CHAPTER VII

FIXED POINTS ON COMPACT SPACES

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FIXED POINTS ON COMPACT SPACES

7.1. In this chapter, we study some fixed point theorems on compact metric spaces and compact spaces.

In recent years various generalizations and extensions of the well known Banach contraction principle has appeared. Let us first state the definition of contractive mapping:

**DEFINITION**: A mapping $T$ of a metric space $(X, d)$ into itself is said to be a contractive mapping if, for all $x, y$ in $X$, $d(Tx, Ty) < d(x, y)$, $x \neq y$.

In 1962, Edelstein, M. [1] extended the Banach's famous theorem for contractive mapping. He has proved the following theorem.

**THEOREM A**: If $T$ is a mapping of a compact metric space $(X, d)$ into itself satisfying the inequality:
\[(7.1.1)\quad d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, x \neq y,\]

then \(T\) has a unique fixed point.


**THEOREM B**: If \(T\) is a continuous mapping of a compact metric space \((X, d)\) into itself satisfying the inequality:

\[(7.1.2)\quad d(Tx, Ty) < \beta_1 d(x, y) + \beta_2 [d(x, Tx) + d(y, Ty)] + \beta_3 [d(x, Ty) + d(y, Tx)]\]

where, \(\beta_1 + 2(\beta_2 + \beta_3) = 1, \beta_2 + \beta_3 < 1, \beta_1 + 2\beta_3 \leq 1, \beta_3 \geq 0\), then \(T\) has a unique fixed point.

Das, P. [1], in 1980, further generalized the result of Fisher [3] in his following two theorems.

**THEOREM C**: Let \(T\) be a self-map of a compact metric space \((X, d)\) such that for some positive integer \(m\), \(T^m\) is continuous and for every \(x, y \in X\) with \(x \neq y\), \(T^m x \neq T^m y\):
(7.1.3) \[ d(T^m x, T^m y) < \alpha_1 \frac{d(x, T^m x) \cdot d(y, T^m y)}{d(x, y)} + \alpha_2 \frac{d(x, T^m x) \cdot d(y, T^m x)}{d(T^m x, T^m y)} + \alpha_3 \frac{d(x, T^m y) \cdot d(y, T^m y)}{d(T^m x, T^m y)} \]

\[ + \beta_2 d(x, T^m x) + \beta_3 d(y, T^m y) \]

\[ + \beta_4 d(x, T^m y) + \beta_5 d(y, T^m x) \]  

where \( \alpha_1 + \alpha_3 + \beta_3 + \beta_4 < 1 \), \( \alpha_3 \geq 0 \), \( \beta_4 \geq 0 \),

\( \alpha_1 + 2\alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\beta_4 = 1 \) and \( \beta_1 + \beta_4 + \beta_5 < 1 \).

Then \( T \) has a unique fixed point.

He [Das, [1]] also proved,

**Theorem D:** Let \( T \) be as in Theorem C. Also, in addition,

let \( \alpha_2 \geq 0 \), \( \alpha_2 + \beta_2 + \beta_5 < 1 \), \( \beta_2 \geq 0 \), \( 2\alpha_2 + \beta_1 + 2\beta_2 + \beta_4 + \beta_5 = 1 \).

If \( d(T^m x, u) < d(x, u) \) for every \( x \in X \) with \( x \neq u \), where \( u \) is the unique fixed point of \( T \) which exists by 

Theorem C, then for every \( x \in X \), \( \lim_{n \to \infty} T^{mn} x = u \).
In 1981, Fisher, B. [7] obtained the following fixed point theorem for a pair of mappings in a compact metric space.

**THEOREM E**: If $S$ and $T$ are continuous mappings of a compact metric space $(X, d)$ into itself satisfying either the inequality

$$(7.1.4) \quad [d(Sx, Ty)]^2 < \alpha d(x, Ty) d(x, Sx) + \beta d(y, Sx) d(y, Tx),$$

if $\alpha d(x, Ty) d(x, Sx) + \beta d(y, Sx) d(y, Tx) \neq 0$, or the equality $d(Sx, Ty) = 0$. Otherwise, for all $x, y \in X$, where $\alpha, \beta > 0$ and

$$[\alpha + (\alpha^2 + 4\alpha)^{1/2}] [\beta + (\beta^2 + 4\beta)^{1/2}] < 4,$$

then $S$ and $T$ have a unique common fixed point $z$.

