CHAPTER V

MULTIPLIERS FOR SEGAL ALGEBRAS ON A COMPACT

ABELIAN GROUP

5.1 Let $G$ be a compact abelian group and $\Gamma$ be its dual. Following Kree*, we define a real-valued continuous even function on satisfying the condition

$$\omega(Y_1 + Y_2) \leq \omega(Y_1) \omega(Y_2)$$

for all $Y_1, Y_2 \in \Gamma$.

We call any $\omega(Y)$ as a weight function on $\Gamma$. We define the Banach space $L^p_{\omega^\infty}(\Gamma)$ on $\Gamma$, $1 \leq p < \infty$, $0 \leq \infty \leq p$, to be the set of all functions $\hat{f}$ such that

$$\|\hat{f}\|_p = \left( \int_{\Gamma} |\hat{f}(Y)|^p \omega^\infty(x) \right)^{1/p} < \infty$$

for all $Y \in \Gamma$.

* P. Kree (36).
We now introduce $A^p_{\omega^\infty}(G)$ to be the set of all functions $f \in L^1(G)$ such that $f \in L^p(\Gamma)$. We define a norm on $A^p_{\omega^\infty}(G)$ by

$$\|f\|_{A^p_{\omega^\infty}} = \|f\|_1 + \|f\|_{L^p(\omega^\infty)}.$$ 

On the lines of Reiter* it can be easily seen that $A^p_{\omega^\infty}(G)$ is a Segal algebra.

Keshava Murthy and Unni** have studied in detail the multiplier problems on the algebras $A^p_{\omega}(G)$ and $S^p(G)$, where $G$ is a non-compact locally compact abelian group. They have proved the following two theorems:

Theorem A: Let $G$ be a non discrete noncompact locally compact abelian group and $1 \leq p < \infty$. If $T \in M(A^p_{\omega^\infty}(G))$, then there exists a unique measure $\mu \in M(G)$ such that

* H. Reiter (49)

** G.N. Keshava Murthy & K.R. Unni. (32)
\[ Tf = \mathcal{M} \ast f \]

for all \( f \in \mathcal{A}_\infty^p(G) \). Further \( M(\mathcal{A}_\infty^p(G)) \) is isometrically isomorphic to \( M(G) \).

Theorem B. Let \( G \) be a non-discrete noncompact locally compact abelian group and \( 1 \leq p < \infty \). If \( T \in M(\mathcal{S}_\infty^p(G)) \), then there exists a unique measure \( \mathcal{M} \in M(G) \) such that

\[ Tf = \mathcal{M} \ast f \]

for all \( f \in \mathcal{S}_\infty^p(G) \). Further \( M(\mathcal{S}_\infty^p(G)) \) is isometrically isomorphic to \( M(G) \).

In case when \( G \) is a compact abelian group, K. Nagarjan* has studied in detail the multiplier problems on the spaces \( \mathcal{A}_\infty^p(G) \), \( 1 \leq p < \infty \), which correspond to those obtained by Larsen** for the algebras \( \mathcal{A}_\infty^p(G) \) as the dual of Banach spaces.

Throughout this chapter we suppose that \( G \) is a compact abelian group. We propose to study the characterizations of the multiplier spaces \( M(\mathcal{A}_\infty^p(G)) \), which generalize the theorems given by Larsen (loc. cit.)

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* K. Nagarjan (44), Chapter II.
** R. Larsen (38), Chapter VI.
and Nagarajan (loc. cit.) for the Algebras $A^p_c(G)$ and $A^p_{\omega_c}(G)$ respectively, as the dual of certain Banach spaces.

Precisely, we shall prove the following

**Theorem I.** If $1 \leq p < 2$, then there exists a continuous algebra isomorphism of $M(A^p_{\omega_c}(G))$ onto the dual space of a Banach space of continuous functions.

If may be mentioned here that Theorems I and II reduces to the known results of Larsen* for $\lambda = 0$. Our theorems also include the corresponding theorems of Nagarajan** for $\lambda = p$. The proof of the theorems follows on the lines of Nagarajan (loc. cit.). Since we have introduced a new parameter $\lambda$, it is necessary to give the proofs in detail.

