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Dear Mr. Pandey,

Your paper entitled "Multipliers for weighted spaces on a locally compact abelian group" has been accepted for publication in "Vikram Mathematical Journal", Vol. VIII, 1988. You will receive 25 reprints free of charge. If you need more reprints, please inform the undersigned immediately.

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MULTIPLIERS FOR THE CONVERGENCE OF JACOBI SERIES IN BANACH SPACES

By

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1. The Jacobi polynomials $P_n^{(a, \beta)}(x)$; $n = 0, 1, 2, \ldots; a, \beta > -1$, are defined by the Rodrigues formula

$$
(1 - x)^{a} (1 + x)^{\beta} P_n^{(a, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^n (1+x)^{n+\beta},
$$

$$
a, \beta > -1.
$$

The nth ultraspherical polynomial $P_n^{(\lambda)}(x)$ is the particular case of the Jacobi polynomial for $a = \beta = \lambda - \frac{1}{2}$. Also, the ultraspherical polynomial reduces to the well known Legendre polynomial $P_n(x)$ for $\lambda = \frac{1}{2}$.

We suppose that $X$ is the Banach space of all continuous functions on $[0, \pi]$ with respect to the norm

$$
||f||_0 = \sup_{0 \leq \theta \leq \pi} |f(\theta)| < \infty.
$$

Let $[\lambda]$ be the Banach algebra of all bounded linear operators of $X$ into itself.

We write

$$
R_k(\theta) \equiv R_k^{(a, \beta)}(\cos \theta) = P_n^{(a, \beta)}(\cos \theta) / P_k^{(a, \beta)}(1).
$$

We now define mutually orthogonal projections $\{B_k\}$ by

$$
B_k f(\theta) = \left( \int_0^\pi f(\theta) R_k(\theta) \, du(\theta) \right) h_k R_k(\theta),
$$

where $du(\theta) = (\sin \theta/2)^{2a+1} (\cos \theta/2)^{2\beta+1}; a, \beta > -\frac{1}{2}$

and $h_k \equiv h_k^{(a, \beta)} = \left( \int_0^\pi (R_k(\theta))^2 \, du(\theta) \right)^{-1}$. 


\[
\frac{(2k + \alpha + \beta + 1) \Gamma(k + \alpha + \beta + 1) \Gamma(k + \alpha + 1)}{\Gamma(k + \beta + 1) \Gamma(k + 1) \Gamma(\alpha + 1) \Gamma(\alpha + 1)}.
\]

It is well known that the sequence \( \{B_k\} \) is total and fundamental in \( X \).

The formal expansion of a function in the form of Fourier-Jacobi series is given by

\[
f \sim \sum_{k=0}^{\infty} B_k f.
\]

(1.3)

\[
= \sum_{k=0}^{\infty} \frac{n^k}{h_k} R(n) d\theta.
\]

where \( n = \int_{0}^{\pi} f(\theta) R_k(\theta) \, d\theta \).

We now suppose that \( S \) is the set of all sequences of scalars. A sequence

\[
\eta = (\eta_k)_{k=0}^{\infty} \in S
\]

is called a multiplier sequence for \( X \) with respect to \( \{B_k\} \), if \( \forall f \in X \exists \) an element \( \eta f \in X \) such that

\[
\eta_k B_k f = B_k f \eta ; \ k = 0, 1, 2, ...
\]

Form this definition it follows that

\[
f \eta \sim \sum_{k=0}^{\infty} \eta_k B_k f.
\]

On account of totality of the sequence \( \{B_k\} \), the element \( f \eta \) is uniquely determined for every \( f \in X \).

We denote by \( M = M(\{x; B_k\}) \) the set of all multipliers for \( X \) corresponding to \( \{B_k\} \). Trebels [7, p. 10] has shown that the set \( M \) is a Commutative Banach algebra with respect to vector addition, coordinatewise multiplication and the norm.
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\[ \| \eta \|_m = \sup \{ \| f \eta \| : f \in X, \| f \| < 1 \}. \]

It is known that the identity sequence \( \{1\} \in M \).