In 1980, Khan and Khan [1] generalized the result of Edelstein [1] [Theorem A] and proved the following theorem:

**THEOREM F**: Let $S$ and $T$ be mappings of a set $X$ into itself and $F$ a real-valued function on $X \times X$ satisfying, for all $x, y$ in $X$, the inequalities:
(7.1.5) \( F(STx, TSy) < F(x, y) \), \( x \neq y \),

and

(7.1.6) \( F(TSx, STy) < F(x, y) \), \( x \neq y \).

Suppose there exists a non-empty subset \( K \) of \( X \) such that \( ST(K) \subseteq K \) and \( TS(K) \subseteq K \), and that the functions \( x \rightarrow F(x, Tx) \) and \( x \rightarrow F(x, Sx) \) attain the minimum values at some points of \( K \). Then \( S \) and \( T \) each have a fixed point. Further if \( S \) and \( T \) have a common fixed point, then it is unique.

7.2 : In this section, we shall prove three fixed point theorems on compact metric spaces, for a pair of mappings. One is the generalized version of Fisher [1] and other two are the generalization of the result of Jaggi [1]. First two theorems of this section are accepted for publication in Journal of Scientist of Physical Sciences Vol. 4, No. 2, 1992.

First, we prove the following generalization of Fisher [1].
THEOREM 1: Let $S$ and $T$ be continuous mappings of a compact metric space $(X,d)$ into itself satisfying:

\[(7.2.1) \quad [d(Sx,Ty)]^2 \leq \alpha \left\{ d(x,Sx) d(y,Ty) + d(x,Ty)d(y,Sx) \right\} + \beta \left\{ d(x,Sx) d(y,Sx) + d(x,Ty)d(y,Ty) \right\} + \gamma d(x,y) d(Sx, Ty)\]

for all distinct $x,y \in X$, where $\alpha$, $\beta$, $\gamma \geq 0, \alpha + 2 \beta + \gamma < 1$. Then either $S$ or $T$ has a fixed point. If $S$ and $T$ have a common fixed point, then it is unique.

PROOF: Let $\phi(x) = d(Sx, x)$ for all $x \in X$. $\phi$ being the composite of two continuous mappings $d$ and $S$, is continuous on the compact space $X$ and hence attains its infimum say $z$ in $X$, i.e. $\phi(z) = \inf \{ \phi(x) : x \in X \}$.

Let $Sz \neq z$, $TSz \neq Sz$. By applying (7.2.1), we have

\[[d(STSz, TSz)]^2 \leq \alpha \left\{ d(TSz, STSz) d(Sz, TSz) + d(TSz, TSz) d(Sz, STSz) \right\} + \beta \left\{ d(TSz, STSz) d(Sz, STSz) \right\} \]
\[ + d(TSz, TSz) \ d(Sz, TSz) \]
\[ + \gamma \ d(TSz, Sz) \ d(STSz, TSz) \]
\[ \leq \alpha \ d(TSz, STSz) \ d(Sz, TSz) \]
\[ + \beta \ d(TSz, STSz) \ [d(Sz, TSz) \]
\[ + d(TSz, STSz) ] \]
\[ + \gamma \ d(TSz, Sz) \ d(TSz, STSz) . \]
\]

implies \, d(STSz, TSz) \leq (\alpha + \beta + \gamma) \ d(Sz, TSz) \]
\[ + \beta \ d(TSz, STSz) \]

This gives \, d(STSz, TSz) \leq \frac{\alpha + \beta + \gamma}{1 - \beta} \ d(Sz, TSz) \]
\[ < d(Sz, TSz) . \]

Again, by applying (7.2.1), we can get easily
\[ [d(Sz, TSz)]^2 \leq (\alpha + \gamma) \ d(z, Sz) \ d(Sz, TSz) \]
\[ + \beta \ d(z, TSz) \ d(Sz, TSz) . \]

i.e. \, d(Sz, TSz) \leq (\alpha + \gamma) \ d(z, Sz) + \beta [d(z, Sz) + d(Sz, TSz)] \]
\[ \]

implies \, d(Sz, TSz) \leq \frac{\alpha + \beta + \gamma}{1 - \beta} \ d(z, Sz) < d(z, Sz). \]
Therefore, \( d(STSz, TSz) < d(z, Sz) \), i.e. \( \phi (TSz) < \phi (z) \).

This contradicts the definition of \( z \) and hence \( Sz = z \), showing that \( z \) is a fixed point of \( S \) or, \( TSz = Sz \), showing that \( Sz \) is a fixed point of \( T \). Hence either \( S \) or \( T \) must have a fixed point.