5.2 In the proof of our theorems we shall use the following lemmas:

**Lemma.** Let $A$ be a semi-simple commutative Banach algebra and $T$ a multiplier on $A$. Then there exists a unique bounded continuous function $\phi$ on the maximal ideal space $\Delta(A)$ of $A$ such that:

* Larsen (38), pp. 207.
** K. Nagarajan (44), Chapter II.
i) \((T\hat{x})^\wedge = \phi \hat{x}\) for all \(x \in A\).

ii) \(\|\phi\|_\infty \leq \|TW\|\),

where \(\hat{x}\) denotes the Gelfand transform of \(x\). For the proof of the lemma see Larsen*, p. 19.

5.3 Proof of Theorem I. Since \(M(A^P_{\omega^m}(G))\) is a Segal algebra, it must be a semi simple continuous Banach algebra** with maximal ideal space \(\Gamma\). Hence, on account of the above lemma, if \(T \in M(A^P_{\omega^m}(G))\), then there exists \(T \in L^\infty(\Gamma)\) such that

\[ \hat{Tf} = \hat{T} \hat{f} \quad \text{for all} \quad f \in A^P_{\omega^m}(G) \]

\[ \|\hat{T}\|_\infty \leq \|T\|. \]

We observe that the map

\[ \Phi : T \mapsto \hat{T} \]

From the space \(M(A^P_{\omega^m}(G))\) into \(L^\infty(\Gamma)\) is well defined, linear and continuous.

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* R. Larsen (38), pp. 19.
** K. Nagarjan (44), Theorem 2.3.
If we suppose that

\[ \hat{T_1} = \hat{T_2} \]

for any two multipliers \( T_1 \) and \( T_2 \) on \( \mathbb{A}_\omega^p(G) \), then we have

\[ \hat{T_1}^f = \hat{T_2}^f \quad \forall f \in \mathbb{A}_\omega^p(G), \]

\[ \Rightarrow \quad T_1 f = T_2 f, \]

\[ \Rightarrow \quad T_1 = T_2, \]

which ensures that \( \beta \) is one to one mapping from \( M(\mathbb{A}_\omega^p(G)) \) into the space \( L^\infty(\Gamma) \).

Now we shall show that \( \beta \) is an onto map.

Let \( \Phi \) be an element of \( L^\infty(\Gamma) \). Then for any \( f \in \mathbb{A}_\omega^p(G) \), we have

\[ \Phi f \in \mathbb{L}_\omega^p(\Gamma). \]

Since \( L^p(\Gamma) \subset L^p(\Gamma) \), it follows that
\[ \phi \hat{f} \in L^p_{\omega^n}(\Gamma). \]

Hence it is clear that \( \phi \hat{f} \in L^p(\Gamma) \cap L^\infty(\Gamma). \)

\[ \Rightarrow \phi \hat{f} \in L^2(\Gamma) \quad \text{for } 1 \leq p \leq 2. \]

\[ \Rightarrow \exists g \in L^2(G) \quad \text{such that} \]

\[ \hat{g} = \phi \hat{f}. \]

\[ \Rightarrow \hat{g} \in L^p_{\omega^n}(\Gamma). \]

\[ \Rightarrow \hat{g} \in A^p_{\omega^n}(G). \]

Thus we have

\[ \| g \|_{A^p_{\omega^n}} = \| g \|_1 + \| \hat{g} \|_{L^p_{\omega^n}} \]

\[ \leq \| g \|_p + \| \hat{g} \|_{L^p_{\omega^n}} \]

\[ \leq 2 \| \hat{g} \|_{L^p_{\omega^n}}. \]
\[ = 2 \| \phi \|_\infty \| f \|_\omega^n \]
\[ \leq 2 \| \phi \|_\infty \| f \|_\omega^n \]
\[ \leq 2 \| \phi \|_\infty \| f \|_\omega^n \]

We now consider the map

\[ T : A^p (G) \longrightarrow A^p_{\omega^n} (G) \]

such that

\[ \hat{T} f = g. \]

We observe that \( T \) is a bounded linear translations invariant map. Also since \( T \) is \( M (A^p_{\omega^n} (G)) \) and \( \hat{T} = \phi \),

the map \( \beta \) is onto.

Hence the Theorem \( I \) holds.

5.4 On the lines of Larsen* and Nagarjan**

we give an example to demonstrate that there exists a

multiplier in \( M (A^p_{\omega^n} (G)) \) for \( p > 2 \), which has no corres-

pondance with any measure in \( M (G) \).

\* R. Larsen (38)
\** K. Nagarjan (44).
Example: We suppose that \( EC \gamma \) is an infinite sidon set.

We write

\[ m = \frac{p}{2}, \quad n = \frac{m}{m-1} \]

and choose \( s \) such that \( 0 < s < 2 \) and \( 8m < 2 \).

