get
\[ d(STz, z) \leq \frac{c \cdot d(z, STz) \cdot \xi(Tz, w)}{\max \{ \xi(Tz, w), d(z, STz) \} } \]

Then we have either,

\[ d(STz, z) \cdot \xi(Tz, w) \leq c \cdot d(z, STz) \cdot \xi(Tz, w) \]

gives \( c \geq 1 \), a contradiction.

Or

\[ [d(STz, z)]^2 \leq c \cdot d(z, STz) \cdot \xi(Tz, w) \]

implies \( d(STz, z) \leq c \cdot \xi(Tz, w) = 0 \ldots \) by (v).

Thus \( STz = z \) and hence \( z = STz = Sw \), \( TSW = Tz = w \).

To prove uniqueness, let \( z' \) be another fixed point of \( ST \),
then by using inequality (2.2.5) we have

\[ d(z, z') = d(STz, STz') \]

\[
\leq \frac{c \max \{ d(z, z') \cdot \xi(Tz, Tz'), d(z, STz) \cdot \xi(Tz, Tz') \}}{\max \{ \xi(Tz, Tz'), d(z, STz), d(z', STz) \}}
\]

\[
\leq \frac{c \max \{ d(z, z') \cdot \xi(Tz, Tz'), d(z', z) \cdot \xi(Tz, Tz') \}}{\max \{ \xi(Tz, Tz'), d(z', z) \}}
\]

\[
\leq \frac{c \cdot \xi(Tz, Tz') \cdot d(z, z')}{\max \{ \xi(Tz, Tz'), d(z, z') \}}
\]
which gives either

\[ d(z, z') \leq c \, d(z, z') \leq c \, \varepsilon(Tz, Tz'), \]

a contradiction, since \( c < 1 \).

or

\[ d(z, z') \leq c \, \varepsilon(Tz, Tz') \quad \text{(vi)} \]

Now, using inequality (2.2.6) we get

\[ \varepsilon(Tz, Tz') = \varepsilon(TSTz, TSTz') \]

\[ \leq c \, \max \left\{ \varepsilon(Tz, Tz') \, d(STz, STz'), \right. \]

\[ \left. \varepsilon(Tz', TSTz') \, d(STz, STz'), \right\} \]

\[ \leq \frac{\varepsilon(Tz', TSTz) \, d(STz', STz') \left\{ \right.}{\max \left\{ d(STz, STz'), \varepsilon(Tz', TSTz'), \varepsilon(Tz', TSTz) \right\}} \]

\[ \leq \frac{c \, \max \left\{ \varepsilon(Tz, Tz') \, d(z, z'), 0, \varepsilon(Tz', Tz) \right\} \, d(z, z')}{\max \left\{ d(z, z'), 0, \varepsilon(Tz', Tz) \right\}} \]

i.e.

\[ (Tz, Tz') \leq \frac{c \, \varepsilon(Tz, Tz') \, d(z, z')}{\max \left\{ d(z, z'), \varepsilon(Tz', Tz) \right\}} \]

implies

\[ \varepsilon(Tz, Tz') \leq c \, d(z, z') \quad \text{(vii)} \]

On applying the inequalities (vi) and (vii) we have

\[ d(z, z') \leq c \, \varepsilon(Tz, Tz') \leq c^2 \, d(z, z') \]
gives \( z' = z \) since \( c < 1 \). This shows that \( ST \) has unique fixed point \( z \). Let \( TS \) has another fixed point \( w' \), then \( TSw' = w' \implies STSw' = Sw' \) and hence \( Sw' = z \). Thus \( w = TSw = Tz = TSw' = w' \), shows that \( TS \) has unique fixed point \( w \).

This completes the proof of the theorem.