CHAPTER III

THE ROLE OF FREELY OPEN SETS IN

BITOPOLOGICAL SPACES

The theory of bitopological spaces was first initiated by J.C. Kelly [36] in 1963. Infact, he observed that with every quasi metric $d$ on $X$ there is associated another quasi metric $p$ on $X$. The $d$-open spheres and the $p$-open spheres generate the topologies on $X$ say $P_1$ and $P_2$ respectively. Similarly, with every quasi uniformity $\mathcal{U}$ on a set $X$ there is associated another quasi uniformity $\mathcal{U}^{-1}$ on $X$. These uniformities on $X$ give rise to unique topologies on $X$ say $P$ and $Q$ respectively. If one studies $X$ with two topologies $\Gamma_1$ and $\Gamma_2$ some of the symmetry of the classical metric situation is regained and symmetric generalization of some of the standard results such as Urysohn's lemma, Urysohn's metrisation theorem, Tietze's extension theorem etc. can be obtained. These observations led Kelly [36] to introduce the concept of a bitopological space. It is a triple $(X,P,Q)$ where a nonempty set $X$ is equipped with two arbitrary topologies $P$ and $Q$. 
Soon after their inception the bitopological spaces started attracting the workers in topology. Many mathematicians such as Fervin [77], Patty [75], Fletcher [24], Datta [22], Kim [37], Murdeshwar and Naimpally [63], Reilly [30 to 33], Singal and his associates [39 to 91], Noiri [62], Zinovie [110], Thampuran [90], Swart [97], Su [96] and others have contributed to the theory of bitopological spaces. In almost all cases one finds that the aim has been to generalise results from the theory of general topological spaces, in such a way that classical theorems in general topology become particular cases of the analogous theorems for bitopological spaces provided one accepts the convention that a topological space \((X, \Gamma)\) is also a bitopological space \((X, \Gamma_1, \Gamma_2)\).

Concerning some notations in a bitopological space \((X, \mathcal{P}, \mathcal{Q})\) we mention that if \(\mathcal{P}\) be any topological concept then \(P = \mathcal{P}(\text{resp.} Q = \mathcal{Q})\) means the property \(\mathcal{P}\) with respect to the topology \(P\) (resp. the property \(\mathcal{Q}\) with respect to the topology \(Q\)). And by \(i, j = 1, 2, i \neq j\), we mean that \(i, j\) assume the values 1 or 2 and at the same time in such a way that they
remain unequal. For example, if \( i \) takes the value 1 this implies that \( j = 2 \) and conversely. Throughout this chapter \( X \setminus A \) denotes the complement of \( A \) in \( X \).

This chapter deals with feebly open sets and concerns with their role in connection with the separation axioms in bitopological spaces. It contains three sections.

3.1. IMPACT OF FEEBLY OPEN SETS ON PAIRWISE SEPARATION AXIOMS.

After the publication of Kelly's [36] paper in 1963, several papers have appeared in the literature so far concerning the separation axioms in bitopological spaces. Kelly himself initiated the study of separation axioms for bitopological spaces and introduced the concepts of pairwise Hausdorff (i.e., pairwise \( T_2 \)), pairwise regular and pairwise normal spaces. The terms pairwise \( T_0 \) and pairwise \( T_1 \) owe to Murschewar and Kaimpally [63] who discussed these notions in the context of bi-quasi uniform spaces. In recent years there has been a
considerable revival of interest not only in the study of separation axioms but also in the search for new such axioms. Consequently, various separation axioms [refer, [89 to 91], [94], [96], [97]] have been introduced in the recent past for bitopological spaces. In the present section we introduce and study some new separation axioms in bitopological spaces. At the same time it also presents the role of feebly open sets in bitopological spaces.

**Definition 3.1.1.** A bitopological space \((X, T_1, T_2)\) is pairwise feebly T\(_0\) if for each pair of distinct points of \(X\), there is a set which is either \(T_1\)-feebly open or \(T_2\)-feebly open containing one of the points but not the other.

**Remark 3.1.1.** It is evident that every pairwise T\(_0\) space is pairwise feebly T\(_0\) (for every open set is feebly open). However, the converse need not be true. For,

**Example 3.1.1.** Let \(X = \{a, b, c\}\),

\[P = \{\emptyset, X\}\, \text{ and } \]

\[Q = \{\emptyset, \{a\}, X\},\]
then \((X, P, Q)\) is pairwise feebly \(T_0\) but it is not pairwise \(T_0\).

The following definition owes to Maheshwari and Prasad [55].

**DEFINITION 3.1.** A bitopological space \((X, P, Q)\) is said to be pairwise semi \(T_0\) if for each pair of distinct points \(x, y\) of \(X\) there exists a \(P\)-semiopen set containing \(x\) but not \(y\) or a \(Q\)-semiopen set containing \(y\) but not \(x\).

**REMARK 3.1.2:** Since every feebly open set is semi-open it is clear that every pairwise feebly \(T_0\) space is pairwise semi \(T_0\). But we do not know about the converse.

**THEOREM 3.1.1:** A bitopological space \((X, P, Q)\) is pairwise feebly \(T_0\) if and only if given two distinct points of \(X\) either their \(P\)-feebly closures or \(Q\)-feebly closures are distinct.