Let \( u \) is a common fixed point of \( S \) and \( T \).

Suppose \( v \) be another fixed point of \( S \), then by (7.2.1) we get

\[
[d(u,v)]^2 = [d(Su, Tv)]^2
\leq \alpha \left\{ d(u, Su) \ d (v, Tv) + d(u, Tv) \ d(v, Su) \right\}
+ \beta \left\{ d(u, Su) \ d(v, Su)+d(u, Tv) \ d(v, Tv) \right\}
+ \gamma \ d(u,v) \ d( Su, Tv)
\]

implies, \( [d(u,v)]^2 \leq (\alpha+\gamma) [d(u,v)]^2 < [d(u,v)]^2 \), giving a contradiction which shows that \( u \) is the unique fixed point of \( S \).

Similarly \( u \) is the unique fixed point of \( T \).

This completes the proof of the theorem.
**COROLLARY 1**: Let $T$ be continuous mapping of a compact metric space $(X,d)$ into itself satisfying the following condition:

$$[d(Tx,Ty)]^2 \leq \alpha \left\{ d(x,Tx) d(y,Ty) + d(x,Ty)d(y,Tx) \right\}$$

$$+ \beta \left\{ d(x,Tx) d(y,Tx) + d(x,Ty)d(y,Ty) \right\}$$

$$+ \gamma \ d(x,y) \ d(Tx,Ty) \ .$$

for all $x \neq y \in X$, where $\alpha, \beta, \gamma \geq 0$, $\alpha + 2\beta + \gamma < 1$.

Then $T$ has a fixed point which is unique.

**PROOF**: Putting $S = T$ in Theorem 1, then the proof of corollary follows easily.

Now, we prove

**THEOREM 2**: Let $S$ and $T$ be two continuous commuting mappings of a compact metric space $(X,d)$ into itself satisfying:

$$(7.2.2) \ d(Sx, STy) < \alpha_1(x,Ty) + \alpha_2\left\{ d(x,Sx) + d(Ty,STy) \right\}$$

$$+ \alpha_3 \left\{ d(x,STy) + d(Ty, Sx) \right\}$$
\[ + \alpha_4 \left\{ \frac{d(x, Sx) \ d(Ty, STy)}{d(x, Ty)} \right\} \\
+ \alpha_5 \left\{ \frac{d(x, STy) \ d(Sx, STy)}{d(x, Sx) + d(Sx, STy)} \right\} \]

for all \( x, y \) in \( X \) with \( x \neq Ty \), \( \alpha_1 \geq 0 \), \( i = 1, 2, 3, 4, 5 \), such that \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 < 1 \), then \( S \) and \( T \) have a common fixed point in \( X \) which is unique whenever \( \alpha_1 + 2\alpha_3 + \alpha_5 \leq 1 \).

**Proof:** Let \( x_0 \) be any point of \( X \), the sequence \( \{x_n\} \) defined by \( Sx_{2n} = x_{2n+1} \), \( Tx_{2n+1} = x_{2n+2} \), for all \( n \in I^+ \), \( I^+ \); is the set of all positive integers).

Let \( f(x) = d(Tx, STx) \), for all \( x \in X \). \( f \) being the composite of continuous mappings \( d, T \) and \( S \), is continuous on the compact space \( X \) and hence attains its infimum say \( u \) in \( X \), i.e., \( f(u) = \inf \{ f(x) : x \in X \} \).

Let us suppose that \( Su \neq u \), then applying \((7.2.2)\), we get
\[ f(Su) = d(TSu, STSu) \]

\[ = d(STu, STSu) \]

\[ < \alpha_1 d(Tu, TSu) + \alpha_2 \left\{ d(Tu, STu) + d(TSu, STSu) \right\} \]

\[ + \alpha_3 \left\{ d(Tu, STSu) + d(TSu, STu) \right\} \]

\[ + \alpha_4 \left\{ \frac{d(Tu, STu) d(TSu, STSu)}{d(Tu, TSu)} \right\} \]

\[ + \alpha_5 \left\{ \frac{d(Tu, STSu) d(STu, STSu)}{d(Tu, STu) + d(STu, STSu)} \right\}. \]

\[ < (\alpha_1 + \alpha_2 + \alpha_3) d(Tu, STu) \]

\[ + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) d(STu, STSu) \]

i.e. 

