 CHAPTER IV

CONNECTEDNESS OF STRONG TYPE III

TOPOLOGICAL SPACES

Connectedness is perhaps the simplest which a topological space may have and yet it is one of the most important for the application of topology to analysis and geometry. A closer study of connectivity belongs to algebraic topology. For instance the number of components has a simple meaning in homology theory. Connectedness is also a basic notion in complex analysis, for the regions on which analytic functions are studied are generally taken to be connected open subspaces of the complex plane. Spaces which are not connected are also interesting. One of the outstanding characteristics of the Cantor set is the extreme degree in which it fails to be connected. Much the same is true of the subspace of the real line which consists of all rational numbers.

Like many other mathematical concepts of a basic nature, connectedness had only an intuitive meaning. And from this point of view, a connected space is a topological space
which consists of a single piece. The increasingly subtle
demands of Analysis and Topology forced formulation of a
satisfactory definition. Two subsets $A$ and $B$ form a separa-
tion of a set $E$ in a topological space $X$ if and only if $E$
is the union of two nonempty disjoint sets $A$ and $B$, neither
which contains a limit point of the other. A set is called
connected if and only if it has no separation. Equivalently,
a set is connected if and only if whenever it is written as
the union of two nonempty disjoint sets, at least one of them
must contain a limit point of the other. The requirements
that $A$ and $B$ be disjoint sets and neither contain a limit
point of the other may be combined in the condition $(A \cap \text{cl}(B))$
$ \cup (\text{cl}(A) \cap B) = \emptyset$. This is often called the Hausdorff
Lennes separation condition. The above situation can be
rephrased as follows:

**DEFINITION 4.A**: [35] Let $X$ be a topological space,$A$ and $B$ be two nonempty subsets of $X$. Then $A$ and $B$ are said
to be separated if and only if $A \cap \text{cl}(B) = \emptyset$ and $\text{cl}(A) \cap B = \emptyset$.

**DEFINITION 4.B**: [35] A topological space $X$ is said
not to be connected if and only if $X$ can be expressed as the union
of two nonempty separated sets. A set $A \subset X$ is said to be
connected if it is a connected subspace of \( X \).

The empty set and every set consisting of exactly one point is a connected set. In a general topological space it is possible that these would be the only connected sets for example in the case of the discrete topology on any set having at least two points. There are several papers of several types so far available in the literature on this topic.

The absence of any nonconstant continuous real-valued functions is a very strong form of connectedness which can not occur in Urysohn space. (A Urysohn function for \( A \) and \( B \) disjoint subsets of a space \( X \), is a continuous function \( f : X \to [0,1] \) such \( f|A = 0 \) and \( f|B = 1 \). A space with a Urysohn function for any two points is a Urysohn space). Some other stronger forms of connectedness have been considered by Leuschen and Sims [45] and Levine [46]. In the present chapter we introduce and study some new other stronger types of connectedness. It contains two sections.
4.1: c-CONNECTED SPACES*

DEFINITION 4.1.1: Two sets A and B of a topological space X are termed semiseparated if and only if $A \cap \text{cl}(B) = \emptyset = \text{cl}(A) \cap B$.

REMARK 4.1.1: Since $A \subseteq \text{cl}(A) \subseteq \text{cl}(A)$, it follows that any two separated sets are semiseparated and any two semiseparated sets are disjoint. However the converse may be false. For, consider the space of example(2.2.1). The sets $\{a,d\}, \{b,c\}$ are semiseparated but not separated whereas $\{a,b\}, \{c,d\}$ are disjoint but not semiseparated.

THEOREM 4.1.2: If $A, B$ are semiseparated and if $C \subseteq A, D \subseteq B$ then $C, D$ are semiseparated.

PROOF: Since $C \subseteq A$, therefore $\text{cl}(C) \subseteq \text{cl}(A)$ and similarly $\text{cl}(D) \subseteq \text{cl}(B)$.

THEOREM 4.1.2: Any two semiopen sets are semiseparated if and only if they are disjoint.

PROOF: Necessity: This follows because by remark (4.1.1), any two semiseparated sets are disjoint.

Sufficiency: Let $A$ and $B$ be disjoint semiopen sets. Suppose that they are not semi separated. Without any loss in the generality, suppose that $A \cap \text{scl}(B) \neq \emptyset$. Then,

$A \cap \text{scl}(B) \neq \emptyset \implies p \in A \cap \text{scl}(B) \implies p \in A$ and $p \in \text{scl}(B)$.

Since $A$ is semiopen and $p \in \text{scl}(B)$, we obtain $A \cap B \neq \emptyset$. This is a contradiction. Hence $A$ and $B$ are semiseparated.

THEOREM 4.1.3: If $A,B$ are semiseparated and $A \cup B$ open, then $A$ and $B$ are semiopen.

PROOF: Since $A,B$ are semiseparated and $B \subseteq \text{scl}(B)$, it follows that $(A \cup B) \cap (X-\text{scl}(B)) = A$. Now, $A \cup B$ being open and $\text{scl}(B)$ semiopen it follows by lemma (2.3) that $A$ is semiopen. Similarly, $B$ is semiopen.