**PROOF:** Sufficiency. For, let \(x, y \in X\), \(x \neq y\) and that either \(P\text{-fcl}(x) \neq P\text{-fcl}(y)\) or \(Q\text{-fcl}(x) \neq Q\text{-fcl}(y)\). For the first case assume that \(p \not\in P\text{-fcl}(x)\) but \(p \in P\text{-fcl}(y)\). We
assert that $y \notin P\text{-fcl}\{x\}$. For, $y \in P\text{-fcl}\{x\}$ implies that $P\text{-fcl}\{y\} \subseteq P\text{-fcl}\{x\} = P\text{-fcl}\{x\}$ i.e. $p \in P\text{-fcl}\{x\}$.

Thus $x \in P\text{-fcl}\{x\}$ is a $P$-feebly open set containing $y$ but not $x$. The case second is analogous. Consequently, the space $(X,P,Q)$ is feebly $T_0$.

\textbf{Necessity}: For let $(X,P,Q)$ is pairwise feebly $T_0$ and $x,y \in X$, $x \neq y$. Suppose that $U$ is a $P$-feebly open set containing $x$ but not $y$. Then $y \in P\text{-fcl}\{y\} \subseteq X - U$, and so $x \notin P\text{-fcl}\{y\}$. Since $x \in P\text{-fcl}\{x\}$, it follows that $P\text{-fcl}\{x\} \neq P\text{-fcl}\{y\}$.

\textbf{Remark 3.1.3}: Not every subspace of a pairwise feebly $T_0$ space is pairwise feebly $T_0$. For, $(b,c)$ as a subspace of a pairwise feebly $T_0$ space $X$ of example (3.1.1), is not pairwise feebly $T_0$. However, we have the following theorem:

\textbf{Theorem 3.1.2}: Every biopen subspace of a pairwise feebly $T_0$ space is pairwise feebly $T_0$.

\textbf{Proof}: Let $(Y,P^*,Q^*)$ be a biopen subspace of a pairwise feebly $T_0$ space $(X,P,Q)$. Let $a,b \in Y$ and $a \neq b$. Since $Y \subseteq X$ and $X$ is pairwise feebly $T_0$ there exists a $P$-feebly open
set or a \( Q \)-feebly open set containing one of the points but
not the other. Without any loss in the generality let \( U \) be a
\( P \)-feebly open set such that \( a \in U \) but \( b \notin U \). Since \( Y \) is \( P \)-open
by Theorem (2.2.6) \( U \cap Y \) is \( P \)-feebly open, contains \( a \) and to
which \( b \) does not belong. Hence \( (Y,P^*,Q^*) \) is pairwise feebly \( T_0 \).

The following definition gives the concepts of
continuity, irresolutability and homeomorphism in their bitopo-
logical forms.

**DEFINITION 3.1.** A function \( f : (X,P,Q) \to (X^*,P^*,Q^*) \)
is said to be pairwise open [30] (resp. bitopological irresolute;
resp. pairwise homeomorphism[30]) if the induced functions \( f_1 : (X,P) \to (X^*,P^*) \) and \( f_2 : (X,Q) \to (X^*,Q^*) \) are open (resp. irresolve-
lute ; resp. homeomorphisms).

**THEOREM 3.1.3.** Let \( f : (X,P,Q) \to (X^*,P^*,Q^*) \) be
one-one, onto pairwise open and bitopological irresolute. If \( X \)
is pairwise feebly \( T_0 \) then so is \( X^* \).

**PROOF :** Let \( a,b \in X^* \) and \( a \neq b \). Since \( f \) is one-one
and onto, there exist \( x,y \in X, x \neq y \) such that \( f(x) = a, f(y) = b \).
\[ f(y) = b. \] Since \( X \) is pairwise feebly \( T_\sigma \), without any loss in the generality, suppose that there is a \( p \)-feebly open set \( G \) containing \( x \) but not \( y \). Then \( f(G) \) is \( p^* \)-feebly open set by Theorem (2.2.10). Evidently, \( a \notin f(G) \) but \( b \notin f(G) \). Hence, \( X \) is pairwise feebly \( T_\sigma \).

**Theorem 3.1.4:** Pairwise homeomorphic image of a pairwise feebly \( T_\sigma \) space is pairwise feebly \( T_\sigma \).

The proof runs similar to that of Theorem (3.1.3) and utilizes Theorem (2.2.11).

**Definition 3.1.2:** A bitopological space \((X, P, Q)\) is pairwise feebly \( T_2 \) if for each pair of distinct points \( x, y \) of \( X \), there exist a \( P \)-feebly open set \( U \) and a \( Q \)-feebly open set \( V \) such that \( x \in U \), \( y \notin U \) and \( y \in V \), \( x \notin V \).

**Remark 3.1.4:** It is evident that every pairwise feebly \( T_2 \) space is feebly \( T_\sigma \). However, the converse may be false. For the space of example (3.1.1) is pairwise feebly \( T_\sigma \) but it is not pairwise feebly \( T_2 \).
REMARK 3.1.8: Since every open set is feasibly open it follows that every pairwise $T_1$ space is pairwise feasibly $T_1$. The converse may be false as shown by:

EXAMPLE 3.1.2: Let $X = \{a, b, c\}$, $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{Q} = \{\emptyset, \{c\}, X\}$. Then $(X, \mathcal{P}, \mathcal{Q})$ is pairwise feasibly $T_1$ but it is not pairwise $T_1$.

The following definition is due to Maheshwari and Prasad [55].

