CHAPTER IV

SOME APPLICATIONS OF REGULAR OPEN SETS

Regular open sets play an interesting role in topology. Several applications of these sets have appeared in the fields of separation axioms, covering axioms and mappings. Work of many mathematicians such as Singh, Carnahan, Herrington, Thompson, Noiri and others may be cited in this connection. The concept of regular open sets in topology is as follows:

**Definition (4.A):** A subset $A$ of a topological space $X$ is termed regular open if $A = \text{Int} \ Cl A$.

**Remark (4.A):** Every regular open set is open but the converse may be false.

**Remark (4.B):** Intersection of any finite number of regular open sets is regular open, but the union of any two regular open sets need not be regular open.

**Remark (4.C):** For any subset $A$ of a space $X$, the set $\text{Int} \ Cl A$ is regular open.

**Definition (4.B):** Complement of a regular open set is called regular closed.
**Remark (4.1.1)**: It is clear that union of any finite number of regular closed sets is regular closed. But, the intersection of two regular closed sets may not be regular closed. Also, that for any subset $A$ of a space $X$, the set $\text{Cl} \text{ Int} A$ is always regular closed.

The purpose of the present chapter is to explore and study some more applications of regular open sets in the fields of separation axioms and mappings.

### 4.1. A NOTE ON REGULAR OPEN SETS

Singal and Arya [143,144,145] have introduced the concepts of almost regular and almost normal spaces by employing regular open sets in the axioms of regularity and normality respectively. In this section the impact of employing regular open sets in the axioms of $T_0, T_1$ and $T_2$ has been studied.

**Definition (4.1.1)**: A topological space $X$ is said to be regularly $T_0$ if for $x, y \in X$ and $x \neq y$ there exists a regularly open set $U$ containing $x$ but not $y$ or containing $y$ but not $x$.

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* Journal of the Institute Jorge Juan de Matematica (To appear).
THEOREM (4.1.1): Every regularly $T_0$ space is $T_0$.

PROOF: Because every regular open set is open //

REMARK (4.1.1): The converse of Theorem (4.1.1) may be false. For,

EXAMPLE (4.1.1): If $X = \{a,b,c\}$, $J = \{\emptyset, \{a\}, \{a,b\}, x\}$ then the space $X$ is $T_0$ but it is not regularly $T_0$.

REMARK (4.1.2): The property of regularly $T_0$ is not even closed hereditary. For, the space $X$ of Example (2.2.2) is regularly $T_0$ and $Y = \{a,c\}$ is a closed subspace of $X$ but it is not regularly $T_0$. However, we have the following theorem.

THEOREM (4.1.2): Every regular open and dense subspace of a regularly $T_0$ space is regularly $T_0$.

The following lemma will be useful.

LEMMA (4.1.1): If $(Y,T_Y)$ be a dense subspace of a space $(X,T)$ and $A \subseteq Y$ then $T_Y - \text{Int} T_Y \sqcap A = \text{Int} C_l A \cap Y$ [4].
PROOF OF THEOREM (4.1.2): Let $X$ be a regularly $T_0$ space and $(Y,T_Y)$ be a dense and regular open subspace of $X$. Let $a, b \in Y$ and $a \neq b$. Since $X$ is regularly $T_0$, without any loss of the generality, suppose that $A$ is a regular open set such that $a \in A$ but $b \notin A$. Since $Y$ is regular open, $A \cap Y$ is regular open. Now $Y$ being dense in $X$, by Lemma (4.1.1) we have, $T_Y = \text{Int}_Y T_Y = \text{Cl}(A \cap Y) = \text{Int} \text{Cl}(A \cap Y) \cap Y = (A \cap Y) \cap Y = A \cap Y$. And so, $A \cap Y$ is regular open in $Y$ such that $a \notin A \cap Y$ but $b \notin A \cap Y$. Consequently, $(Y,T_Y)$ is regularly $T_0$. //

THEOREM (4.1.3): If $X$ and $Y$ be two regularly $T_0$ spaces then so is the product space $X \times Y$.

The following lemma will be required.

LEMMA (4.1.1): Let $X, Y$ be two spaces. If $A$ is regular open in $X$ and $B$ is regular open in $Y$ then $A \times B$ is regular open in the product space $X \times Y$.

PROOF OF LEMMA (4.1.1): Since $A$ is regular open in $X$ we have, $A = \text{Int}_X \text{Cl}_X A$. Similarly, $B = \text{Int}_Y \text{Cl}_Y B$. Now, $\text{Int}_X (X \times Y) \text{Cl}_X (X \times Y)(A \times B) = \text{Int}_X (X \times Y) \text{Cl}_X A \times \text{Cl}_Y B = \text{Int}_X \text{Cl}_X A \times \text{Int}_Y \text{Cl}_Y B = A \times B$. Consequently $A \times B$ is regular open in $X \times Y$. //
**Proof of Theorem (4.1.3):** Let \((x_1, y_1), (x_2, y_2)\in X \times Y\) such that \((x_1, y_1) \neq (x_2, y_2)\). Suppose that \(x_1 \neq x_2\).