THEOREM 4.1.4: Let $Y$ be an open subspace of a topological space $X$, and $A,B$ are subsets of $Y$. Then $A,B$ are semiseparated in $X$ if and only if they are semiseparated in $Y$. 
**Proof:** By lemma (2A) it follows that,\\((\text{semicl}\text{-}closure \text{ of } A \text{ in } Y) \cap B) \cup (A \cap (\text{semicl}\text{-}closure \text{ of } B \text{ in } Y)) = (\text{scI}(A) \cap Y \cap B) \cup (A \cap \text{scI}(B)).\\The theorem is now evident.

**Remark 4.1.2:** In the above theorem, Y to be open is essential. For, consider, Y = \{b, c, d\} in the space of example (2.2.1). Then Y is semiclosed but not open in X. The sets \{b, c\} and \{d\} are semiseparted in X but not semiseparted in Y.

**Theorem 4.1.5:** If the sets A and B are both semiclosed or both semiclosed in a space X then the sets A ∩ B and B ∩ A are semiseparted.

**Proof:** If A is semiclosed, then we have,
\[
\text{scI}(A-B) \cap (B-A) = \text{scI}(A \cap (X-B)) \cap (B \cap (X-A)) \subset \text{scI}(A)
\]
\[
\text{scI}(X-B) \cap B \cap (X-A) \subset A \cap (X-A) = \emptyset. \text{ And, if B is semiclosed then we have scI}(A-B) \cap (B-A) \subset (X-B) \cap B = \emptyset, \text{ consequently the theorem follows.}
\]

We introduce now the connectedness of a stronger type and study some of its basic properties:

**Definition 4.1.8:** A topological space X is s-connected if and only if it is not the union of two
nonempty semiseparated sets.

**Theorem 4.1.6**: Every $s$-connected space $X$ is connected.

**Proof**: For, if $X$ is not connected it is a union of two nonempty separated sets. Since any two separated sets are semiseparated by Remark (4.1.1) it follows that $X$ is not $s$-connected.

**Remark 4.1.3**: The converse of Theorem (4.1.6) need not be true. For, the space of example (2.2.1) is connected but it is not $s$-connected.

**Theorem 4.1.7**: A space $X$ is $s$-connected if and only if no nonempty proper subset of $X$ is both semiopen and semiclosed.

**Proof**: Necessity - Let $A$ be a nonempty proper subset of $X$ which is both semiopen and semiclosed. Then $X - A$ is nonempty and semiopen. By Theorem (4.1.2), we obtain $A$ and $X - A$ are semiseparated. Consequently, $X$ is not $s$-connected.
Sufficiency: Suppose that $X$ is not $s$-connected.

Let $G, H$ be two nonempty semiseparated sets such that $X = G \cup H$.

By Theorem (4.1.3), $G$ and $H$ are semiopen and being semiseparated they are disjoint by remark (4.1.1). Thus $G = X \setminus H$. Hence, $G$ is a nonempty proper subset of $X$ which is both semiopen and semiclosed.

**Theorem 4.1.6**: A space $X$ is not $s$-connected if and only if it is the union of two nonempty disjoint semiopen (resp. semiclosed) sets.

This is a consequence of Theorem (4.1.7).

**Definition 4.1.3**: In a topological space $X$, a set is $s$-connected if and only if it is a $s$-connected subspace of $X$.

**Remark 4.1.4**: By Theorem (4.1.7) it follows that each singleton set and any indiscrete space are $s$-connected. Any discrete space, which contains more than one point, is not $s$-connected.

**Remark 4.1.5**: A non $s$-connected space may have an open $s$-connected set. For, $\{a\}$ is an open $s$-connected set in the space of example (2.2.1).
**Theorem 4.1.9**: Let the sets \( A, B \) be nonempty and semiseparated. Then \( A \cup B \) is not \( s \)-connected if it is open.

This follows from Theorem (4.1.4).

**Theorem 4.1.10**: If \( B \) be an open \( s \)-connected set and \( A, B \) are semiseparated in \( X \) such that \( B \subseteq A \cup B \), then \( B \subseteq A \) or \( B \subseteq B \).

**Proof**: Since \( B \subseteq A \cup B \), we obtain, \( B = (E \cap A) \cup (E \cap B) \). Assume, \( E \cap A \neq \emptyset \neq E \cap B \). By Theorem (4.1.1) \( E \cap A \) and \( E \cap B \) are semiseparated in \( X \) and since \( E \) is open, they are semiseparated in \( E \) by Theorem (4.1.4). Therefore \( E \) is not \( s \)-connected. This is a contradiction. And so, \( E \cap A = \emptyset \) or \( E \cap B = \emptyset \). If \( E \cap B = \emptyset \) then \( B = E \cap A \). And so, \( B \subseteq A \). Similarly, if \( E \cap A = \emptyset \) then \( E \cap B = \emptyset \). And so, \( B \subseteq B \). Consequently, \( B \subseteq A \) or \( B \subseteq B \).

**Theorem 4.1.11**: Let \( B \) be open, \( F \) semiopen in \( X \) and \( B \subseteq F \). If \( B \) is \( s \)-connected in \( X \) then so it is in \( F \).
The following lemmas will be needed and these are known [51].