DEFINITION 3.6: A bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is pairwise semi $T_2$ if for each pair of distinct points $x, y$ of $X$ there exists a $\mathcal{P}$-semi open set containing $x$ but not $y$ and a $\mathcal{Q}$-semiopen set containing $y$ but not $x$.

REMARK 3.1.6: Since every feasibly open set is semiopen it follows that every pairwise feasibly $T_1$ space is pairwise semi $T_1$. However, the converse may be false. Consider:

EXAMPLE 3.1.3: Let $X = \{a, b, c, d\}$,
\[ P = \{ \emptyset, \{a\}, \{b\}, \{a,b\}, X \} \] and
\[ Q = \{ \emptyset, \{a\}, \{c\}, \{a,c\}, X \}. \]

Then \((X,P,Q)\) is pairwise semi \(T_2\) but it is not pairwise feebly \(T_1\).

**Remark 3.1.7:** Not every subspace of a pairwise feebly \(T_2\) space is pairwise feebly \(T_1\). For, \{a,b\} as a subspace of the pairwise feebly \(T_2\) space \(X\) of example (3.1.2) is not pairwise feebly \(T_1\). However, we have the following theorem:

**Theorem 3.1.5:** Every biopen subspace of a pairwise feebly \(T_2\) space is pairwise feebly \(T_1\).

The proof uses Theorem (2.2.6) and is similar to that of Theorem (3.1.2).

**Theorem 3.1.6:** Let \( f : (X,P,Q) \to (X',P',Q') \) be one-one, onto, pairwise open and hitopological irresolute. If \( X \) is pairwise feebly \( T_2 \) then so is \( X' \).

The proof is analogous to that of Theorem (3.1.3).
THEOREM 3.1.7: Pairwise homeomorphic image of a pairwise feebly $T_1$ space is pairwise feebly $T_1$.

The proof is similar to Theorem (3.1.3) and utilizes Theorem (2.2.11).

DEFINITION 3.1.3: A bitopological space $(X, P, Q)$ is pairwise feebly $T_2$ if for each pair of distinct points $x, y$ of $X$, there exists a $P$-feebly open set $U$ and a $Q$ feebly open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

REMARK 3.1.8: Evidently, every pairwise $T_2$ space is pairwise feebly $T_2$. The converse may be false as shown by example (3.1.3) which is pairwise feebly $T_2$ but it is not pairwise $T_2$.

REMARK 3.1.9: Every pairwise feebly $T_2$ space is pairwise semi $T_2$. But the converse may be false. For, the space of example (3.1.3) is pairwise semi $T_2$ but it is not pairwise feebly $T_2$. 
REMARK 3.1.10: Every pairwise feebly $T_2$ space is pairwise feebly $T_1$. In general the converse is false, as shown by,

EXAMPLE 3.1.4: Let $X$ be the set of all the real numbers, $P$ be the countable complement topology and $Q$ be the usual topology on $X$. Then the space $(X,P,Q)$ is pairwise $T_1$ and consequently it is pairwise feebly $T_1$. However, it is not pairwise feebly $T_2$. For, every $Q$-open set is uncountable and so intersects every nonempty $P$-open set. Consequently, every $P$-feebly open set intersects every $Q$-feebly open set.

THEOREM 3.1.9: In a bitopological space $(X,P_1,P_2)$ the following conditions are equivalent:

(a) $(X,P_1,P_2)$ is pairwise feebly $T_2$.

(b) Let $x \in X$. For each $y \in X$ and $y \neq x$, there exists a $P_1$-feebly open set $U$ such that $x \in U$ and $y \notin P_i$-fcl($U$), $i \neq j$, $i,j = 1,2$.

(c) For each $x \in X$, $x = \bigcap \{P_j$-fcl($U$) $| x \in U$, $U$ is $P_j$-feebly open $\}$, $i \neq j$, $i,j = 1,2$. 
**Proof**: (a) $\Rightarrow$ (b) : Since $X$ is pairwise feebly $T_2$ and $x \neq y$, there exist a $P_j$-feebly open set $U$ containing $x$ and a $P_j$-feebly open set $V$ containing $y$ such that $U \cap V = \emptyset$ where $i \neq j$, $i,j = 1,2$. Now $X \setminus V$ is $P_j$-semiclosed containing $U$ and so $P_j$-scl($U$) $\subseteq X \setminus V$. It is evident that $y \notin P_j$-scl($U$).

(b) $\Rightarrow$ (c) : Let $y$ be any point distinct from $x$. Then by (b), there exists a $P_i$-feebly open set $U$ such that $x \in U$ but $y \notin P_j$-scl($U$), where $i \neq j$, $i,j = 1,2$. Therefore, the set $\cap \{ P_j$-scl($U$) $| x \in U, U$ is $P_j$-feebly open $\}$ contains no point of $X$ other than $x$, where $i \neq j$, $i,j = 1,2$. Consequently (c) holds.

c) $\Rightarrow$ (a) : Let $x,y \in X$ and $x \neq y$. Then by (c), there exists a $P_j$-feebly open set $U$ such that $x \in U$ but $y \notin P_j$-scl($U$) where $i \neq j$, $i,j = 1,2$. Since $U \subseteq P_j$-scl($U$), $V = X \setminus P_j$-scl($U$) is a $P_j$-feebly open set containing $y$. Evidently, $U \cap V = \emptyset$. Consequently, $X$ is pairwise feebly $T_2$.