Let \( \phi \) be a bounded function on \( \Gamma \) such that

\[ \phi(Y) = 0 \quad \text{for} \quad Y \notin \xi, \]

\[ \sum_Y |\phi(Y)|^2 = \infty \]

and

\[ \sum_Y |\phi(Y)|^{8m} < \infty \]

Let \( f \in A^{\omega,\gamma} (G) \). Then we have

\[
\sum_Y |\phi(Y) \hat{f}(Y) \omega^{\gamma}(\infty)|^2 \leq \left( \sum_Y |\phi(Y)|^{2\frac{p}{p-2}} \right)^{1-\frac{2}{p}} \left( \sum_Y \hat{\phi}(Y) \omega^{\gamma}(Y)^{2\frac{p}{2}} \right)^{2/p} \]

\[
\leq \left( \sum_Y |\phi(Y)|^{2\gamma} \right)^{1/\gamma} \left\| \hat{f} \right\|_{A^{\omega,\gamma}} \]

\[
\leq \left\| \phi \right\|_{\infty} (\sum_Y |\phi(Y)|^{8m})^{1/\gamma} \left\| \hat{f} \right\|_{A^{\omega,\gamma}} < \infty.
\]
Thus we infer that

\[
(5.4.1) \quad \sum_{Y} |\phi(Y) \hat{f}(Y) \omega^\eta(Y)|^2 \leq K \|f\|^2 \omega^\eta \quad A^\phi,
\]

where $K$ is a constant which depends only on $\phi$.

It is well known that Fourier transformation is an isometry on $L^2$. Hence there exists a function $g$ in $L^2(G)$ such that

\[
(5.4.2) \quad \hat{g} = \phi \hat{f} \omega^\eta/\phi.
\]

Further, since $L^2(G) \subset L^1(G)$, we have

\[
\|g\|_1 \leq \|g\|_q \leq \|g\|_2 \quad \text{for} \quad 1 < q < 2.
\]

\[
= \|\hat{g}\|_2
\]

\[
= \|\phi \hat{f} \omega^\eta/\phi\|_2
\]

\[
< K \|f\| \omega^\eta \quad A^\phi.
\]

Since $g \in L^q(G)$ and $\hat{g} \in L^p(G)$, using Hausdorff-Young inequality*, we get

\[
\|\hat{g}\|_p \leq \|g\|_q
\]

(5.4.3) \[ \sum_{Y} |\phi(Y) \hat{f}(Y)|^2 \leq K \| \hat{f} \|^2 \] 

In a similar way it can be easily seen that

\[ \sum_{Y} |\phi(Y) \hat{f}(Y)|^2 \leq K \| \hat{f} \|^2 \]

\[ \leq K \| \hat{f} \|^2 \]

\[ \leq K \| \hat{f} \|^2 \]

Hence there exists \( g' \in L^1(G) \subset L^2(G) \) such that

(5.4.4) \[ \hat{g}' = \phi \hat{f} \]

and

\[ \| g' \|_1 \leq \| g' \|_2 \]

\[ = \| \hat{g}' \|_2 \]

\[ = \| \phi \hat{f} \|_2 \]

(5.4.5) \[ \leq K^{1/2} \| \hat{f} \|_{L^p \omega^m} \]

By virtue of (5.4.2) and (5.4.4) we infer that

\[ \hat{g} = \hat{g}' \omega^m/\phi \]
Also, on account of (6.4.3) and (6.4.5) we find that
\[ g' \in A^p_\omega(G) \]
and
\[ \| g' \|_{A^p_\omega} = \| g' \|_1 + \| \hat{g} \|_L^p_\omega \]
\[ = \| g' \|_1 + \| \hat{g} \|_p \]
\[ \leq 2 k^{1/2} \| g \|_{L^p_\omega} \]
\[ \leq 2 k^{1/2} \| f \|_{A^p_\omega} \]

Thus we see that the mapping
\[ F : A^p_\omega(G) \rightarrow A^p_\omega(G) \]
\[ \text{such that } \ T f = g' \text{ is a multiplier on } \ A^p_\omega(G) \text{ and } \ T = \phi . \]

But
\[ \sum_Y | \phi(Y) |^2 = \infty \]
which implies that \( \phi \neq \hat{\lambda} \) for any \( \mu \in M(G) \).