**Lemma 4.1.** If $U$ be open and $A$ semiopen in $X$ then $U \cap A$ is semiopen in $U$.

**Lemma 4.2.** Let $Y$ be a subspace of a space $X$. Then $A$ is semiopen in $Y$ is semiopen in $X$ if and only if $Y$ is semiopen in $X$.

**Proof of Theorem 4.1.11.** Suppose that $E$ is not s-connected in $F$. Let $A$ and $B$ be nonempty semi-separated sets in a subspace $E$ of $F$ such that $A \cup B = E$. Since $E$ is open in $F$, $A$ and $B$ are semi-separated in $F$ by Theorem (4.1.4). And so, by Theorem (4.1.3), $A$ and $B$ are semiopen in $F$. Since $F$ is semiopen it results by lemma (4.2) that $A, B$ are semiopen in $X$. Now by lemma (4.1), $E \cap A = A$ and $E \cap B = B$ are semiopen in the subspace $E$ of $X$. Evidently $A, B$ are disjoint. Hence by Theorem (4.1.3), $E$ is not s-connected in $X$. This is a contradiction. Hence, $E$ is s-connected in $F$. 

**THEOREM 4.1.12**: Let \( E \) be an open \( \ast \)-connected set and \( E \subset F \subset \text{cl}(E) \). Then \( F \) is \( \ast \)-connected.

**PROOF**: Suppose that \( F \) is not \( \ast \)-connected. Let \( A \) and \( B \) be two nonempty semiseparated sets in \( F \) such that \( A \cup B = F \). Since \( B \subset F \subset \text{cl}(E) \subset \text{cl}(E) \) it follows that \( F \) is semiopen. Therefore, by Theorem (4.1.11), \( E \) is \( \ast \)-connected in \( F \). Evidently, \( E \) is open in \( F \). Hence by Theorem (4.1.10), \( E \subset A \) or \( E \subset B \). Let \( E \subset A \). Now since \( B \subset \text{cl}(E) \), \( \text{cl}(B) \subset \text{cl}(A) \) and \( B \subset F \) it follows by utilizing lemma (2.1.18) that \( B = \text{cl}(B) \cap B \subset \text{cl}(A) \cap F \cap B \subset (\text{semiclosure of } A \text{ in } F) \cap B = \emptyset \), since \( A \) and \( B \) are semiseparated in \( F \). Consequently \( B = \emptyset \). This is a contradiction. Hence, \( F \) is \( \ast \)-connected.

**COROLLARY 4.1.1**: If \( E \) be an open \( \ast \)-connected set then \( \text{cl}(E) \) is \( \ast \)-connected and the interior of \( \text{cl}(E) \) is open \( \ast \)-connected.

**THEOREM 4.1.13**: If \( A, B \) be open, \( \ast \)-connected and non-semiseparated sets in \( X \) then \( A \cup B \) is \( \ast \)-connected.

**PROOF**: Assume that \( A \cup B \) is not \( \ast \)-connected. Let \( C, D \) be nonempty and semiseparated sets in \( A \cup B \) such that
A \cup B = C \cup D. Since A \cup B is open by Theorem (4.1.4), C and D are semiseparated in X. Now A \cap C, A \cap D are semiseparated in A by Theorems (4.1.1) and (4.1.4). Since A = (A \cap C) \cup (A \cap D), we have, A \cap D = \emptyset or A \cap C = \emptyset. Therefore, A \subset C or A \subset D. In the same way, B \subset C or B \subset D. If A \subset C then B \subset C \Rightarrow A \cup B = C \Rightarrow C \cup D = C \Rightarrow D = \emptyset, and we reach a contradiction. Thus, A \subset C \Rightarrow B \subset D. Similarly, A \subset D \Rightarrow B \subset C. Consequently, by Theorem (4.1.1), A and B are semiseparated in X. This is contrary to the hypothesis. Hence, A \cup B is s-connected.

**THEOREM 4.1.14:** If \{D_\lambda \mid \lambda \in \Lambda \} be a family of open s-connected sets such that one of them, D_{\lambda_0}, is not semiseparated from every other member, then \bigcup D_\lambda is s-connected.

**PROOF:** Let E = \bigcup D_\lambda. Assume that E is not s-connected. Then there exist nonempty sets A and B, which are semiseparated in E such that E = A \cup B. Since E is open, A, B are semiseparated in X by Theorem (4.1.4). Now by Theorem (4.1.10), let D_{\lambda_0} \subset A or D_{\lambda_0} \subset B. Since D_{\lambda_0} and D_{\lambda_0} are not semiseparated for any \lambda, D_{\lambda_0} \cup D_\lambda are open s-connected by Theorem(4.1.13).

We assert that for each \lambda, D_{\lambda_0} \cup D_\lambda \subset A. (For, if for some \lambda = \lambda_0 \neq D_{\lambda_0} \cup D_\lambda \subset B then D_{\lambda_0} \subset A, D_{\lambda_0} \subset B imply that...
$D_\alpha$ and $D_\beta$ are semi-separated. This is a contradiction. Hence, $E \subseteq A$. Therefore, $D = \emptyset$. This is contrary to the fact that $D$ is nonempty. Hence $E$ is $s$-connected.

