CHAPTER IV

UNIFORM MATRIX SUMMABILITY & ALMOST MATRIX SUMMABILITY
4.1 Uniform Harmonic Summability of Fourier Series

The concept of uniform Harmonic summability was first defined by Saxena\(^{37}\) in 1965 in the following manner:

Let

\[
(4.1.1) \quad u_0(x) + u_1(x) + u_2(x) + \ldots
\]

be any infinite series; and

\[
U_r(x) = u_0(x) + u_1(x) + \ldots + u_r(x).
\]

If there exists a function \( U = U(x) \), such that,

\[
(4.1.2) \quad \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{k+1} \left\{ U_{n-k} - U \right\} = o(1)
\]

uniformly in a set \( E \) in which \( U = U(x) \) is bounded, then we shall say that the series \((4.1.1)\) is summable by Harmonic means uniformly in \( E \) to the sum \( U \).

Let the Fourier series corresponding to a function \( f(x) \), periodic with period \( 2\pi \) and integrable \((L)\), be

\[
(4.1.3) \quad \frac{a_0}{2} + \sum_n (a_n \cos nx + b_n \sin nx)
\]

Let \( S_n(x) \) denote the partial sum of \((4.1.3)\). Let \( \Phi(t) = \Phi(x,t) \) and \( \Phi(t) \) be denoted as in \((3.3.1)\) and \((3.3.2)\) respectively, then Saxena established the following theorem:

**Theorem A.** If

\[
(4.1.4) \quad \Phi(t) = o\left( t/\log \left(1/t\right) \right)
\]

uniformly in a set \( E \) in which \( S = S(x) \) is bounded, as \( t \to +0 \).

\(^{37}\) Saxena, Ashok
then the Fourier series (4.1.3) is summable by Harmonic means uniformly in E to the sum $S$.

Saxena himself generalised the theorem A by replacing the special sequence of coefficients $p_n = \frac{1}{n+1}$ by a more general sequence of coefficients.

4.2 Uniform Nörlund Summability of Fourier Series

Let $\Sigma a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$p_n = p_0 + p_1 + \ldots + p_n$$

The sequence-to-sequence transformation

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu}}{p_n} S_\nu = \sum_{\nu=0}^{n} \frac{p_\nu S_{n-\nu}}{p_n}, \quad (P_n \neq 0)$$

(4.2.1)

defines the sequence $\{t_n\}$ of Nörlund means (Nörlund 1919) of the sequence $\{S_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\Sigma a_n$ is said to be summable $(N, p_n)$ to the sum $S$ if

$$\lim_{n \to \infty} t_n \text{ exists and equals } S.$$ 

The conditions for the regularity of the method of summability $(N, p_n)$ have already been discussed in Chapter II.

Definition: Let

$$u_0(x) + u_1(x) + \ldots$$

be any infinite series and

$$u_\nu(x) = u_0(x) + \ldots + u_\nu(x),$$

If there exists a function $U = U(x)$, such that,
\[(4.2.3) \quad \frac{1}{\mathcal{P}_n} \sum_{v=0}^{n} p_v \{ u_{n-v}(x) - u \} = o(1) \]

uniformly in a set \(E\) in which \(U = U(x)\) is bounded, then we shall say that the series \((4.2.2)\) is summable \((N,p_n)\) uniformly in \(E\) to the sum \(U\).

The conditions for regularity of the method of uniform \((N,p_n)\) summability defined by \((4.2.3)\) are the same as they are in the case of ordinary \((N,p_n)\) summability because they are independent of \(x\). Saxena\(^{38}\) has established the following theorem:

**Theorem:** Let \([p_n]\) be a real, non-negative monotonic non-increasing sequence, such that \(P_n \to \infty\), as \(n \to \infty\).

If \(\alpha(t)\) denotes a function of \(t\), \(\alpha(t)\) and \(\frac{t}{\alpha(t)}\) ultimately increase steadily with \(t\), (where \(\gamma = [1/t]\), denotes the integral part of \(1/t\)).

\[(4.2.4) \quad \log n = O(\alpha(P_n))\], as \(n \to \infty\); and

\[(4.2.5) \quad \Phi(t) = o\left(\frac{t}{\alpha(P_n)}\right),\]

uniformly in a set \(E\) in which \(S = S(x)\) is bounded, as \(t \to \infty\) then the Fourier series \((4.1.3)\) is summable \((N,p_n)\) uniformly in \(E\) to the sum \(S\).

4.3 Uniform Matrix Summability

Let \(T = (d_{n,k})\) be an infinite matrix, satisfying Silverman Toeplitz conditions of regularity. Let

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38) Saxena, Ashok (2)
(4.3.1) \[ u_0(x) + u_1(x) + \ldots. \]
be any infinite series, and

(4.3.2) \[ u_k(x) = u_0(x) + u_1(x) + \ldots + u_k(x) \]

If there exists a function \( u = u(x) \), such that,

(4.3.3) \[ \lim_{n \to \infty} \sum_{k=0}^{n} d_{n,k} [u_k(x) - u] = o(1) \]

uniformly in a set \( E \) in which \( u = u(x) \) is bounded, then we shall say that the series (4.3.1) is summable (T) uniformly in \( E \) to the sum \( U \).