Next, let \( T \) be an operator form \( X \) into itself. We say that \( T \) is a multiplier operator provided there exists a sequence \( \tau \in S \) such that

\[ B_k T f = \tau_k B_k f \]

\[ \forall f \in X; k = 0, 1, 2, \ldots \]

Hence we see that corresponding to any multiplier operator \( T \) we have the expansion

\[ T f \sim \sum_{k=0}^{\infty} \tau_k P_k f. \]

From the above discussion it is clear that with respect to each multiplier operator \( T \exists \) a multiplier sequence \( \tau \in M \) and vice versa.

2. Let \( L_1 [0, \pi] \) be the space of all measurable functions such that

\[ \| f \|_1 = \int_{0}^{\pi} | f(\theta) | \, du(\theta) < \infty \]

On the lines of Askey and Wainger [1], we now define the convolution of any two functions \( f_1, f_2 \in L_1 \) in the following way:

\[ (f_1 \ast f_2)(\theta) = \int_{0}^{\pi} f_1(\theta) (T_\phi f_2)(\theta) \, du(\phi). \]

\[ = \int_{0}^{\pi} \int_{0}^{\pi} f_1(\phi) f_2(\psi) K(\theta, \phi, \psi) \, du(\phi) \, du(\psi), \]

where \( T_\phi f(\theta) = \int_{0}^{\pi} f(\psi) K(\theta, \phi, \psi) \, du(\psi) \),

\( K(\theta, \phi, \psi) \) is a non-negative symmetric function such that
\[ R_n(\theta) R_n(\phi) = \int_0^\pi R_n(\psi) K(\theta, \phi, \psi) \, du(\psi), \]

and \[ \int_0^\pi K(\theta, \phi, \psi) \, du(\psi) = 1. \]

We write

\[ \omega(\phi, f, x) = \sup_{0 \leq \psi \leq \phi} ||T_\phi f(\theta) - f(\theta)||_X \]

and \( V_p = \{ \eta \in S : \sum_{k=0}^{\infty} (k+1)^p |\Delta^k \eta| < \infty \}. \)

If \( \omega(\phi, f, x) \leq C \phi^\gamma, \)

where \( 0 < \gamma \leq 1 \) and \( C \) is any positive constant then we say that \( f \) belongs to the Lipschitz class of order \( \gamma \), i.e., \( f \in \text{Lip } \gamma \equiv X_\gamma \), say. It is well known that the space \( X_\gamma \) is a Banach space with respect to the norm

\[ ||f||_{\text{Lip } \gamma} = ||f||_X + \sup_{n \in \mathbb{Z}^+} (n^\gamma \omega(n^{-1} f, X)), \]

where \( \mathbb{Z}^+ \) is the set of all positive integers.

We write

\[ Q(\psi) = T_\psi f(\theta) - f(\theta). \]

The multiplier problems for Fourier-Jacobi expansions in Banach spaces have been studied in detail by Butzer, Nessel and Trebels [3], Connett and Schwartz [4], and Gasper and Trebels [5]. It may be mentioned here that all the results in this line depend on a well known theorem of Szego [6] on the (C, \( \delta \)) summability of Jacobi series for \( \delta > a + \frac{1}{2}; a \geq - \frac{1}{2}. \) In the present paper we intend to prove two theorems on the multipliers for the convergence of Fourier-Jacobi series in Banach spaces. Precisely, we prove the following theorems:

**Theorem 1:** If \( |a| < \frac{1}{2} \) and \( \beta \geq a \), then every \( \eta \in S \) such that
\[ \sum_{k=0}^{\infty} (k+1)^{a+\frac{1}{2}} | \triangle \eta | < \infty. \]

is a multiplier sequence for the Banach space C.

**Theorem 2**: If \( |a| < \frac{1}{2} \) and \( \beta \geq a \), then every \( \eta \in S \) such that

\[ \sum_{k=0}^{\infty} (k+1)^{-\delta} | \triangle \eta_k | < \infty, \quad 0 \leq \delta \leq \frac{1}{2} - a, \]

is a multiplier sequence for the convergence of the series (1.3) provided

\[ |Q(\psi)| = O(\psi^{a+\delta+\frac{1}{2}}) \text{ as } \psi \to 0. \]

3. The Proof of theorem 1 depends on the following:

**Lemma.**

If \( f \in C \), then

\[ \|S_n f(\theta) - f\|_0 = O(n^{a+\frac{1}{2}}). \]