**Corollary 4.1.2:** If $\{ D_\lambda \mid \lambda \in \Lambda \}$ be a nonempty family of open $s$-connected sets such that $\bigcap D_\lambda \neq \emptyset$, then $\bigcup D_\lambda$ is $s$-connected.

**Proof:** It follows from remark (4.1.1) that any two non-disjoint sets are not semi-separated. Theorem (4.1.14) is now applicable.

**Corollary 4.1.3:** If $(A_n)_{n \geq 0}$ be an infinite sequence of open $s$-connected sets such that $A_{n+1} \cap A_n \neq \emptyset$ for each $n \geq 0$ then $\bigcup_{n=0}^{\infty} A_n$ is $s$-connected and open.

**Proof:** By induction on $n$, the set $B_n = \bigcup_{i=0}^{n} A_i$ is open $s$-connected for all $n$ by corollary (4.1.2). The sets $B_n$ have a nonempty intersection. Hence $\bigcup_{n=0}^{\infty} A_n$ is $s$-connected by corollary (4.1.2). It is evidently open.
It is known [40, Theorem 3, p. 200] that if $C$ is connected and $C \cap A \neq \emptyset \neq C - A$ then $C \cap \text{Fr}(A) \neq \emptyset$, where $\text{Fr}(A)$ denotes the frontier of $A$. We have,

**Theorem 4.1.15:** If $C$ is open $c$-connected in $X$ and $C \cap A \neq \emptyset \neq C - A$ then $C \cap \text{scl}(A) \cap \text{scl}(X - A) \neq \emptyset$.

**Proof:** By virtue of the $c$-connectedness of the set $C$ and $C = (C \cap A) \cup (C - A)$ it follows that the sets $C \cap A$ and $C - A$ are not semi-separated in $C$. $C$ being open these sets are not semi-separated in $X$ by Theorem (4.1.4). That is,

$$[\text{scl}(C \cap A) \cap (C - A)] \cup [(C \cap A) \cap \text{scl}(C - A)] \neq \emptyset \text{ i.e.}$$

$$C \cap [(\text{scl}(C \cap A) \cap (X - A)) \cup (\text{scl}(C - A) \cap A)] \neq \emptyset.$$ Since $\text{scl}(C \cap A) \subset \text{scl}(A)$, $(X - A) \subset \text{scl}(X - A)$, $\text{scl}(C - A) \subset \text{scl}(X - A)$ and $A \subset \text{scl}(A)$ it follows that $C \cap \text{scl}(A) \cap \text{scl}(X - A) \neq \emptyset$.

The following definitions are well known:

**Definition 4.1:** [101, 36]. A function $f$ from a topological space $X$ into a topological space $Y$ is termed continuous if the inverse image by $f$ of every open set of $Y$ is open in $X$. 


DEFINITION 4.5. [44]: A function \( f \) from a topological space \( X \) into a topological space \( Y \) is semicontinuous if the inverse image by \( f \) of every open set of \( Y \) is semi-open in \( X \).

Loving [44] remarked that every continuous function is semicontinuous but the converse may be false.

So we have,

THEOREM 4.1.16: Let \( f : X \rightarrow Y \) be onto and semicontinuous. If \( X \) is \( s \)-connected then \( Y \) is connected.

PROOF: If \( Y \) is not connected then it is the union of two nonempty disjoint open sets \( A \) and \( B \). Since \( f \) is semicontinuous, \( f^{-1}(A) \), \( f^{-1}(B) \) are semiopen, nonempty, disjoint and their union is \( X \). Hence by Theorem (4.1.8), \( X \) is not \( s \)-connected.

THEOREM 4.1.17: Let \( f : X \rightarrow Y \) be onto and irresolute. If \( X \) is \( s \)-connected then \( Y \) is \( s \)-connected.

The proof is analogous to that of Theorem (4.1.16) and utilizes Theorem (4.1.8).
THEOREM 4.1.10: s-connectedness is a topological property.

The proof is similar to that of Theorem (4.1.16) and utilizes Theorem (4.1.6) and Lemma given below which is known [51].

**Lemma 4.10:** If \( f : X \rightarrow Y \) be one-one, open and continuous and if \( B \) is semiopen in \( Y \) then \( f^{-1}(B) \) is semiopen in \( X \).

The following definition is well known [76]:

**Definition 4.1.1:** A topological space \( X \) is said to be locally connected if for every point \( x \in X \) and every open set \( O \) containing \( x \) there exists an open connected set \( G \) such that \( x \in G \subseteq O \).

We introduce the following concept:

**Definition 4.1.4:** A space \( X \) is locally \( s \)-connected if and only if for every point \( x \in X \) and every open set \( O \) containing \( x \), there exists an open \( s \)-connected set \( G \) such that \( x \in G \subseteq O \).
**Theorem 4.1.19**: Every locally $s$-connected space is locally connected.

This follows from Theorem (4.1.6).

**Remark 4.1.6**: The converse of Theorem (4.1.19) may be false. For the space of example (2.3.1) is locally connected but not locally $s$-connected.