\[ (1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) d(STu, STSu) \]

\[ < (\alpha_1 + \alpha_2 + \alpha_3) d(Tu, STu) \]

\[ \implies d(STu, STSu) < \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5} d(Tu, STu) \]

implies, \( d(STu, STSu) < d(Tu, STu) \), i.e. \( f(Su) < f(u), \)

this contradicts the definition of \( u \). Therefore \( Su = u \)

and hence \( u \) is a fixed point for \( S \), we have \( STu = TSu = Tu \).
Now, we shall prove that $Tu = u$. Let us suppose that $Tu \neq u$, then we have by (7.2.2),

$$d(u, Tu) = d(Su, STu)$$

$$< \alpha_1 d(u, Tu) + \alpha_2 \left\{ d(u, u) + d(Tu, Tu) \right\}$$

$$+ \alpha_3 \left\{ d(u, Tu) + d(Tu, u) \right\} + \alpha_4 \left( \frac{d(u, u) d(Tu, Tu)}{d(u, Tu)} \right)$$

$$+ \alpha_5 \left\{ \frac{d(u, Tu) d(u, Tu)}{d(u, u) + d(u, Tu)} \right\}$$

i.e. $d(u, Tu) < (\alpha_1 + 2\alpha_3 + \alpha_5) d(u, Tu)$

implies $d(u, Tu) < d(u, Tu)$, this contradiction proves that $Tu = u$.

To prove the uniqueness, let $\alpha_1 + 2\alpha_3 + \alpha_5 \leq 1$, holds and let $v$ be another fixed point of $S$ and $T$, $(u \neq v)$, then by (7.2.2), we have

$$d(u, v) = d(Su, STv)$$

$$< \alpha_1 d(u, v) + \alpha_2 \left\{ d(u, u) + d(v, v) \right\}$$

$$+ \alpha_3 \left\{ d(u, v) + d(v, u) \right\} + \alpha_4 \left( \frac{d(u, u) d(v, v)}{d(u, v)} \right)$$

$$+ \alpha_5 \left\{ \frac{d(u, v) d(u, v)}{d(u, v) + d(u, v)} \right\}.$$
i.e. \( d(u,v) < (\alpha_1 + 2\alpha_3 + \alpha_5) \) \( d(u,v) < d(u,v) \), a contradiction. This shows that \( S \) and \( T \) have unique common fixed point. This completes the proof of the theorem.

Now, we further generalize the above result and prove the following theorem.

**Theorem 3**: Let \( S \) and \( T \) be two continuous commuting mappings of a compact metric space \((X,d)\) into itself satisfying:

\[
(7.2.3) \quad [d(Sx, STy)]^2 \leq \alpha_1 \{ d(Sx, STy) + d(Ty, STy) \} d(x, Sx) \\
+ \alpha_2 \{ d(Sx, STy) + d(Ty, STy) \} d(x, Ty) \\
+ \alpha_3 \{ d(x, STy) + d(Ty, Sx) \} d(Sx, STy) \\
+ \alpha_4 d(x, STy) d(Ty, STy) \\
+ \alpha_5 d(Sx, STy) d(Ty, STy) \\
+ \alpha_6 [d(Ty, STy)]^2
\]

for all \( x, y \) in \( X \) with \( x \neq Ty, \alpha_1 \geq 0, (1 \leq i \leq 6) \), such that \( 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 < 1 \), then \( S \) and \( T \) have a common fixed point in \( X \), which is unique whenever \( \alpha_2 + 2\alpha_3 \leq 1 \).
PROOF: Define the sequence \( \{x_n\} \) as in theorem 2.
Let \( f(x) = d(Tx, STx) \) for all \( x \in X \), \( f \) being the composite of continuous mappings \( d, T \) and \( S \), is continuous on the compact space \( X \) and hence attains its infimum say \( u \) in \( X \), i.e. \( f(u) = \inf \{ f(x) : x \in X \} \).

Let us suppose that \( Su \neq u \), then applying (7.2.3), we get

\[
f(Su) = [d(TSu, STSu)]^2
\]

\[
= [d(STu, STSu)]^2
< \alpha_1 \left\{ d(STu, STSu) + d(TSu, STSu) \right\} d(Tu, STu)
+ \alpha_2 \left\{ d(STu, STSu) + d(TSu, STSu) \right\} d(Tu, TSu)
+ \alpha_3 \left\{ d(Tu, STSu) + d(TSu, STSu) \right\} d(STu, STSu)
+ \alpha_4 d(Tu, STSu) d(TSu, STSu)
+ \alpha_5 d(STu, STSu) d(TSu, STSu)
+ \alpha_6 [d(TSu, STSu)]^2
< (2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) d(Tu, STu) d(STu, STSu)
+ (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) [d(TSu, STSu)]^2
\]
i.e. \[ d(STu, STSu) < (2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) d(Tu, STu) + (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) d(TSu, STSu) \]

implies, \[ d(STu, STSu) < \frac{2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4}{1-\alpha_3-\alpha_4-\alpha_5-\alpha_6} d(Tu, STu) \]

i.e. \[ d(STu, STSu) < d(Tu, STu) \implies f(Su) < f(u), \]
this contradicts the definition of u. Therefore Su = u and hence u is a fixed point for S, we have \[ STu = TSu = Tu. \]