**Remark 3.2.11** : Not every subspace of a pairwise feebly $T_2$ space is pairwise feebly $T_2$. For $\{a,b\}$ as a subspace of the pairwise feebly $T_2$ space of example (3.1.3) is not pairwise feebly $T_2$. However,
THEOREM 3.1.2: Every biopen subspace of a pairwise feebly $T_2$ space is pairwise feebly $T_2$.

PROOF: Let $(Y, P, Q)$ be a biopen subspace of a pairwise feebly $T_2$ space $(X, P, Q)$. Let $a, b \in Y$ and $a \neq b$.

Since $a, b \in X$, by hypothesis let $U$ be a $P$-feebly open set, $V$ be a $Q$-feebly open set such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

Since $Y$ is $P$-open, $U \cap Y$ is $P^*$-feebly open by Theorem 2.2.6. Similarly, $V \cap Y$ is $Q^*$-feebly open. Evidently $(U \cap Y) \cap (V \cap Y) = \emptyset$.

Hence, $(Y, P, Q)$ is pairwise feebly $T_2$.

THEOREM 3.1.10: Let $f : (X, P, Q) \rightarrow (X^*, P^*, Q^*)$ be one-one, onto, pairwise open and bitopological irresolute. If $X$ is pairwise feebly $T_2$ then so is $X^*$.

PROOF: Let $a, b \in X^*$ and $a \neq b$. Since $f$ is one-one onto, there exist $x, y \in X$, $x \neq y$, such that $f(x) = a$, $f(y) = b$. $X$ being pairwise feebly $T_2$ there exist a $P$-feebly open set $G$ containing $x$ and a $Q$-feebly open set $H$ containing $y$ such that $G \cap H = \emptyset$. Then by Theorem 2.2.10, $f(G)$ is $P^*$-feebly open, $f(H)$ is $Q^*$-feebly open, $a \in f(G)$ and $b \in f(H)$.

Since $f$ is one-one we get $f(G) \cap f(H) = f(G \cap H) = f(\emptyset) = \emptyset$.

Hence, $X^*$ is pairwise feebly $T_2$. 
**THEOREM 3.1.11**: Pairwise homeomorphic image of a pairwise feebly $T_0$ space is pairwise feebly $T_0$.

The proof is similar to Theorem (3.1.10) and utilises Theorem (2.2.11).

**DIAGRAM**

**REMARK 3.1.12**: The space of example (3.1.4) is pairwise feebly $T_1$ but not pairwise semi $T_2$ and the space of example (3.1.1) is pairwise feebly $T_0$ but is not pairwise semi $T_1$.

**REMARK 3.1.13**: The space of example (3.1.3) is pairwise semi $T_2$ but not pairwise feebly $T_1$.

**REMARK 3.1.14**: The space of example (3.1.4) is pairwise $T_1$ but it is not pairwise feebly $T_2$. The space of the following example is pairwise $T_0$ but it is not pairwise feebly $T_1$.

**EXAMPLE 3.1.5**: Let $X = \{a, b, c\}$,

$P = \mathcal{P}(X)$ and $Q = \text{the discrete topology on } X.$
**Example 2.1.6:** Let \( X = \{a, b, c, d\} \)
\[
P = \{\emptyset, \{a\}, X\} \text{ and } q = \{\emptyset, \{b\}, X\}.
\]

Thus we arrive at the following diagram where the implications \( \rightarrow \) and \( \twoheadrightarrow \) stand for 'implies' and 'strictly implies' respectively.

\[
\begin{array}{ccc}
\text{Pairwise } T_2 & \rightarrow & \text{Pairwise feebly } T_2 \\
\downarrow & & \downarrow \\
\text{Pairwise } T_1 & \rightarrow & \text{Pairwise feebly } T_1 \\
\downarrow & & \downarrow \\
\text{Pairwise } T_0 & \rightarrow & \text{Pairwise feebly } T_0 \quad \text{Pairwise semi } T_0
\end{array}
\]

The above study raises a problem of the existence of a pairwise semi \( T_2 \) space which is not pairwise feebly \( T_0 \).
3.2. PAIRWISE FEEBLY $\alpha$-SPACES

It has been observed in the previous section that every pairwise feebly $T_1$ space is pairwise feebly $T_\alpha$ but the converse need not be true. The problem arises as to which restrictions should be imposed on the axiom of pairwise feebly $T_\alpha$ in order that it should imply pairwise feebly $T_1$. The present section aims in this direction and we investigate and study a pairwise axiom which is independent of both the above cited axioms but together with it the axiom of pairwise feebly $T_\alpha$ implies the axiom of pairwise feebly $T_1$.

**DEFINITION 3.2.1.** A bitopological space $(X, T_1, T_\alpha)$ is pairwise feebly $T_\alpha$, if for every $T_1$-feebly open set $G$, $x \in G$ implies that $T_\alpha$-cl$\{x\} \subseteq G$.

**REMARK 3.2.1.** Pairwise feebly $T_\alpha$ is independent of both pairwise feebly $T_1$ and pairwise $T_\alpha$, as shown by the following examples:
EXAMPLE 3.2.1: Let \( X = \{a, b, c\} \), 
\[ P_1 = \{ \emptyset, \{a\}, X \} \] and 
\[ P_2 = \text{the discrete topology} \]
on \( X \). Then \((X, P_1, P_2)\) is pairwise feebly \( T_2 \) and consequently pairwise feebly \( T_0 \) also but it is not pairwise feebly \( T_0 \).