5.5. Proof of Theorem II.

We suppose that
\[ A(G) = \{ f : f \in L^1(\Gamma) \} . \]

Now for each \( T \in M(\mathbb{A}^p_{\omega \infty}(G)) \) we define the map

\[(5.5.1) \quad \beta_T(f) = \int_\Gamma \hat{T}(\gamma) \hat{f}(\gamma) d(\gamma) \quad \forall f \in A(G). \]

From the definition it follows that

is linear on \( A(G) \) and

\[ |\beta_T f| \leq \|T\|_{\infty} \|f\|_{\infty} . \]

We define a norm on \( A(G) \) by

\[(5.5.2) \quad \|f\|_{\mathbb{B}^p_{\omega \infty}} = \sup \{ |\beta_T(f) : T \in M(\mathbb{A}^p_{\omega \infty}(G)) \|T\|_{\infty} \leq 1 \} . \]

Following Larsen* it can be easily seen that

\[ \|\cdot\|_{\mathbb{B}^p_{\omega \infty}} \]

is a semi-norm on \( A(G) \). Also, if

\[ f \]

is a function in \( A(G) \) satisfying the condition \( \|f\| = 0 \),

then we have

\[ * \text{ R. Larsen} \quad (38), \text{ p. 209}. \]
\[ \beta_T(f) = \int_G \frac{\hat{\mu}(Y)}{\hat{T}} \hat{f}(Y) \, dY \]

\[ \leq \| \frac{\hat{\mu}}{\hat{T}} \|_{\infty} \cdot \| \hat{f} \|_1 \]

\[ \leq \| \frac{\hat{\mu}}{\hat{T}} \|_{\infty} \cdot \| f \|_1 \]

\[ = 0, \text{ for all } T \in M(A^{p}_{\omega^n}(G)). \]

Next, let \( \tau_y \in M(A^{p}_{\omega^n}(G)) \).

Putting \( T = \tau_y \), we get

\[ Tf = \tau_y f = \hat{f}(-y) \]

and

\[ \beta_T(f) = \int_G \frac{\hat{\mu}(Y)}{\hat{T}} \hat{f}(Y) \, dY \]

\[ = \int_G \frac{\hat{\mu}(-y)}{\hat{T}} \hat{f}(Y) \, dY \]

\[ = 0, \quad \forall \, y \in G. \]

\[ \Rightarrow \quad f = 0. \]

Hence, the norm defined by (5.5.2) is, in fact, a norm on \( A(G) \). We denote this normed linear space
by $\mathcal{B}_{\omega^n}^p (G)$. 

Thus from the above discussion it is clear that for each $T \in M(A_{\omega^n}^p (G))$ the linear form $\beta_T$ is well defined on $\mathcal{B}_{\omega^n}^p (G)$ and

$$\left| \beta_T(f) \right| = \frac{\| \mathcal{B}_T f \|}{\| T \|} \leq \| T \| \| f \|$$

This implies that $\beta_T \in \left[ \mathcal{B}_{\omega^n}^p (G) \right]^*$, i.e., $\beta_T$ is an element of the $\mathcal{B}_{\omega^n}^p (G)$. 

We now define a mapping $\beta$ on $M(A_{\omega^n}^p (G))$ into $\left[ \mathcal{B}_{\omega^n}^p (G) \right]^*$, i.e.,

$$\beta : M(A_{\omega^n}^p (G)) \rightarrow \left[ \mathcal{B}_{\omega^n}^p (G) \right]^*$$

such that

$$\beta (T) = \beta_T$$
It is evident that $\beta$ is well defined and linear. Further since

$$\| \beta(T) \| = \| \beta_T \|$$

$$\leq \| T \| \text{ for all } T \in M(A_{\omega_0}(G))$$

hence $\beta$ is a continuous map.

We shall now show that $\beta$ is one-one and onto map.

Let us suppose that

$$\beta_{T_1} = \beta_{T_2}.$$ 

This implies that

$$\int_{\Gamma} \hat{T}_1(y) \hat{f}(y) \, dy = \int_{\Gamma} \hat{T}_2(y) \hat{f}(y) \, dy \quad \forall f \in L^1(\Gamma)$$

$$\hat{T}_1 = \hat{T}_2 \quad \text{as function of } L^\infty(\Gamma).$$

is one-one map.