**Proof of the lemma.** The nth partial sum of the series (1.3) is given by

\[ S_n f(\theta) = \sum_{\nu=0}^{n} \hat{f}(\nu) h_\nu R_\nu C(\cos \theta). \]

\[ = \sum_{\nu=0}^{n} h_\nu \int_{0}^{\pi} f(\phi) R_\nu C(\cos \phi) du(\phi) R_\nu (\cos \theta). \]

Now, using the orthogonal property, we have

\[ S_n f(\theta) - f(\theta) = \sum_{\nu=0}^{n} \int_{0}^{\pi} [f(\phi) - f(\theta)] h_\nu R_\nu (\cos \phi) R_\nu (\cos \theta). \]
\[
= \sum_{\nu=0}^{n} h_{\nu} \int_{0}^{\pi} [f(\phi) - f(\theta)] \, d\nu(\phi).
\]
\[
= \int_{0}^{\pi} K(\theta, \phi, \psi) R_{\nu}(\cos \psi) \, d\nu(\psi).
\]
\[
= \sum_{\nu=0}^{n} h_{\nu} R_{\nu}(\cos \psi) \, d\mu(\psi)
\]
\[
= L_{n}(a, \beta) \int_{0}^{\pi} P_{n}(\alpha + 1, \beta)(\cos \psi) \, d\nu(\psi) \int_{0}^{\pi} [f(\phi) - f(\theta)] K(\theta, \phi, \psi) \, d\mu(\psi)
\]
\[
= L_{n}(a, \beta) \int_{0}^{\pi} [T_{\theta} f(\theta) - f(\theta)] P_{n}(\alpha + 1, \beta)(\cos \psi) \, d\mu(\psi)
\]
\[
= L_{n}(a, \beta) \int_{0}^{\pi} Q(\phi) P_{n}(\alpha + 1, \beta)(\cos \psi) \, d\mu(\psi),
\]
where
\[
L_{n}(a, \beta) = \frac{\Gamma(n + a + \beta + 2)}{\Gamma(a + 1) \Gamma(a + \beta + 1)} \sim n^{a + 1}
\]
and
\[
Q(\phi) = T_{\theta} f(\theta) = f(\theta).
\]
Hence, we obtain
\[
||S_{n} f(\theta) - f(\theta)||_{0} = \left\| \int_{0}^{\pi} Q(\psi) L_{n}(\psi) - (\sin \psi/\lambda) \right\|_{c}
\]
\[
\leq C \left[ \left\| \int_{0}^{\pi} ||_{c} + \left\| \int_{\lambda_{n}}^{\pi} \right\|_{c} \right] + \left\| \int_{0}^{\pi} ||_{c} \right\|_{c}.
\]
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(3.2) \[ = I_1 + I_2 + I_3, \text{ Say.} \]

where \[ \lambda_n = n^{-(a+\beta)/2} (2a+2)^{-1} \]

Now using the order estimates for Jacobi polynomials [see [6], p. 167], we have

\[ I_1 = \| \int_0^\lambda_n Q(\psi) \psi^{2a+1} P_n(\alpha+1) L_n(\alpha, \beta) \, d\psi \|_C \]

\[ = O \left( n^{\alpha+1} \cdot n^{-2a-1} \right) \cdot O \left( n^{\alpha+1} \right) \| \int_0^{\lambda_n} Q(\psi) \, d\psi \|_C \]

\[ = O \left( n^{2a+2} \right) \cdot \lambda_n^{2a+1} \| \int_0^{\lambda_n} Q(\psi) \, d\psi \|_C \]

\[ = O \left( n^{2a+2} \right) \cdot \lambda_n^{-2a-2} \]

\[ = O \left( n^{\alpha+\frac{1}{4}} \right) \]

We now consider \( I_2 \).

\[ I_2 = \| \int_{\frac{\pi - \gamma}{n}}^{\gamma} Q(\psi) (\sin \psi/\alpha)^{2a+1} (\cos \psi/\alpha)^{2\beta+1} \, d\psi \cdot P_n(\alpha+1, \beta) (\cos \psi) L_n(\alpha, \beta) \, d\psi \|_C \]