Now, we shall prove that Tu = u. Let us suppose that Tu \neq u, then by (7.2.3), we have

\[ [d(u, Tu)]^2 = [d(Su, STu)]^2 \]

\[ < \alpha_1 \{ d(u, Tu) + d(Tu, Tu) \} d(u, u) + \alpha_2 \{ d(u, Tu) + d(Tu, Tu) \} d(u, Tu) + \alpha_3 \{ d(u, Tu) + d(Tu, u) \} d(u, Tu) + \alpha_4 d(u, Tu) d(Tu, Tu) + \alpha_5 d(u, Tu) d(Tu, Tu) + \alpha_6 [d(Tu, Tu)]^2 \]
which gives, $d(u, Tu) < (\alpha_2 + 2\alpha_3) d(u, Tu) < d(u, Tu)$,

this contradiction proves that $Tu = u$.

Let $\alpha_2 + 2\alpha_3 \leq 1$ holds and $v$ be another fixed point of $S$ and $T$, $(u \neq v)$ then by (7.2.3), we have

$[d(u, v)]^2 < (\alpha_2 + 2\alpha_3) [d(u, v)]^2 < [d(u, v)]^2$,

a contradiction. This shows that $S$ and $T$ have unique common fixed point. This completes the proof.

7.3. In this section, we generalize the fixed point theorem of Popa [1] and Khan and Khan [1] on compact spaces. [Accepted for publication in Journal of Acta Ciencia Indica, Meerut].


Here, first we obtain the generalization of Popa's result on compact spaces.

**Theorem 4:** Let $S$ be a continuous mapping of a compact space $X$ into itself and $f$ be continuous symmetric mapping of $X \times X$ into $\mathbb{R}^+$ such that for all $x \neq y \in X$. 
(7.3.1) \( f(x, y) \neq 0 \)

(7.3.2) \( f^2(x, y) \geq f(x, x) f(y, y) \)

(7.3.3) \( f(Sx, Sy) < \max \left\{ \frac{f(x, Sx) f(y, Sy)}{f(x, y)}, \frac{f(y, Sx) f(Sy, S^2x)}{f(Sx, Sy)} \right\} \)

Then \( S \) have a unique fixed point.

**Proof:** Let \( \phi(x) = f(x, Sx) \) for all \( x \in X \), \( \phi \) being the composite of continuous mappings \( f \) and \( S \), is continuous on the compact space \( X \) and hence it will attain its infimum say \( z \) in \( X \), i.e., \( \phi(z) = \inf \left\{ \phi(x) : x \in X \right\} \).

If \( z \neq Sz \), then by applying (7.3.3), we get

\[
f(Sz, S^2z) < \max \left\{ \frac{f(z, Sz) f(Sz, S^2z)}{f(z, Sz)} \right\}.
\]

Then we have either,

\[
f^2(Sz, S^2z) < f(Sz, Sz) f(S^2z, S^2z) < f^2(Sz, S^2z) \quad \text{by (7.3.2)}
\]

i.e., \( f^2(Sz, S^2z) < f^2(Sz, S^2z) \), a contradiction.
or \[ f(Sz, S^2z) < f(z, Sz), \] i.e. \( \phi(Sz) < \phi(z). \]

This contradicts the definition of \( z \) and hence \( z = Sz. \)

To prove uniqueness, let \( w \) be another fixed point of \( S. \)

Then we have by (7.3.3)

\[
f(z, w) = f(Sz, Sw) < \max \left\{ \frac{f(z, z) f(w, w)}{f(z, w)}, \frac{f(w, z) f(w, z)}{f(z, w)} \right\}.
\]

Then we have either

\[ f^{2}(z, w) < f(z, z) f(w, w) \leq f^{2}(z, w) \] \text{ by (7.3.2)}

i.e. \( f^{2}(z, w) < f^{2}(z, w) \), a contradiction.