EXAMPLE 3.2.2: Let \( X = \{a, b, c\} \) and 
\[ P_1 = P_2 = \{ \emptyset, \{a\}, \{b, c\}, X \} \]. Then \((X, P_1, P_2)\) is pairwise feebly \( T_0 \) but it is not pairwise feebly \( T_0 \) and so it is not pairwise feebly \( T_1 \).

THEOREM 3.2.1: Every pairwise feebly \( T_0 \) pairwise feebly \( T_0 \) space \((X, P_1, P_2)\) is pairwise feebly \( T_1 \).

PROOF: Let \( x, y \in X \) and \( x \neq y \). Since \( X \) is pairwise feebly \( T_0 \), there is a set which is either \( P_1 \)-feebly open or \( P_2 \)-feebly open containing one of the points but not the other. Let \( U \) be \( P_1 \)-feebly open and \( x \in U \) but \( y \not\in U \) since \( X \) is pairwise feebly \( T_0 \), \( P_2 \)-fcl \( \{x\} \subset U \). Then \( X \setminus P_2 \)-fcl \( \{x\} \) is a \( P_2 \)-feebly open set containing the point \( y \) but not \( x \). Consequently \( X \) is pairwise feebly \( T_1 \).
**Definition 3.2.2:** In a bitopological space $(X, P_1, P_2)$, for any $x \in X$, bi-$\text{cl}(x) = P_1$-$\text{cl}(x) \cap P_2$-$\text{cl}(x)$ and bi-$\text{ker}(x) = P_1$-$\text{ker}(x) \cap P_2$-$\text{ker}(x)$.

The following theorem gives several characterizations of pairwise feebly $P_0$-spaces.

**Theorem 3.2.3:** In a bitopological space $(X, P_1, P_2)$, the following conditions are equivalent:

(a) $(X, P_1, P_2)$ is pairwise feebly $P_0$.

(b) If $x \in X$, $P_1$-$\text{cl}(x) \subset P_2$-$\text{ker}(x)$.

(c) If $x, y \in X$, $y \in P_1$-$\text{ker}(x)$ if and only if $x \in P_2$-$\text{ker}(y)$.

(d) If $x, y \in X$, $y \in P_1$-$\text{cl}(x)$ if and only if $x \in P_2$-$\text{cl}(y)$.

(e) If $F$ is $P_1$-feebly closed and $x \notin F$ then there exists a $P_2$-feebly open set $G$ such that $x \notin G$ and $F \subseteq G$.

(f) If $F$ is $P_1$-feebly closed, then $F = \bigcap \{G | G \text{ is } P_2$-feebly open and $F \subseteq G \}$. 
(a) If $G$ is $P_\frac{1}{2}$-feebly open then $G = \bigcup\{F| F$ is $P_\frac{1}{2}$-feebly closed, $F \supset G\}.

(b) If $F$ is $P_\frac{1}{2}$-feebly closed and $x \notin F$ then $P_\frac{1}{2}$-fel{$x$} $\cap F = \emptyset$.

PROOF: 
(a) $\implies$ (b): For any $x \in X$, we have $P_\frac{1}{2}$-febr{$x$} = $\bigcap\{P_\frac{1}{2}$-feebly open set $G| x \in G\}$. By (a), each $P_\frac{1}{2}$-feebly open set $G$ containing $x$ contains $P_\frac{1}{2}$-fel{$x$}. Hence, $P_\frac{1}{2}$-fel{$x$} $\subset$ $P_\frac{1}{2}$-febr{$x$}.

(b) $\implies$ (a): For any $x, y \in X$, if $y \in P_\frac{1}{2}$-febr{$x$} then $x \in P_\frac{1}{2}$-fel{$y$}. Now by (b), $x \in P_\frac{1}{2}$-febr{$y$}. Similarly if $x \in P_\frac{1}{2}$-febr{$y$} then $y \in P_\frac{1}{2}$-febr{$x$}.

(a) $\implies$ (c): For any $x, y \in X$, if $y \in P_\frac{1}{2}$-fel{$x$} then $x \in P_\frac{1}{2}$-febr{$y$}. By (c), $x \in P_\frac{1}{2}$-fel{$y$}. Similarly if $x \in P_\frac{1}{2}$-fel{$y$} then $y \in P_\frac{1}{2}$-fel{$x$}.

(c) $\implies$ (d): Let $F$ be a $P_\frac{1}{2}$-feebly closed set and $x \notin F$. Then for any point $y \in F$ implies $P_\frac{1}{2}$-fel{$y$} $\supset F$ implies $x \notin P_\frac{1}{2}$-fel{$y$}. Now by (d), $x \notin P_\frac{1}{2}$-fel{$y$} implies $y \notin P_\frac{1}{2}$-fel{$x$}.
That is there exists a $P_j$-feebly open set $G_y$ such that $y \in G_y$ and $x \notin G_y$. Let $G = \bigcup_{y \in F} \{ G \mid G$ is $P_j$-feebly open, $y \in G_y$ and $x \notin G_y \}$. Then $G$ is $P_j$-feebly open such that $F \subseteq G$ and $x \notin G$.