Next, we suppose that $\lambda \in \left[ B_{\omega_0}(G) \right]_1^*$ and $B(G)$ is the set of all functions in $L^1(\Gamma)$ whose
Fourier transform has compact support. Let \( f, g \in B(G) \). Then we have
\[
f \ast g \in B_{\omega_{\kappa}}^\rho (G)
\]
and
\[
\left| \mathcal{F}_T \left( f \ast g \right) \right| = \left| \int T(Y) \hat{f}(Y) \hat{g}(Y) \, dY \right|
\]
\[
= \left| \int T(Y) \hat{f}(Y) \hat{g}(Y) \, dY \right|
\]
\[
= \left| (T \ast f \ast g)(0) \right|
\]
\[
\leq \| T \|_1 \cdot \| f \|_\infty \cdot \| g \|_\infty
\]
\[
\leq \| T \|_1 \cdot \| f \|_A^\rho \cdot \| g \|_\infty
\]

Hence we obtain
\[
(5.5.3) \quad \| f \ast g \|_{B_{\omega_{\kappa}}^\rho} \leq \| f \|_{A_{\omega_{\kappa}}^\rho} \cdot \| g \|_\infty
\]

Also, we see that
\[
\mathcal{F}_T (f \ast g) = \int T(Y) \hat{f}(Y) \omega(Y) \cdot \frac{T(Y) \hat{g}(Y)}{\omega(Y)} \, dY,
\]
which implies that

\[(5.5.4) \quad \| B_f (f \ast g) \| \leq \| \hat{f} \|_{\omega} \| \hat{g} \|_{\omega} \| f \|_{B_{\omega}^p} \| g \|_{L_{\omega}^{\gamma/1-p}} \],

where

\[\gamma_p + \gamma_{p'} = 1\]

and \(L_{\omega}^{\gamma/1-p}(\Gamma)\) is the conjugate space of \(L_{\omega}^\gamma(G)\).

In virtue of (5.5.4) it is obvious that

\[(5.5.5) \quad \| f \ast g \|_{B_{\omega}^p} \leq \| \hat{f} \|_{\omega} \| \hat{g} \|_{\omega} \| f \|_{B_{\omega}^p} \| g \|_{L_{\omega}^{\gamma/1-p}}\]

We now suppose that \(f \in B(G)\) is fixed and define a linear form on \(B(G)^{\wedge}\) by

\[F_f(\hat{g}) = \langle f \ast g \rangle \quad \text{for} \quad \hat{g} \in B(G)^{\wedge}.\]

Thus using (5.5.5)

\[\| F_f(\hat{g}) \| = \| \langle f \ast g \rangle \| \leq \| \hat{f} \|_{\omega} \| f \|_{B_{\omega}^p} \| g \|_{L_{\omega}^{\gamma/1-p}}.\]
(5.5.6) \[ \| \lambda \| \cdot \| \hat{f} \|_{L^p_{\omega^\gamma}} \cdot \| \hat{g} \|_{L^p_{\omega^{\gamma/1-p}}} \cdot \frac{1}{p} + \frac{1}{1-p} = 1. \]

On account of (5.5.6) we see that \( F_f \) is a linear form on \( B(G)^\wedge \), which is bounded in the norm of \( L^{\gamma/1-p}_{\omega^{\gamma/1-p}} (\Gamma) \). Hence, by Hahn–Banach extension theorem, \( F_f \) can be extended as a continuous linear functional to the whole space \( L^p_{\omega^{\gamma/1-p}} (\Gamma) \), there exists \( h \in L^p_{\omega^{\gamma}} (\Gamma) \) such that

\[ (5.5.7) \quad F_f (g) = \lambda (f \ast g) = \int_{\Gamma} \hat{g}(\gamma) \overline{\hat{f}}(\gamma) \, d\gamma \quad \forall \, \hat{g} \in B(G)^\wedge \]

and

\[ (5.5.8) \quad \| h \|_{L^p_{\omega^{\gamma}}} = \| F_f \| \leq \| \lambda \| \cdot \| \hat{f} \|_{A^p_{\omega^\gamma}}. \]

Next, we consider another linear form \( G_f \) on \( B(G) \) such that

\[ G_f(g) = \lambda (f \ast g) \quad \forall \, g \in B(G). \]
Thus using (5.5.3), we get

\[ |G_f(g)| \leq \|x\| \cdot \|x^*g\|_{B_{\omega_1}} \]

\[ \leq \|x\| \cdot \|x^*g\|_{A_{\omega_1}} \cdot \|g\|_{\omega_1} \]