**Remark 4.1.7**: Locally $s$-connected space need not be $s$-connected. For, any finite discrete space is locally $s$-connected but it is not $s$-connected.

**Theorem 4.1.20**: Every open subspace $Y$ of a locally $s$-connected space $X$ is locally $s$-connected.

**Proof**: Let $p \in Y$, $O$ is open in $Y$ and $p \in O$. Since $X$ is locally $s$-connected there is an open $s$-connected set $G$ in $X$ such that $x \in G \subset O$. Now $G \subset O \subset Y$ and so by Theorem (4.1.11), $O$ is $s$-connected in $Y$. Evidently $G$ is open in $Y$. Consequently, $Y$ is locally $s$-connected.
DEFINITION 4.1.5: Let $X$ be a locally $s$-connected space and $p \in X$. The $s$-component of $p$ is the union of all open $s$-connected sets which contain the point $p$.

THEOREM 4.1.21: Each $s$-component is open and $s$-connected.

This follows from corollary (4.1.2).

THEOREM 4.1.22: Each $s$-component of an open set of a locally $s$-connected space is open.

This is a consequence of Theorems (4.1.20) and (4.1.21).

THEOREM 4.1.23: Two distinct $s$-components are semiseparated.

PROOF: If the $s$-components $S_1$ and $S_2$ are not semiseparated then by Theorems (4.1.21) and (4.1.13), $S_1 \cup S_2$ is open $s$-connected. And so, $S_1 \cup S_2 \subset S_1$ and $S_1 \cup S_2 \subset S_2$. That is, $S_1 = S_2$.

COROLLARY 4.1.4: The family of all the $s$-components in a locally $s$-connected space $X$ is a partition of $X$. 
PROOF: This follows from Theorem (4.1.23) and the fact that each point of $X$ is contained in some $s$-component.

**Theorem 4.1.24:** Each $s$-component is closed.

**Proof:** Each $s$-component is open by Theorem (4.1.21). The theorem now follows in view of corollary (4.1.4).

**Theorem 4.1.26:** If $G$ is open in a locally $s$-connected space $X$ and if $A$ be a $s$-component of $G$ then $Fr(A) \cap G = \emptyset$.

**Proof:** $A$ is open in $X$ by Theorem (4.1.22). Since $A$ is closed in $G$ by Theorem (4.1.24) and $G$ being open in $X$ it follows that $G - A$ is open in $X$. Now, $Fr(A) \cap G = cl(A) \cap cl(X - A) \cap G = cl(A) \cap (X - A) \cap G = cl(A) \cap (G - A) \subset cl(A \cap (G - A)) = \emptyset$.

### 4.2. $t$-Connected Spaces

In this section we introduce another stronger form of connectedness which is strictly weaker than the notion

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of $n$-connectedness.

**Definition 4.2.1**: Two sets $A$ and $B$ of a topological space $X$ are termed feebly separated if $A \cap \text{fcl}(B) = \emptyset = \text{fcl}(A) \cap B$.

**Remark 4.2.1**: Since $A \subseteq \text{fcl}(A) \subseteq \text{cl}(A)$, it follows that any two separated sets are feebly separated and any two feebly separated sets are disjoint. However, the converse may be false. For, consider the space of example (2.1.3). Then the sets $\{a\}, \{d\}$ are feebly separated but not separated, whereas the sets $\{b, c\}, \{a, d\}$ are disjoint but not feebly separated.

**Theorem 4.2.1**: If $A, B$ are feebly separated and if $C \subseteq A$, $D \subseteq B$ then $C, D$ are feebly separated.

**Proof**: This follows because if $P \subseteq Q$ then $\text{fcl}(P) \subseteq \text{fcl}(Q)$.

**Theorem 4.2.2**: Any two feebly open sets are feebly separated if and only if they are disjoint.

**Proof**: The necessity is evident for by remark (4.2.1) any two feebly separated sets are disjoint.
The condition is sufficient. For, if \( A \) is feebly open then \( A \cap \text{fcl}(B) \neq \emptyset \implies p \in A \cap \text{fcl}(B) \implies A \cap B \neq \emptyset \), for \( p \in \text{fcl}(B) \) implies that each feebly open set containing \( p \) meets \( B \). This is a contradiction. And so, \( A \cap \text{fcl}(B) = \emptyset \). Similarly if \( B \) is feebly open, \( \text{fcl}(A) \cap B = \emptyset \). Hence, \( A \) and \( B \) are feebly separated.

**Theorem 4.2.3:** If \( A, B \) are feebly separated and \( A \cup B \) open then \( A \) and \( B \) are feebly open.

**Proof:** Since \( A \) and \( B \) are feebly separated and \( B \subseteq \text{fcl}(B) \), it follows that \((A \cup B) \cap (X \setminus \text{fcl}(B)) = A\). Now \( A \cup B \) being open and \( \text{fcl}(B) \) feebly closed it results by Theorem (2.2.6) that \( A \) is feebly open. Similarly \( B \) is feebly open.

**Theorem 4.2.4:** If \( Y \) be an open subspace of a topological space \( X \) and if \( A, B \) are subsets of \( Y \) then \( A, B \) are feebly separated in \( X \) if and only if they are feebly separated in \( Y \).