Varma\(^{39}\) has proved the following theorem on uniform matrix summability of Fourier series.

**Theorem:** Let \( T = (a_{n,k}) \) be an infinite triangular regular matrix such that the elements \( a_{n,k} \) be non-negative, non-decreasing sequence with respect to \( k \) such that

\[ A_{n,v} = \sum_{k=v}^{n} a_{n,k}, \]

and if

(4.3.4) \[ \tilde{b}(t) = o \left( \frac{t}{B(t/t)} \right) \]

uniformly in a set \( E \) in which \( S = S(x) \) is bounded, as \( t \to +0 \), then the Fourier series (4.1.3) is summable (T) uniformly in \( E \) to the sum \( S \) where \( B(t) \) is positive non-decreasing with respect to \( t \) such that

(4.3.5) \[ \int_{1}^{\infty} \frac{A_{n,u}}{uB(u)} \, du = o(1) \]

and

\[ B(n) \to \infty \text{ as } n \to \infty \]

39) Varma, S.K.
4.4 It is interesting to note that the above theorem includes several old and new results. Some particular cases are given below:

(a) If
\[ a_{n,k} = \frac{1}{(n-k+1)} (\log n)^{-1} \quad \text{and} \quad \beta(x) = \log x \]
result of Saxena ⁴⁰) becomes the particular case of this theorem.

(b) If
\[ a_{n,k} = \frac{p_{n-k}}{p_n}, \quad (0 \leq k \leq n) \]
\[ = 0, \quad (k > n) \]
\[ p_n = p_0 + p_1 + \ldots + p_n \]
and
\[ \beta(x) = \alpha(p(x)) \]
result of Saxena ⁴¹) becomes the particular case of this theorem.

(c) If
\[ \beta(x) = \log x \]
then result of Prasad & Saxena ⁴²) becomes the particular case of this theorem.

(d) If the set E contains only one point and further if \( S(x) = f(x) \) and \( a_{n,k} \) is defined as in (b) and \( \beta(x) \) as in (a), the result of Rajagopal ⁴³) becomes the particular case of this theorem.

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⁴⁰) Saxena, Ashok
⁴¹) Ibid
⁴²) Prasad, R. & Saxena, A.
⁴³) Rajagopal, C.T.
4.5 Almost Matrix Summability of Fourier Series

Lorentz\textsuperscript{44}) has given following definition and proved a theorem.

Definition: A sequence \( \{S_n\} \) is said to be almost convergent to a limit \( S \), if

\[
(4.5.1) \quad \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=p}^{n+p} S_k = S
\]

uniformly with respect to \( p \).

Theorem: A sequence \( \{x_n\} \) has unique Banach limit if and only if it is almost convergent.

This idea of Banach limit has been applied to Fourier series and conjugate Fourier series. Sharma, Dixit and Shukla\textsuperscript{45}) have proved following theorem on almost Borel summability.

Theorem: If

\[
\left( \int_0^t \left| \Phi(u) \right| \, du = o \left( \frac{t}{(\log 1/t)A} \right) \right. \quad \text{as} \quad t \to +0
\]

and

\[
\left. \int_{1/(y+p+1)^{A/2}} \frac{1}{u} \left| \Phi(u) \right| \, du = o(1) \right. \quad \text{as} \quad y \to \infty
\]

hold uniformly with respect to \( p \), where \( 0 < A < 1 \), then the Fourier series is almost Borel summable to \( f(x) \) at the point \( t = x \).

While reviewing this result of Sharma, Dixit & Shukla

\textsuperscript{44}) Lorentz, G.G. \hspace{1cm} (1)
\textsuperscript{45}) Sharma, P.L., Dixit, S.S. & Shukla, Y.B. \hspace{1cm} (1)
late Prof. B.N. Sahney\(^{46}\) pointed out to convert the above result to matrix method. Hence Varma\(^{47}\) defined for the first time almost matrix summability and applied it to Fourier series.

**Definition**

A series \( \sum_{n=0}^{\infty} a_n \) with the sequence of partial sums \( \{S_n\} \) is said to be almost matrix summable to \( S \) if

\[
\sigma_{n,p} = \lim_{n \to \omega} \sum_{k=0}^{n} a_{n,k} S_{k,p} \to S,
\]

uniformly with respect to \( p \), where

\[
S_{n,p} = \frac{1}{(n+1)} \sum_{k=p}^{n+p} S_k
\]

Varma has proved the following theorem:

**Theorem**: If

\[
\int_0^t |\phi(u)| \, du = o\left(\frac{t}{(\log 1/t)^\Delta}\right), \quad t \to +\infty
\]

and

\[
\frac{1}{(n+p)\Delta} \int_0^t \frac{|\phi(u)|}{u} \, du = o(1), \quad \text{as} \quad n \to \infty
\]

hold uniformly with respect to \( p \), where \( 0 < \Delta < 1 \), then the Fourier series (4.1.3) is almost \( T \) summable to \( f(x) \) at the point \( t = x \) where \( T \) is a regular matrix and \( (a_{n,k}) \) being non-negative and non-decreasing with respect \( k \).

\(46\) Sahney, B.N. \hspace{1cm} (1)

\(47\) Varma, S.K. \hspace{1cm} (2)