Since \( B(G) \) is dense in \( C_0(G) \), by Hahn–Banach extension theorem, \( G_f \) can be extended to a continuous linear functional on \( C_0(G) \). Hence there exists \( \mu \in M(G) \) such that

\[(5.5.9) \quad G_f(g) = \chi(x^*g) = \int_g g(\omega) \, d\mu(\omega) \]

\[ \forall g \in B(G). \]

and

\[(5.5.10) \quad \|\mu\| \leq \|x\| \cdot \|x^*g\|_{[B_{\omega_1}]^*} \cdot \|g\|_{A_{\omega_1}} \]

We put

\[ T_f = \mu \quad \forall f \in B(G) \]

Then we get
\[(Tf * g)(0) = \lambda (f * g)(0) ; \quad f, g \in B(G),\]

which implies that

\[\left( T (\tau_y f) * g \right)(0) = \lambda (\tau_y f * g)(0)\]

\[= \lambda (f * \tau_y g)(0)\]

\[= (Tf * \tau_y g)(0).\]

\[= (\tau_y (Tf) * g) \quad \forall y \in G_n.\]

\[T (\tau_y f) = \tau_y (Tf) \quad \forall y \in G_n.\]

Thus we infer that the map \( G \rightarrow M(G) \) such

that \( y \rightarrow \tau_y Tf \) is continuous. But this ensures

that \( \mu \in L^1(G) \), i.e. \( Tf \in L^1(G) \).

If \( f, g \in B(G) \), then we have

\[F_f (\hat{g}) = \lambda (f * g) = G_f (g).\]

Hence, using (5.5.7) and (5.5.9), we get

\[\int_{G_n} \hat{g}(y) \overline{\hat{h}(y)} \, dy = \int_{G_n} g(x) \, d\mu(x).\]

\[\star \quad W. \text{ Rudin} \quad (59).\]
Also, it is known that

$$\int \hat{g}(y) \hat{h}(y) \, dy = \int \hat{g}(y) \hat{\Lambda}(y) \, dy$$

holds for every $\hat{g} \in \mathcal{B}(G)$.

Thus we have

$$h(y) = \hat{\Lambda}(y)$$

almost everywhere on $\Gamma$, which implies that

$$\hat{\Lambda} \in L^p_{\omega^m}(\Gamma)$$.

Hence by (5.5.10), we get

$$(5.5.11) \quad \|\hat{\Lambda}\|_{L^p_{\omega^m}} = \|\hat{h}\|_{L^p_{\omega^m}} \leq \|\lambda\|_{\mathcal{B}^p_{\omega^m}} \ast \|\hat{f}\|_{L^p_{\omega^m}} A_{\omega^m}$$

end

$$(5.5.12) \quad \|Tf\|_{L^1_{\omega^m}} = \|\hat{\Lambda}\|_{L^p_{\omega^m}} \leq \|\lambda\|_{\mathcal{B}^p_{\omega^m}} \ast \|\hat{f}\|_{L^p_{\omega^m}} A_{\omega^m}$$

Since $Tf = \mu$, in virtue (5.5.11) and (5.5.12), we obtain

$$\|Tf\|_{L^1_{\omega^m}} = \|Tf\|_{L^1_{\omega^m}} + \|\hat{Tf}\|_{L^p_{\omega^m}}$$

* K. Nagarjan (44) Chapter II.
\[ \leq \| \lambda \| \|A^p_{\omega^m}\| \|B^p_{\omega^m}\|^* + \| \lambda \| \|A^p_{\omega^m}\| \|B^p_{\omega^m}\|^* \| \|A^p_{\omega^m}\| \|B^p_{\omega^m}\|^* \]

\[ = 2 \| \lambda \| \|B^p_{\omega^m}\|^* \| \|A^p_{\omega^m}\| \]

Hence by Hahn–Banach theorem the map

\[ T : B(G) \rightarrow A^p_{\omega^m}(G). \]

can be extended to a map from \( A^p_{\omega^m}(G) \) into \( A^p_{\omega^m}(G) \)

which is bounded linear and translation invariant.

In other words, \( T \) can be extended to an

element of \( M(A^p_{\omega^m}(G)) \) with norm

\[ \| T \| \leq 2 \| B \| . \]

Moreover, we have seen earlier

\[ p_T(f \ast g) = \int_{\Gamma} \hat{f}(\gamma) \hat{g}(\gamma) \delta(\gamma) \, d\gamma . \]

\[ = \lambda(f \ast g) , \]

and also \( B(G) * B(G) \) is dense in \( A(G) \).
This implies that
\[ P_T(f) = \lambda(f) \quad \forall f \in A(G) \]

\[ \Rightarrow P_T = \lambda \]

Hence \( \beta \) is an onto map.