$(a) \Rightarrow (f)$: Let $F$ be a $P_1$-feebly closed set and suppose that $H = \bigcap \{ P_j$-feebly open set $G \mid F \subseteq G \}$. Clearly, $F \subseteq H$. Let $x \notin F$ then by $(c)$ there exists a $P_j$-feebly open set $G$ such that $x \notin G$ and $F \subseteq G$. Hence, $x \notin H$. And so, $F = H$.

$(f) \Rightarrow (g)$: Evident.

$(g) \Rightarrow (h)$: Let $F$ be a $P_1$-feebly closed set and $x \notin F$. Then $X - F = G$ (say) is a $P_1$-feebly open set containing $x$. By $(g)$, there exists a $P_j$-feebly closed set $H$ such that $x \in H \subseteq G$. Therefore, $P_j$-$\text{fcl}(x) \subseteq G$. Hence, $P_j$-$\text{fcl} \{ x \} \cap F = \emptyset$.

$(h) \Rightarrow (a)$: Let $G$ be $P_1$-feebly open and $x \in G$. Then $x \notin X - G$ which is $P_1$-feebly closed. By $(h)$, $P_j$-$\text{fcl}(x) \cap (X - G) = \emptyset$. This implies that $P_j$-$\text{fcl} \{ x \} \subseteq G$. Thus $(a)$ holds.

**THEOREM 3.2.3:** If $(X, P_1, P_2)$ is pairwise feebly $P_1$ and $x, y \in X$, then either bi-$\text{fcl}(x)$ equals bi-$\text{fcl}(y)$ or they are disjoint.
PROOF: Suppose that, \((\text{bi-fcl}(x) \cap \text{bi-fcl}(y)) \neq \emptyset\). Let \(p \in P_1\)-\text{fcl}(x) \cap P_2\)-\text{fcl}(x) \cap P_1\)-\text{fcl}(y) \cap P_2\)-\text{fcl}(y).

Then, \(P_1\)-\text{fcl}(p) \subseteq P_1\)-\text{fcl}(x) \cap P_2\)-\text{fcl}(y) and \(P_2\)-\text{fcl}(p) \subseteq P_2\)-\text{fcl}(x) \cap P_2\)-\text{fcl}(y). Now by (d) in Theorem (3.2.2), we get

\[p \in P_1\)-\text{fcl}(x) \implies x \in P_2\)-\text{fcl}(p) \implies P_2\)-\text{fcl}(x) \subseteq P_2\)-\text{fcl}(y), and so, \(p \in P_1\)-\text{fcl}(x) \implies P_2\)-\text{fcl}(x) \subseteq P_2\)-\text{fcl}(y).

Similarly, \(p \in P_2\)-\text{fcl}(x) \implies P_1\)-\text{fcl}(x) \subseteq P_1\)-\text{fcl}(y).

\[p \in P_1\)-\text{fcl}(y) \implies P_2\)-\text{fcl}(y) \subseteq P_2\)-\text{fcl}(x) \text{ and}
\[p \in P_2\)-\text{fcl}(y) \implies P_1\)-\text{fcl}(y) \subseteq P_1\)-\text{fcl}(x) \text{ consequently,}
\[\text{bi-fcl}(x) = \text{bi-fcl}(y).

**Theorem 3.2.6:** If \((x, P_1, P_2)\) is pairwise feebly \(P_0\) and \(x, y \in X\) then either bi-\text{fker}(x) equals bi-\text{fker}(y) or they are disjoint.

In view of (c) in Theorem (3.2.2), this follows as Theorem (3.2.3).

**Theorem 3.2.8:** Every biopen subspace of a pairwise feebly \(P_0\)-space \((X, P_1, P_2)\) is pairwise feebly \(P_0\).
\textbf{Proof}: Let \((Y, T_1, T_2)\) be a biopen subspace of \((X, T_1, T_2)\). Let \(A\) be \(T_1\)-feebly closed and \(x \notin Y\), \(x \notin A\). Then \(Y \sim A\) is \(T_1\)-feebly open. So \(Y \sim A\) is \(T_2\)-feebly open by Theorem (2.2.4). Now \(X \sim (Y \sim A) = (X \sim Y) \cup A\), is \(T_2\)-feebly closed and \(x\) does not belong to it. Therefore by (e) in Theorem (3.2.2) there is a \(T_3\)-feebly open set \(G\) such that \(x \notin G\), and \((X \sim Y) \cup A \subseteq G\). Since \(Y\) is \(T_3\)-open and \(G\) is \(T_3\)-feebly open, it follows by Theorem (2.2.6) that \(Y \cap G\) is \(T_3\)-feebly open, and it contains \(A\) and \(x \notin Y \cap G\). Hence by (e) of Theorem (3.2.2), it results that \((Y, T_1, T_2)\) is pairwise feebly \(T_0\).

\section{2.8. Pairwise Feebly Regular Spaces}

The study of pairwise regular spaces has been initiated by Kelly [56]. In 1978 Maheshwari and Prasad [57] introduced the concept of pairwise \(s\)-regular spaces in such a way that every pairwise regular space is pairwise \(s\)-regular but the converse may be false. In order to achieve this they have utilized the semiopen sets introduced by Levine [44]. It has been observed in section (3.1) of this chapter that in general the axiom of pairwise feebly \(T_0\) does not imply the axiom of pairwise feebly \(T_2\). In this section we introduce
and study a pairwise axiom which lies strictly between pairwise regular and pairwise s-regular axioms. More over it is such that it is independent of pairwise T₀ and does not imply pairwise feebly T₂ also but with it pairwise T₀ implies pairwise feebly T₂. The axiom of pairwise s-regular [67] is as follows:

**DEFINITION 3.9.** A bitopological space $(X, T₁, T₂)$ is pairwise s-regular if for every $T₁$-closed set $F$ and a point $x \notin F$, there exist a $T_j$-semiopen set $U$ and a $T_i$-semiopen set $V$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$ where $i \neq j$, $i, j = 1, 2$.