Since \[ P_n(\alpha, \beta) (\cos \theta) = n^{-\frac{\beta}{2}} k(\theta) \{ \cos (N \theta + \gamma) + o(n \sin \theta)^{-1} \}, \]

where \[ k(\theta) = \pi^{-\frac{1}{2}} (\sin \theta/\alpha)^{-\frac{1}{2}} (\cos \theta/\alpha)^{-\beta-\frac{1}{2}} \]

and \[ N = n + \frac{a+\beta+1}{2}; \gamma = -(\alpha+\frac{1}{2}) \pi/n. \]
we have

\[ I_2 = \int_{\lambda_n} (\sin \frac{\theta}{2})^{\alpha - \frac{\beta}{2}} (\cos \frac{\theta}{2})^{-\frac{\beta}{2}} \cos \left\{ (n + \frac{\alpha + \beta}{2} + 1) \theta \right\} \]

\[ \int_{\lambda_n} Q(\psi) (\sin \psi_2)^{2\alpha + 1} n^{-\frac{1}{2}} \pi^{-\frac{\alpha}{n}} \]

\[ - (a + \frac{1}{2}) \pi \int_{0}^{\pi} Q(\psi) (\sin \psi_2)^{2\alpha + 1} n^{-\frac{1}{2}} \pi^{-\frac{\alpha}{n}} \]

Finally, we consider \( I_3 \).

\[ I_3 = \int_{\pi}^{\pi} Q(\psi) L_n(\alpha, \beta)(\psi) P_n(\alpha + 1, \beta)(\cos \psi) (\sin \psi_2)^{2\alpha + 1} \]

Now putting \( \psi = \pi - \theta \), we have
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\[ O(n^{a+1+\beta}) \int_0^{s/n} |Q(\pi - \zeta)| \cdot \zeta^{2\beta+1} d\zeta. \]

\[ = O(n^{a+1+\beta})(n^{-2\beta-1}) \sup |Q(\pi - \zeta)| \cdot n^{-1} \]

\[ = O(n^{a-\beta-1}). \]

\[ = O(n^{a+\frac{1}{2}}). \]

4. Proof of Theorem 1

Let \( f \) be any function of the Banach space \( C \).

Then we have

\[ f = \sum_{k=0}^{\infty} \eta_k P_k f. \]

\[ = \sum_{k=0}^{\infty} \Delta \eta_k \bar{S}_k f + \eta_\infty f. \]

where

\[ \Delta \eta_k = \eta_k - \eta_{k+1}; k \in \mathbb{P}. \]

and \( \eta_\infty = \lim_{n \to \infty} \eta_k. \)

Now using lemma 1, we have

\[ \|f^\alpha\| \leq \sum_{k=0}^{\infty} \Delta \eta_k \|S_k f\| + \|\eta_\infty\| \cdot \|f\|. \]

\[ \leq B \|n^{a+\frac{1}{2}}\| \cdot |\eta_k| < \infty. \]

(4.1) \[ \leq B \|\eta\| v_p \cdot \|f\|. \]

Thus it remains to show that
\[ f^\alpha \sim \sum \eta_k B_k f. \]

But since \( B_k \in \mathcal{I}_x \), it follows

\[ B_n S_k f \begin{cases} \in \mathcal{I}_f & \text{if } k \geq n \\ \in \mathcal{I}_o & \text{if } k < n \end{cases} \]

Hence we obtain

\[ B_n f^\alpha = \sum_{k=0}^{\infty} \Delta \eta_k B_n f + \eta_\infty B_n f. \]

\[ = \eta_n B_n f. \]

This completes the proof of theorem 1.

5. We require following lemma for the proof of theorem 2.

Lemma. If \( f \in \mathcal{X}_{\alpha + \frac{1}{2} + \delta} \), then

\[ \| S_n f (\theta) - f (\theta) \| \mathcal{X}_{\gamma} = O (n^{-\delta}), \]

where \( \gamma = \alpha + \frac{1}{2} + \delta; \quad 0 \leq \delta < \alpha + \frac{1}{2} \).