**Proof:** By Theorem (2.2.9), \((\text{feebly closure of } A \text{ in } Y \cap B) \cup (A \cap \text{feebly closure of } B \text{ in } Y) = (\text{fcl } (A) \cap Y \cap B) \cup (A \cap Y \cap \text{fcl } (B)) = (\text{fcl } (A) \cap B) \cup (A \cap \text{fcl } (B)).\) The theorem now follows.
REMARK 4.2.2: In Theorem (4.2.4), $Y$ to be open is essential. For if $Y = \{a,d\}$, in the space of example (3.1.3) then the sets $\{a\}, \{d\}$ are feebly separated in $X$ but not feebly separated in $Y$.

We now introduce below a new type of connectedness and obtain some of its basic properties.

DEFINITION 4.2.2: A space $X$ is $f$-connected if and only if it is not the union of two nonempty feebly separated sets.

THEOREM 4.2.5: Every $f$-connected space $X$ is connected.

PROOF: For, if not, then it is a union of two nonempty separated sets which by remark (4.2.1) are feebly separated. That is $X$ is not $f$-connected, which is absurd.

THEOREM 4.2.6: A space $X$ is $f$-connected if and only if no nonempty proper subset of $X$ is both feebly open and feebly closed.
**Proof:** Necessity: Let $A$ be a nonempty proper subset of $X$ both feebly open and feebly closed. Then $(X - A)$ is feebly open and nonempty. By Theorem (4.2.2), $A$ and $X - A$ are feebly separated. Consequently, $X$ is not $f$-connected.

**Sufficiency:** Suppose that $X$ is not $f$-connected. Let $G, H$ be two nonempty feebly separated sets such that $X = G \cup H$. Since $G \cup H$ is open it follows by Theorem (4.2.3) that $G$ and $H$ are feebly open and $G \cap H = \emptyset$ being feebly separated. Thus $G = X - H$. Hence $G$ is a nonempty proper subset of $X$ both feebly open and feebly closed.

**Theorem 4.2.7:** A space $X$ is not $f$-connected if and only if it is a union of two nonempty disjoint feebly open (resp. feebly closed) sets.

This is a consequence of Theorem (4.2.6).

**Remark 4.2.3:** Since every feebly open set is semiopen it follows that for any set $A$ in a space $X$, in general $\text{col}(A) \subseteq \text{fcl}(A)$. It therefore results that every $s$-connected space is $f$-connected. The converse of this may be false. For, the space of example (2.1.2) is $f$-connected but it is not $s$-connected.
**DEFINITION 4.2.3:** In a space $X$, a set is $f$-connected if and only if it is a $f$-connected subspace of $X$.

**Remark 4.2.4:** By Theorem, it follows that each singleton set and any indiscrete space are $f$-connected. Any discrete space which contains more than one point is not $f$-connected.

**Theorem 4.2.8:** Let $A, B$ be nonempty and feebly separated in $X$. Then $A \cup B$ is not $f$-connected if it is open.

This follows from Theorem (4.2.4).

**Theorem 4.2.9:** If $E$ be an open $f$-connected set and $A, B$ are feebly separated in $X$ such that $E \subseteq A \cup B$ then $E \subseteq A$ or $E \subseteq B$.

**Proof:** Since $E \subseteq A \cup B$ it follows that $E = (E \cap A) \cup (E \cap B)$. Assume $E \cap A \neq \emptyset$ or $E \cap B$. Now, $E \cap A$, $E \cap B$ are feebly separated by Theorem (4.2.1) and $E$ being open they are feebly separated in $E$ by Theorem (4.2.4). Hence, $E$ is not $f$-connected, a contradiction. And so, $E \cap A = \emptyset$ or $E \cap B = \emptyset$. Consequently, $E \subseteq A$ or $E \subseteq B$. 
THEOREM 4.2.10: Let $E, F$ be open in $X$ and $E \subseteq F$. If $E$ is $f$-connected in $X$ then so it is in $F$.

PROOF: Assume that $E$ is not $f$-connected in $F$. In view of Theorem (4.2.7), let $A, B$ be two nonempty disjoint feebly open sets in a subspace $E$ of $F$ such that $A \cup B = F$. Evidently $E$ is open in $F$. And so, $A, B$ are feebly open in $F$ by Theorem (2.2.4) and consequently feebly open in $X$ for $F$ is open in $X$. Now by Theorem (2.2.6), $E \cap A$ and $E \cap B$ are feebly open in a subspace $E$ of $X$. Evidently $E \cap A = A$ and $E \cap B = B$. Hence by Theorem (4.2.7), $E$ is not $f$-connected in $X$.

THEOREM 4.2.11: If $E$ be an open $f$-connected set and $F$ be open such that $E \subseteq F \subseteq \text{fcl}(E)$, then $F$ is $f$-connected.