We introduce now a new axiom as follows:

**DEFINITION 3.3.11** A bitopological space $(X, T₁, T₂)$ is termed pairwise feebly regular if for every $T₁$-closed set $F$ and a point $x \notin F$, there exist a $T_j$-feebly open set $U$ and a $T_i$-feebly open set $V$ such that $U \cap V = \emptyset$, $F \subseteq U$, $x \in V$ where $i \neq j$, $i, j = 1, 2$.

This extends definition (2.4.1) in its bitopological formulations.
REMARK 3.3.1: It is clear that every pairwise regular space is pairwise feebly regular. However, the converse may be false. For,

**EXAMPLE 3.3.1** Let \( X = \{a, b, c\} \),
\[
P = \emptyset, \{a\}, X,
\]
\[
Q = \emptyset, \{b\}, X.
\]

Then the space \((X, P, Q)\) is pairwise feebly regular but it is not pairwise regular.

REMARK 3.3.2: Every pairwise feebly regular space is pairwise \(\sigma\)-regular. But the converse need not be true. For,

**EXAMPLE 3.3.2** Let \( X = \{a, b, c\} \),
\[
P = \emptyset, \{b\}, \{c\}, \{b, c\}, X,
\]
\[
Q = \emptyset, \{a\}, \{b\}, \{a, b\}, X.
\]

Then the space \((X, P, Q)\) is pairwise \(\sigma\)-regular but it is not pairwise feebly regular.

Consider the following examples:
EXAMPLE 3.3.3: Let $X = \{a, b, c, d, e\}$,

$$P = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$$

and $Q = \{\emptyset, \{a, c\}, X\}$.

Then the space $(X, P, Q)$ is pairwise feebly regular but it is neither pairwise $T_0$ nor it is pairwise feebly $T_2$.

EXAMPLE 3.3.4: Let $X = \{a, b, c\}$,

$$P = \{\emptyset, \{a\}, \{b, c\}, X\}$$ and

$$Q = \{\emptyset, \{b\}, \{a, c\}, X\}.$$ 

Then the space $(X, P, Q)$ is pairwise $T_0$ but it is not pairwise feebly regular.

REMARK 3.3.3: The examples (3.3.3) and (3.3.4) show that the axiom of pairwise feebly regular need not imply the axiom of pairwise feebly $T_2$ and it is also independent of the axiom of pairwise $T_0$.

However, we have the following theorem:

THEOREM 3.3.1: Every pairwise $T_0$, pairwise feebly regular space $(X, P_1, P_2)$ is pairwise feebly $T_2$. 
PROOF: Let $x, y \in X$ and $x \not\approx y$. Since $(X, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise $T_0$, without any loss in the generality, suppose that $V$ is a $\mathcal{P}_1$-open set which contains $x$ but does not contain $y$. Then $x \not\approx V$ is $\mathcal{P}_1$-closed containing $y$ to which $x$ does not belong. Now $(X, \mathcal{P}_1, \mathcal{P}_2)$ being pairwise feebly regular there exist a $\mathcal{P}_j$-feebly open set $U$ and a $\mathcal{P}_i$-feebly open set $V$, such that $x \not\approx U$, $x \in V$ and $U \cap V = \emptyset$ where $i \neq j$, $i, j = 1, 2$. Consequently $(X, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise feebly $T_2$.

The following theorem gives several characterisations of pairwise feebly regular spaces:

THEOREM 5.5.2: In a bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ the following conditions are equivalent:

(a) $(X, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise feebly regular.

(b) For each $x \in X$ and each $\mathcal{P}_i$-open set $U$ containing $x$ there exists a $\mathcal{P}_i$-feebly open set $V$ such that $x \in V \subseteq \mathcal{P}_j$-fel($U$) $\subseteq U$, where $i, j = 1, 2$ such that $i \neq j$.

(c) Every $\mathcal{P}_i$-closed set $A$ is identical with the intersection of all the $\mathcal{P}_i$-feebly closed $\mathcal{P}_j$-feebly neighbourhoods of $A$ where $i \neq j$ and $i, j = 1, 2$. 
(d) For every set \( A \) and every \( P_2 \)-open set \( B \) such that \( A \cap B \neq \emptyset \), there exist a \( P_2 \)-feebly open set \( C \) for which \( A \cap C \neq \emptyset \) and \( P_j \text{-fecl}(C) \subseteq B \) where \( i \neq j, i,j=1,2 \).

(e) For every nonempty set \( A \) and \( T_1 \)-closed set \( B \) such that \( A \cap B = \emptyset \), there exist disjoint sets \( G \) and \( H \) such that \( G \) is \( P_2 \)-feebly open and \( H \) is \( P_2 \)-feebly open, \( A \cap G \neq \emptyset \) and \( B \subseteq H \), where \( i \neq j, i,j=1,2 \).