Proof of lemma 2. We have

\[ \| S_n f (\theta) - f (\theta) \| \mathcal{X}_{\gamma} = \| \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+1) \Gamma(n+\beta+1)} \int_0^\pi (\sin \psi \alpha)^{2\alpha+1} \]

\[ \cdot (\cos \psi \alpha)^{2\beta+1} Q (\psi) P_n (\alpha+1, \beta) (\cos \psi) d\psi \| \mathcal{X}_{\gamma}. \]

\[ \leq \int_0^{\lambda_n} \int_{\mathcal{X}_{\gamma}} + \int_{\lambda_n}^{\pi - \sigma/n} \int_{\mathcal{X}_{\gamma}} + \int_{\pi - \sigma/n}^\pi \int_{\mathcal{X}_{\gamma}}. \]

\[ = E_1 + E_2 + E_3, \text{ say} \]

where

\[ \lambda_n = n^{-(2\alpha+2+\delta)} (3\alpha + \frac{3}{2} + \delta)^{-1}. \]
We consider $E_1$ first

$$E_1 = O \left( n^{a+1} \right) \int_0^{\lambda_n} Q(\psi) (\psi^{2a+1} \cdot O(n^{a+1})).$$

$$= O(\lambda_n) \int_0^{\lambda_n} \psi^{a+\frac{3}{2}} \psi^\delta d\psi.\psi^{2a+1}$$

$$= O(\lambda_n) \int_0^{\lambda_n} (\psi^{2a+1} \cdot \psi^\delta) \psi^{a+\frac{3}{2}} d\psi.$$
\[ = \mathcal{O}(\pi) \left( \frac{1}{n^\alpha} \right) \int_0^e \psi(\pi - \theta) | \mathcal{O}(n^\beta) \theta^{2\beta + 1} \psi d\theta \]

\[ = \mathcal{O}(n^{\alpha - 1}) \cdot \mathcal{O}(n^\beta) \cdot n^{-2\beta - 1} \sup | \psi(\pi - \theta) | \]

\[ = \mathcal{O}(n^{\beta + \alpha - 1}) \cdot n^{-2\beta - 1} - \mathcal{O}(1) n^{-1}. \]

\[ = \mathcal{O}(n^{-\delta}). \]

Finally we discuss \( E_2 \)

\[ E_2 = \left\| \frac{\Gamma(n + \alpha + \beta + 1)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)} \int_{\lambda_n}^{\pi - \pi/n} (\sin \psi/2)^{2\alpha + 1} (\cos \psi/2)^{2\beta + 1} Q(\psi) \pi^{-1/n} n^{-1/2} \right\| \]

\[ \leq \left\| \frac{\Gamma(n + \alpha + \beta + 1)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)} \cdot n^{-\frac{1}{2}} \pi^{-\frac{1}{n}} \int_{\lambda_n}^{\pi - \pi/n} (\sin \psi/2)^{\alpha - \frac{1}{2}} (\cos \psi/2)^{\beta + \frac{1}{2}} \right\| \]

\[ Q(\psi), \cos \left\{ n + \frac{\alpha + \beta}{2} + 1 \right\} \psi - (\alpha + \frac{3}{4}) \pi/2 \right\} d\psi \|_{x, \gamma} \]

\[ + \mathcal{O}(n^{\alpha - \frac{1}{2}}) \int_{\lambda_n}^{\pi - \pi/n} (\sin \psi/2)^{\alpha - \frac{1}{2}} (\cos \psi/2)^{\beta + \frac{1}{2}} \| \cdot Q(\psi) \| d\psi. \]

\[ = \left\| \left\{ \Gamma(n + \alpha + 1) \right\}^{-1} n^{\alpha + \frac{1}{2}} R \int_{\lambda_n}^{\pi - \pi/n} (\sin \psi/2)^{\alpha - \frac{1}{2}} (\cos \psi/2)^{\beta + \frac{1}{2}} \right\| \]

\[ Q(\psi), e^{i n + \alpha + \beta/2} (\pi - 1) \psi, e^{-i (\alpha + \frac{3}{4}) \pi} d\psi \|_{x, \gamma} \]
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\[ \pi^{-1/n} \int_{\lambda_n} \psi^{-a + \frac{1}{2}} \psi^{a + \frac{1}{2}} d\psi + O(n^{-\delta}). \]

(5.1) \[ = \| \Gamma (1 + \frac{1}{2}) \frac{\pi^2}{4} \| \frac{1}{n^{a + \frac{1}{2}}} R \{ e^{-i(\alpha + \beta)\pi/2} \}
\]

\[ \pi^{-1/n} \int_{\lambda_n} (\sin \psi/2)^{a - \frac{1}{2}} (\cos \psi/2)^{\beta + \frac{1}{2}} Q(\psi) e^{i(n + a + \beta + 1)\psi} d\psi \]

+ \( O(n^{-\delta}) \); for \(-\frac{1}{2} < \alpha < \frac{1}{2} \).