Finally, in order to complete the proof of the theorem it remains to demonstrate that the completion of \( B^p_{m}(G) \) can be embedded into a space of continuous functions.

We assume that \( f \) is an arbitrary, but fixed, function of \( B^p_{m}(G) \) and let \( T = C_x \) for some \( x \in G \).

Then we have

\[ |\beta_T(f)| = |\int_G \hat{T}(y) \hat{f}(y) d\gamma| \]

Since the norm on \( B^p_{m}(G) \) has been defined by

\[ \|f\|_{B^p_{m}} = \sup \{ |\beta_T(f)| : T \in M(A^p_{m}(G)), \|T\| \leq 1 \} \]
We have
\[ \| f \|_p \geq |B_\mathcal{F}(f)| = \int_{\mathcal{F}} (-x, y) \hat{f}(y) dy \]
\[ \text{for } \hat{f}(y) = (-x, y). \]
\[ \Rightarrow f(x) \]

Hence we get
\[ (5.5.13) \quad \| f \|_\infty \leq \| f \|_p \quad \forall \hat{f} \in B_{\mathcal{F}}(G). \]

Let \( B_{\mathcal{F}}^p(G) \) be the completion of \( B_{\mathcal{F}}(G) \) and let \( \{ f_n \} \) be a Cauchy sequence in \( B_{\mathcal{F}}^p(G) \). Then on account of (5.5.13), there exists a continuous function \( f \) on \( G \) such that \( f_n \rightarrow f \) in the essential supremum norm.

We now define a map
\[ \varphi : B_{\mathcal{F}}^p(G) \rightarrow C(G) \]
such that
\[ \varphi (f_n) = f, \]
where \( C(G) \) is the space of all continuous functions on \( G \).
From the relation (6.5.13) it follows that the map \( \nu \) from \( B_{\infty}^p (G) \) into \( C(G) \) is continuous.

We shall show that the above map is injective. We suppose that \( \{ f_n \} \in B_{\infty}^p (G) \) is a Cauchy sequence such that

\[
\lim_{n \to \infty} \| f_n \|_\infty = 0
\]

Now corresponding to every \( g \in L^1(G) \) we associate an element \( T_g \) of \( M(A_{\infty}^p \mathcal{P}(G)) \) such that

\[
T_g(f) = g \ast f \quad \forall f \in A_{\infty}^p \mathcal{P}(G).
\]

This implies that

\[
\| T_g \| \leq \| g \|_1
\]

and

\[
T_g(Y) = g(Y), \quad Y \in \mathcal{P}.
\]

Again, let \( g \in B(G) \). Then we have

\[
\lim_{n} |B_{Tp} (f_n)| = \lim_{n} \left| \int_{\mathcal{P}} \hat{T}_g(Y) \hat{f}_n(Y) \, dY \right|
\]

\[
= \lim_{n} \left| \int_{\mathcal{P}} \hat{g}(Y) \hat{f}_n(Y) \, dY \right|
\]
\[
= \lim_{n} |g \ast f_n(0)| \\
\leq \lim_{n} \|g\|_1 \cdot \|f_n\|_{\infty} \\
= 0.
\]

Thus we find that

\((5.5.14)\) \(\lim_{n \to \infty} B(f_n) = 0 \quad \forall \ f \in \mathcal{B}(G).\)

We assume that \(\{\varphi_\alpha\}_{j}\) is an approximate identity for \(A^p_{\omega^n}(G)\) such that \(\{\varphi_\alpha\}_{j} \subset \mathcal{B}(G)\) and

\[\|\varphi_\alpha\|_1 \leq 1\]

Then we have

\[T \varphi_\alpha = h_\alpha \in \mathcal{B}(G)\]

and

\[\|Tf - T\varphi_\alpha(\cdot)\| = \|Tf - \varphi_\alpha \ast f\|_{A^p_{\omega^n}} = \|Tf - T\varphi_\alpha \ast f\|_{A^p_{\omega^n}}.\]
\[
\| T \| \cdot \| f - \varphi_\alpha \ast f \|_{A^p_{\omega^n}} \leq \lim_{\alpha \to \infty} \| T_{h_\alpha} (f) \|_{A^p_{\omega^n}} = \| \varphi_\alpha \ast f \|_{A^p_{\omega^n}} = \| \varphi_\alpha \ast T f \|_{A^p_{\omega^n}} \leq \| \varphi_\alpha \|_{L^1} \cdot \| T f \|_{A^p_{\omega^n}} \leq \| T \| \cdot \| f \|_{A^p_{\omega^n}}.
\]