PROOF: Suppose that $F$ is not $f$-connected. Let $A, B$ be nonempty feebly separated sets in $F$ such that $A \cup B = F$. By Theorem (4.2.10), $E$ is $f$-connected in $F$. Since $E$ is open in $F$ it follows by Theorem (4.2.9) that $E \subseteq A$ or $E \subseteq B$. Let $E \subseteq A$. Now since $E \subseteq \text{fcl}(E)$, on using Theorem (2.2.9), we get $E = \text{fcl}(E) \cap E \subseteq \text{fcl}(A) \cap F \cap B = (\text{feebly closure of } A$
in $F \setminus B = \emptyset$, since $A,B$ are feebly separated in $F$. It follows
that $B = \emptyset$, a contradiction. Hence $F$ is $f$-connected.

**COROLLARY 4.2.1**: If $B$ be an open $f$-connected set
in $X$ then $\text{int}(\text{fcl}(B))$ is $f$-connected.

**THEOREM 4.2.12**: If $A,B$ be open $f$-connected and
not feebly separated sets in $X$ then $A \cup B$ is $f$-connected.

**PROOF**: Assume that $A \cup B$ is not $f$-connected. Let
$C,D$ be nonempty feebly separated sets in $A \cup B$ such that $A \cup B =
C \cup D$. Since $A \cup B$ is open, by Theorem (4.2.4) $C,D$ are feebly
separated in $X$. Now by Theorems (4.2.1) and (4.2.4), $A \cap C,
A \cap D$ are feebly separated in $A$. Since $A = (A \cap C) \cup (A \cap D)$
and $A$ is $f$-connected, it follows that $A \cap D = \emptyset$ or $A \cap C = \emptyset$.
Therefore, $A \subset C$ or $A \subset D$. In the same way $B \subset C$ or $B \subset D$.
If $A \subset C$ then $B \subset C \implies A \cup B \subset C \implies C \cup D = C \implies D = \emptyset$,
a contradiction. Thus $A \subset C \implies B \subset D$. Similarly, $A \subset D \implies
B \subset C$. Consequently by Theorem (4.2.1), $A,B$ are feebly separa-
ted in $X$. This is absurd. Hence $A \cup B$ is $f$-connected.

**THEOREM 4.2.13**: If $\{ D_{\lambda} \mid \lambda \in \Lambda \}$ be a family of
open $f$-connected sets such that one of them say, $D_{\lambda_0}$, is not
feebly separated from every other member, then $\bigcup D_\lambda$ is $f$-connected.

**Proof:** Let $E = \bigcup D_\lambda$ and assume that $E$ is not $f$-connected. Let $A, B$ be nonempty and feebly separated sets in $E$ such that $E = A \cup B$. Since $E$ is open, $A, B$ are feebly separated in $X$ by Theorem (4.2.4). Now by Theorem (4.2.9),

$$D_\beta \subseteq A \text{ or } D_\beta \subseteq B.$$ 

Let $D_\beta \subseteq A$. Since $D_\beta$ and $B$ are not feebly separated for any $\beta$, $D_\beta \cup D_\beta$ are open $f$-connected by Theorem (4.2.12). If for some $\lambda = \beta$, $D_\beta \cup D_\beta \subseteq B$ then $D_\beta \subseteq A$ and $D_\beta \subseteq B$ imply that $D_\beta$ and $D_\beta$ are feebly separated which is absurd. Hence for each $\lambda$, $D_\lambda \cup D_\lambda \subseteq A$, by virtue of Theorem (4.2.9). It results that $E \subseteq A$. That is $B = \emptyset$. This is a contradiction. Consequently, $E$ is $f$-connected.

**Corollary 4.2.2:** If $\{D_\lambda \mid \lambda \in \Lambda\}$ be a nonempty family of open $f$-connected sets such that $\bigcap D_\lambda \neq \emptyset$ then $\bigcup D_\lambda$ is $f$-connected and open.

**Proof:** It follows from remark (4.2.1) that any two non-disjoint sets are non-feebly separated. Hence by Theorem (4.2.13), $\bigcup D_\lambda$ is $f$-connected.
COROLLARY 4.2.3: If \((A_n)_n \geq 0\) be an infinite sequence of open \(f\)-connected sets such that \(A_{n+1} \cap A_n \neq \emptyset\) for each \(n \geq 0\), then \(\bigcup_{n=0}^{\infty} A_n\) is \(f\)-connected.

**Proof:** By induction on \(n\), the set \(B_n = \bigcup_{i=0}^{n} A_i\) is open \(f\)-connected for all \(n\) by Corollary (4.2.2). The sets \(B_n\) have a nonempty intersection. Hence \(\bigcup_{n=0}^{\infty} A_n\) is \(f\)-connected by corollary (4.2.2).

The following theorem extends Theorem 3 in [40, p.200] to \(f\)-connected spaces.

**Theorem 4.2.14:** If \(C\) is open \(f\)-connected and \(C \cap A \neq \emptyset \neq C - A\), then \(C \cap \text{fcl}(A) \cap \text{fcl}(X-A) \neq \emptyset\).