The integral in (5.1) may be rewritten in the form

\[ \frac{\pi^{-1/n}}{2} \int_{\lambda_n} (\sin \psi/2)^{a - \frac{1}{2}} (\cos \psi/2)^{\beta + \frac{1}{2}} Q(\psi) e^{i(n + a + \beta + 1)\psi} d\psi \]

\[ = \frac{\pi^{-1/n}}{2} \int_{\lambda_n} (\psi + \mu_n) \left( \frac{\psi + \mu_n}{2} \right)^{a - \frac{1}{2}} \left( \frac{\psi + \mu_n}{2} \right)^{\beta + \frac{1}{2}} Q(\psi + \mu_n) e^{i(n + a + \beta + 1)\psi} d\psi. \]

(5.2) \[ \leq \frac{1}{2} (L_1 + L_2 + L_3 + L_4), \]

where

\[ L_1 = \int_{\lambda_n - \mu_n} (\sin \frac{\psi + \mu_n}{2})^{a - \frac{1}{2}} (\cos \frac{\psi + \mu_n}{2})^{\beta + \frac{1}{2}} Q(\psi + \mu_n) d\psi. \]

\[ L_2 = \int_{n^{-1/n} - \mu_n} (\sin \frac{\psi + \mu_n}{2})^{\beta + \frac{1}{2}} Q(\psi) d\psi. \]

\[ L_3 = \int_{\lambda_n - \mu_n} |Q(\psi + \mu_n) - Q(\psi)| (\sin \frac{\psi + \mu_n}{2})^{a - \frac{1}{2}} (\cos \frac{\psi + \mu_n}{2})^{\beta + \frac{1}{2}} d\psi. \]
and
\[ L_4 = \sum_{\lambda_n} \left( \sin \frac{\psi_1 + \mu_n}{2} \right)^{\alpha - \frac{1}{2}} \left( \cos \frac{\psi_1 + \mu_n}{2} \right)^{\beta + \frac{1}{2}} \left. Q(\psi) \middle| d\psi \right. \]

Form the hypothesis of our theorem, we have
\[ L_1 = O \left\{ \int_{\lambda_n - \mu_n} \left( \psi + \mu_n \right)^{2\alpha + \delta} d\psi \right\} \]
\[ = O(n^{-\delta}). \]

\[ L_2 = O(n^{-\delta}). \]

\[ L_3 = \Theta (\mu_n^{\alpha + \frac{1}{2} + \delta}) \int_{\lambda_n} \left( \psi + \mu_n \right)^{\alpha - \frac{1}{2}} d\psi. \]
\[ = O(n^{-\delta}). \]

and
\[ L_4 = O\left[ \sum_{\lambda_n} \left( \sin \frac{\psi_1}{2} \right)^{\alpha - \frac{3}{2}} \left( \cos \frac{\psi_1}{2} \right)^{\beta + \frac{1}{2}} \left( \psi + \mu_n \right)^{\alpha + \frac{1}{2} + \delta} \right] \]
\[ + O\left[ \mu_n \int_{\lambda_n} \left( \sin \frac{\psi}{2} \right)^{\alpha - \delta} \left( \cos \frac{\psi}{2} \right)^{\beta - \frac{1}{2}} d\psi \right] \]
\[ = O(n^{\alpha - \frac{1}{2}}). \]
6: Proof of Theorem 2.

we have

\[ f^n = \sum_{k=0}^{\infty} \eta_k P_k f. \]

\[ = \sum_{k=0}^{\infty} \triangle \eta_k S_k f + \eta_\infty f, \]

where \( \triangle \eta_k = \eta_k - \eta_{k+1}; k \in \mathbb{P} \).

and \( \eta_\infty = \lim_{n \to \infty} \eta_n. \)

Now using lemma 2, we have

\[ \| f^a \| \leq \sum_{k=0}^{\infty} \| \triangle \eta_k \| \| S_k f \| + \| \eta_\infty \| \cdot \| f \|. \]

\[ \leq O (k+1)^{-\delta} \cdot \| \triangle \eta_\infty \| < \infty. \]

\[ \leq B \| \eta \|_{v_p} \cdot \| f \|. \]

\[ = O (1). \]

This completes the proof of Theorem 2.

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REFERENCES