Hence we infer that

\[
\| T_{h_\alpha} \| \leq \| T \| \text{ for all } \alpha.
\]

Next, let \( f, g \in B(G) \). Then we have \( f \ast g \in B^p_{\omega^n} (G) \) and
\[ B_{T_h}^\alpha (f \ast g) = \int \frac{\hat{T}_{h'}}{\gamma} \hat{f}(\gamma) \hat{g}(\gamma) \, d\gamma \]

\[ = (T_{h'} \hat{f} \ast \hat{g})(0) \rightarrow (Tf \ast g)(0) \]

for \( T_{h'} f \rightarrow Tf \) in the \( L^1 \)-norm.

(5.5.15) \[ = B_T(f \ast g) \]

If \( U \in \mathcal{B}^{\omega^{\infty}}_\omega(G) \), then for any given \( \varepsilon > 0 \) there exist functions \( f, g \in B(G) \) satisfying the condition

\[ \| f \ast g - U \|_{\mathcal{B}^{\omega^{\infty}}_\omega} \leq \frac{\varepsilon}{3 \| T \|} \]

Hence it follows that

\[ |\beta_{T_h^\alpha}(u) - \beta_T(u)| \leq |\beta_{T_h^\alpha}(u) - \beta_{T_h^\alpha}(f \ast g)| + |\beta_{T_h^\alpha}(f \ast g) - \beta_T(f \ast g)| \]

\[ + |\beta_T(f \ast g) - \beta_T(u)| \]
\[
\leq \|T_\alpha\| \cdot \|u - f \ast g\|_{\mathcal{B}^p_{\infty}} + |\beta_{T_\alpha}(f \ast g) - \beta_T(f \ast g)|
\]

\[
\leq 2\|T\| \cdot \|u - f \ast g\|_{\mathcal{B}^p_{\infty}} + |\beta_{T_\alpha}(f \ast g) - \beta_T(f \ast g)|
\]

\[
\rightarrow 0 \quad \text{as} \quad \alpha \rightarrow \infty.
\]

by (5.5.15) and (5.5.16).

Thus we obtain

\[
\lim_{\alpha \rightarrow \infty} \beta_{T_\alpha}(u) = \beta_T(u) \quad \forall u \in \mathcal{B}^p_{\infty}(G)
\]

This implies that \(\beta_{T_\alpha}(f_n)\) converges to \(\beta_T(f_n)\) for every \(n\), i.e.

\[
\lim_{n} \beta_{T_\alpha}(f_n) = \lim_{n} \beta_T(f_n)
\]

Hence, by (5.5.14), we obtain

\[
\lim_{n} \beta_T(f_n) = 0 \quad \forall \quad T \in \mathcal{M}(\mathcal{A}^p_{\infty}(G))
\]
Thus, for any given $\varepsilon > 0$ and every integer $n$, there exists a multiplier $T_n \in M(A_{\omega^n}(G))$ such that

$$\|T_n\| \leq 1$$

and

$$\|f_n\|_{B_{\omega^n}^p} \leq |\beta_T(f_n)| + \varepsilon/3.$$  

But $\{f_n\}$ is a cauchy sequence in $B_{\omega^n}^p(G)$, therefore, there exists a positive integer $N$ such that

$$\|f_m - f_n\|_{B_{\omega^n}^p} < \varepsilon/3 \quad ; \quad m, n \geq N.$$  

Thus, choosing $m \geq N$ we get

$$\|f_N\|_{B_{\omega^n}^p} \leq |\beta_{TN}(f_N - f_m)| + |\beta_{TN}(f_m)| + \varepsilon/3$$

$$\leq \|T_N\| \cdot \|f_N - f_m\| + |\beta_{TN}(f_m)| + \varepsilon/3.$$  


\[ \Rightarrow \quad \| f_n \|_{B^p} \leq \frac{2\varepsilon}{3} \]

because \( \lim_{n \to \infty} |T_N(f_n)| = 0. \)

\[ \Rightarrow \quad \| f_n \|_{B^p} \leq \| f_{n-N} \|_{B^p} + \| f_{n-N} \|_{B^p} + \varepsilon \]

for \( n \geq N. \)

\[ \lim_{n} \| f_n \|_{B^p} = 0. \]

This ensures that the mapping \( \Upsilon \) defined above is injective.

This completes the proof of the theorem.