Or \( f(z, w) < f(z, w) \), contradiction of \( z \neq w \), and hence \( S \) has unique fixed point. This completes the proof of the theorem.

Now, we shall prove another generalization of above theorem 4.
THEOREM 5: Let $S$ be a continuous mapping of a compact space $X$ into itself and $f$ be a continuous symmetric mapping of $X \times X$ into $\mathbb{R}^+$, satisfying condition (7.3.1) and (7.3.2) of Theorem 4 and

\[(7.3.4) \quad f(Sx, Sy) \leq a \frac{f(x, Sx) f(y, Sy)}{f(x, y)} \]

\[\quad + b \frac{f(y, Sx) f(Sy, S^2x)}{f(Sx, Sy)} + c f(x, y),\]

for all $x \neq y \in X$, $a, b, c \in \mathbb{R}^+$ such that $a + b + c < 1$.

Then $S$ has a unique fixed point.

PROOF: Mapping $\Phi$ is defined in the same way as in Theorem 4 and proceeding in the similar manner, by applying (7.3.4), we get

\[f(Sz, S^2z) \leq a f(Sz, S^2z) + b \frac{f(Sz, Sz) f(S^2z, S^2z)}{f(Sz, S^2z)} \]

\[\quad + c f(z, Sz)\]

\[\leq a f(Sz, S^2z) + b f(Sz, S^2z) + c f(z, Sz)\]

..... by using (7.3.2).
i.e. \[(1 - (a+b)) f(Sz, S^2z) \leq c f(z, Sz)\]

implies, \[f(Sz, S^2z) \leq \frac{c}{1 - (a+b)} f(z, Sz) < f(z, Sz)\].

This shows, \(\phi(Sz) < \phi(z)\) and hence \(z = Sz\).

Uniqueness follows easily.

At the end of the chapter, we obtain a fixed point theorem for a pair of mappings on compact spaces which generalizes the theorem 1 of Khan and Khan [1], [Theorem F].

**THEOREM 6**: Let \(S\) and \(T\) be continuous mappings of a compact space \(X\) into itself and \(F\) a real valued continuous mapping defined on \(X \times X\) such that \(F(x, x) = 0\), for all \(x \in X\), satisfying the following inequalities:

(7.3.5) \[F(STx, TSy) < \max \{F(x, y), F(STx, Sy), F(x, Tx), F(Sy, TSy)\}\]

(7.3.6) \[F(TSx, STy) < \max \{F(x, y), F(TSx, Ty), F(x, Sx), F(Ty, STy)\}\].

for all \(x \neq y \in X\). Then \(S\) and \(T\) each have a fixed point. Further, if \(S\) and \(T\) have a common fixed point, then it is unique.
**Proof:** Let \( \phi(x) = F(x, Tx) \) for all \( x \in X \). \( \phi \) being the composite of two continuous mappings \( F \) and \( T \), is continuous on the compact space \( X \) and hence attains its infimum say \( z \) in \( X \), i.e. \( \phi(z) = \inf \{ \phi(x) : x \in X \} \).

If \( z \neq Tz \), then by applying (7.3.5), we get

\[
F(STz, TSTz) < \max \left\{ F(z, Tz), F(STz, STz), F(z, Tz), F(STz, TSTz) \right\}.
\]

implies, \( F(STz, TSTz) < F(z, Tz) \), i.e. \( \phi(STz) < \phi(z) \),

which contradicts the definition of \( z \) and hence \( z = Tz \).

Let \( \psi(x) = F(x, Sx) \) for all \( x \in X \). \( \psi \) being the composite of two continuous mappings \( F \) and \( S \), is continuous on the compact space \( X \) and hence attains its infimum say \( w \) in \( X \), i.e. \( \psi(w) = \inf \{ \psi(x) : x \in X \} \).

If \( w \neq Sw \), then by applying (7.3.6), we have

\[
F(TSw, STSw) < \max \{ F(w, Sw), F(TSw, TSw), F(w, Sw), F(TSw, STSw) \}
\]

implies, \( F(TSw, STSw) < F(w, Sw) \), i.e. \( \psi(TSw) < \psi(w) \),
which contradicts the definition of \( w \) and hence \( w = Sw \).

Let \( S \) and \( T \) have two different common fixed points \( u \) and \( v \), then by applying each of (7.3.5) and (7.3.6), we get

\[ F(u, v) < F(u, v), \] a contradiction, therefore S and T have unique common fixed point.

This completes the proof of the theorem.