**Proof:** By virtue of the \(f\)-connectedness of the set \(C\) and \(C = (C \cap A) \cup (C - A)\), the sets \(C \cap A\) and \(C - A\) are not feebly separated in \(C\). \(C\) being open these sets are not feebly separated in \(X\) by Theorem (4.2.1). That is, \([\text{fcl}(C \cap A) \cap (C-A)] \cup [\text{fcl}(C-A) \cap (C \cap A)] \neq \emptyset\). That is, \(C \cap ([\text{fcl}(C \cap A) \cap (X-A)) \cup (\text{fcl}(C-A) \cap A)] \neq \emptyset\). Since \(\text{fcl}(C \cap A) \subset \text{fcl}(A)\), \(X - A \subset \text{fcl}(X-A)\), \(\text{fcl}(C-A) \subset \text{fcl}(X-A)\) and \(A \subset \text{fcl}(A)\), it follows that \(C \cap \text{fcl}(A) \cap \text{fcl}(X-A) \neq \emptyset\).
**Definition 4.2.4:** A function \( f : X \to Y \) is feebly continuous if the inverse image of every open set is feebly open.

**Remark 4.2.5:** Since every open set is feebly open it is clear that every continuous function is feebly continuous. But the converse may be false. For,

**Example 4.2.1:** Let \( X = \{a, b, c, d\} \),
\[
T = \{\emptyset, \{b\}, \{a\}, \{b, c, d\}, X\}
\]
and \( X^* = \{a, b\} \),
\[
T^* = \{\emptyset, \{a\}, X^*\}.
\]
Define \( f : X^* \to X \) by \( f(a) = f(b) = f(c) = a \) and \( f(d) = b \). Then \( f \) is feebly continuous but it is not continuous.

**Theorem 4.2.16:** Let \( f : X \to Y \) be onto and feebly continuous. If \( X \) is \( f \)-connected then \( Y \) is connected.

**Proof:** If \( Y \) is not connected then it is the union of two nonempty disjoint open sets \( A \) and \( B \). Since \( f \) is feebly continuous, \( f^{-1}(A) \), \( f^{-1}(B) \) are feebly open, nonempty, disjoint and their union is \( X \). Hence by Theorem (4.2.7), \( X \) is not \( f \)-connected.

**Theorem 4.2.18:** \( f \)-connectedness is a topological property.
The proof is similar to Theorem (4.2.15) and follows by virtue of Theorem (4.2.7) and Theorem (2.2.12).

**Definition 4.2.5**: A space $X$ is locally $f$-connected if and only if for every point $x \in X$ and every open set $O$ containing $x$ there exists an open $f$-connected set $G$ such that $x \in G \subset O$.

**Theorem 4.2.17**: Every locally $f$-connected space is locally connected.

This follows from Theorem (4.2.5) and definition (4.2.5).

**Remark 4.2.8**: Locally $f$-connected space need not be $f$-connected. For any finite discrete space is locally $f$-connected but not $f$-connected.

**Theorem 4.2.18**: Every open subspace $Y$ of a locally $f$-connected space $X$ is locally $f$-connected.

**Proof**: Let $p \in Y$ and $O$ is open in $Y$ such that $p \in O$. Since $X$ is locally $f$-connected there is an open $f$-connected set $G$ in $X$ such that $x \in G \subset O$. Now $G \subset O \subset Y$. By Theorem (4.2.10), $G$ is $f$-connected in $Y$. Also $G$ is open in $Y$. Consequently, $Y$ is locally $f$-connected.
**Definition 4.2.6:** Let $X$ be a locally $f$-connected space and $p \in X$. The $f$-component of $p$ is the union of all open $f$-connected sets which contain the point $p$.

**Theorem 4.2.19:** Each $f$-component is open $f$-connected.

This follows from corollary (4.2.2).

**Theorem 4.2.20:** Each $f$-component of an open set of a locally $f$-connected space is open.

This is a consequence of Theorems (4.2.19) and (4.2.19).

**Theorem 4.2.21:** Two distinct $f$-components are feasibly separated.

**Proof:** If the $f$-components $S_1$ and $S_2$ are not feasibly separated then by Theorems (4.2.19) and (4.2.12), $S_1 \cup S_2$ is open $f$-connected. And so, $S_1 \cup S_2 \subseteq S_1$ and $S_1 \cup S_2 \subseteq S_2$. That is, $S_1 = S_2$.

**Corollary 4.2.24:** The family of all the $f$-components in a locally $f$-connected space $X$ is a partition of $X$. 
This follows from Theorem (4.2.21) since by definition (4.2.6) each point \( p \) of \( X \) belongs to some \( f \)-component.

**Theorem 4.2.22:** Each \( f \)-component is closed.

**Proof:** Each \( f \)-component is open by Theorem (4.2.19). The theorem now follows in view of corollary (4.2.4).

**Theorem 4.2.23:** If \( G \) be open in a locally \( f \)-connected space \( X \) and if \( A \) be an \( f \)-component of \( G \) then \( \text{Fr}(A) \cap G = \phi \).

**Proof:** \( A \) is open in \( X \) by Theorem (4.2.20). \( A \) is closed in \( G \) by Theorem (4.2.22) and so \( G \cap A \) is open in \( X \) since \( G \) is open in \( X \). Now, \( \text{Fr}(A) \cap G = \text{cl}(A) \cap \text{cl}(X - A) \cap G = \text{cl}(A) \cap (X - A) \cap G \subseteq \text{cl}[A \cap (G - A)] = \text{cl}(\phi) = \phi. \)