**Proof:** \( (a) \implies (b) \): Let \( U \) be a \( P_2 \)-open set such that \( x \in U \). Then \( x \sim U \) is \( P_2 \)-closed and \( x \notin x \sim U \). By (a), there exists a \( P_2 \)-feebly open set \( V \) and \( P_2 \)-feebly open set \( W \) such that \( x \in W \), \( x \sim U \subseteq V \) and \( W \cap V = \emptyset \). And so, \( x \in W \subseteq P_j \text{-fecl}(W) \subseteq x \sim V \subseteq U \).

\( (b) \implies (c) \): Let \( A \) be \( P_2 \)-closed and \( x \notin A \). Then \( x \sim A \) is \( P_2 \)-open and contains \( x \). By (b), there is a \( P_2 \)-feebly open set \( W \) such that \( x \in W \subseteq P_j \text{-fecl}(W) \subseteq x \sim A \). Therefore, \( A \subseteq x \sim P_j \text{-fecl}(W) \subseteq x \sim W \). And so, \( x \sim W \) is \( P_2 \)-feebly closed \( P_j \)-feebly neighbourhood of \( A \) to which \( x \) does not belong. Hence, (c) holds.
Let $A \cap B \neq \emptyset$ and $B$ be $P_2$-open.

Let $x \in A \cap B$. Since $x \notin X \setminus B$ which is $P_2$-closed, by (c), there exists a $P_2$-feebly closed $P_j$-feebly neighbourhood $V$ of $X \setminus B$ such that $x \notin V$. Let $U$ be a $P_j$-feebly open set such that $X \setminus B \subseteq U \subseteq V$. Then $O = X \setminus V$ is $P_j$-feebly open such that $A \cap O \neq \emptyset$, $P_j$-$\text{fcl}(O) \subseteq B$.

$(a) \implies (b)$: Let $A \cap B = \emptyset$, $A$ be nonempty and $B$ be $P_2$-closed. Then, $A \cap (X \setminus B) \neq \emptyset$ and $X \setminus B$ is $P_2$-open.

By (d), there exists a $P_2$-feebly open set $G$ such that $A \cap G \neq \emptyset$, $G \subseteq P_j$-$\text{fcl}(G) \subseteq X \setminus B$. Put $H = X \setminus P_j$-$\text{fcl}(G)$. Then $H$ is $P_j$-feebly open, $B \subseteq H$ and $G \cap H = \emptyset$.

$(e) \implies (a)$: Obvious.

**Remark 3.3.4:** The property of being pairwise feebly regular is not hereditary. For, $\{b, c\}$ as a subspace of the pairwise feebly regular space of example (3.3.1) is not pairwise feebly regular.

We have the following theorem:

**Theorem 3.3.5:** Every biopen subspace of a pairwise feebly regular space is pairwise feebly regular.
**NOTE:** Let \((Y, P_1^*, P_2^*)\) be a biopen subspace of a pairwise feebly regular space \((X, P_1, P_2)\). Let \(A\) be \(P_1^*\)-open and \(x \in A\). Since \(Y\) is \(P_1^*\)-open it follows that \(A\) is \(P_1^*\)-open. Now \((X, P_1, P_2)\) being pairwise feebly regular, there exists a \(P_1^*\)-feebly open set \(U\) such that \(x \in U \subseteq \bigcap_j P_j^*\text{-}\text{cl}(U) \subseteq A\). This implies that, \(x \in U \subseteq \bigcap_j P_j^*\text{-}\text{cl}(U) \subseteq A\).

By Theorem (2.2.8), this reduces to \(x \in U \subseteq \bigcap_j P_j^*\text{-}\text{cl}(U) \subseteq A\).

By Theorem (2.2.6) it follows that \(U\) is \(P_1^*\)-feebly open.

Hence by (b) of Theorem (3.3.3) it results that \((Y, P_1^*, P_2^*)\) is pairwise feebly regular.

**THEOREM 3.3.4:** Pairwise homeomorphic image of a pairwise feebly regular space is pairwise feebly regular.

**NOTE:** Let \((X, P_1, P_2)\) be a pairwise feebly regular space and \((X^*, P_1^*, P_2^*)\) be a pairwise homeomorphic image of \((X, P_1, P_2)\) under a pairwise homeomorphism \(f\). Let \(F\) be \(P_1^*\)-closed and \(q\) be any point of \(X^*\) such that \(q \not\in F\). Since \(f\) is one-one and onto there exists \(p \in X\) such that \(f(p) = q\) if and only if \(f^{-1}(q) = p\). Now \(f\) being a pairwise homeo-
morphism, \( f^{-1}(F) \) is \( P_1 \)-closed and evidently \( p \not\in f^{-1}(F) \).

Since \((X, P_1, P_2)\) is pairwise feebly regular there exist a \( P_1 \)-feebly open set \( G \) and a \( P_j \)-feebly open set \( H \) such that \( p \in G, f^{-1}(F) \subseteq H \) and \( G \cap H = \emptyset \). These imply that \( q = f(p) \in f(G) \), and \( f \) being onto \( F = f(f^{-1}(F)) \subseteq f(H) \) and \( f \) being one-one, \( f(G) \cap f(H) = f(G \cap H) = f(\emptyset) = \emptyset \). By Theorem (2.2.11) it results that \( f(G) \) is \( P_2 \)-feebly open and \( f(H) \) is \( P_j \)-feebly open. Consequently, \((X, P_1^*, P_2^*)\) is pairwise feebly regular.