Since \(X\) is regularly \(T_0\), there is a regular open set in \(X\) containing \(x_1\) but not \(x_2\) or containing \(x_2\) but not \(x_1\). Let \(A\) be regular open in \(X\) containing \(x_1\) but not \(x_2\). Since \(Y\) is regular open in itself by Lemma (4.1.1) \(A \times Y\) is regular open in \(X \times Y\) such that \((x_1, y_1) \in A \times Y\) but \((x_2, y_2) \notin A \times Y\).

Consequently, the product space \(X \times Y\) is regularly \(T_0\).

**Definition (4.1.2):** A topological space \(X\) is said to be regularly \(T_1\) if for \(x, y \in X\) and \(x \neq y\) there exist regular open sets \(U\) and \(V\) such that \(x \in U\), \(y \notin U\) and \(x \notin V\), \(y \in V\).

Since every regular open set is open, it follows that,

**Theorem (4.1.4):** Every regularly \(T_1\) space is \(T_1\).

**Remark (4.1.3):** The converse of Theorem (4.1.4) may be false. For,

**Example (4.1.2):** Let \(X\) be an infinite set equipped with the cofinite topology. Then \(X\) is a \(T_1\)-space. But it is not regularly \(T_1\) for the only regular open sets are the empty set and \(X\) itself. 
**Theorem (4.1.5):** Every regularly $T_1$ space is regularly $T_0$.

**Proof:** Follows from Definitions (4.1.1) and (4.1.2).//

**Remark (4.1.4):** The converse of Theorem (4.1.5) need not be true. For, the space $X$ of Example (2.2.2) is regularly $T_0$ but it is not regularly $T_1$.

**Remark (4.1.5):** The axioms of $T_1$ and regularly $T_0$ are independent. For, the space $X$ of Example (4.1.2) is $T_1$ but it is not regularly $T_0$. On the other hand, the space $X$ of Example (2.2.2) is regularly $T_0$ but it is not $T_1$.

**Remark (4.1.6):** The regularly $T_1$ property is hereditary or not, is not known. However, similar to Theorem (4.1.2) we have the following theorem.

**Theorem (4.1.6):** Every regular open dense subspace of a regularly $T_1$ space is regularly $T_1$.

**Theorem (4.1.7):** Product of any two regularly $T_1$ spaces is regularly $T_1$. 
The proof is analogous to that of Theorem (4.1.3).

**DEFINITION (4.1.3):** A topological space $X$ is said to be regularly $T_2$ if for $x, y \in X$ and $x \neq y$, there exist regularly open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Similar to Theorem (4.1.4), we have,

**THEOREM (4.1.8):** Every regularly $T_2$ space is regularly $T_1$.

**REMARK (4.1.7):** We expect that the converse of Theorem (4.1.8) should be false. But, we couldn't assert it.

**THEOREM (4.1.9):** A space is regularly $T_2$ if and only if it is $T_2$.

**PROOF:** The necessity is obvious because every regular open set is open.

**Sufficiency:** Suppose that a space $X$ is $T_2$. Let $x, y \in X$ and $x \neq y$. Then there exist two disjoint open sets $U$ and $V$ containing $x$ and $y$ respectively. Since $U$ and $V$ are
disjoint open, we have $U \cap \text{Cl} \ V = \emptyset$ and hence $U \cap \text{Int} \ \text{Cl} \ V = \emptyset$. Similarly we obtain $\text{Int} \ \text{Cl} \ U \cap \text{Int} \ \text{Cl} \ V = \emptyset$. It can be easily seen that $x \in \text{Int} \ \text{Cl} \ U$ and $y \in \text{Int} \ \text{Cl} \ V$. Thus, $\text{Int} \ \text{Cl} \ U$ and $\text{Int} \ \text{Cl} \ V$ are two disjoint regular open sets in $X$ containing $x$ and $y$ respectively. Consequently, $X$ is regularly $T_2$. 

\[ \text{Note: For Theorem (4.1.9) one can also refer[15].} \]

Let us denote by '--->' and '--->' the expressions 'strictly stronger than' and 'stronger than' respectively. Then, the study in this section leads us to the following diagram of implications.

\[
\begin{array}{ccc}
\text{Regularly } T_2 & \longrightarrow & \text{Regularly } T_1 \\
\downarrow & & \downarrow \\
T_2 & \longrightarrow & T_1 \\
\end{array}
\]

\[ \quad \quad \quad \quad \downarrow \]

\[ \quad \quad \quad \quad T_0 \]

4.2. DECOMPOSITION OF REGULAR OPENNESS OF SETS*

There are many weak forms of continuity in the literature. In [69], Levine introduced a decomposition of continuity. In fact, he introduced two weak notions of continuity in the forms of $\omega$-continuity and $\omega^*$-continuity.

in such a way that both these notions are independent to each other but together imply continuity. By analogy with the \(\omega\)-continuous mappings, Popa [124] has defined the notion of rarely continuous mappings and obtained a decomposition for continuity which generalized the decomposition of continuity given by Levine. Popa [125] in his another paper has studied a decomposition of quasicontinuous mappings in topological spaces. Singal and Arya [145a] has obtained a decomposition of paracompactness also. The present section aims to obtain a decomposition of regular openness of sets and to study the new notion thus introduced.

**Definition (4.2.1):** A subset \(A\) of a topological space \(X\) is termed regular semiopen if there exists a regular open set \(O\) such that \(O \subseteq A \subseteq \text{Cl} O\).

**Remark (4.2.1):** We have come to know in the year 1980, that this notion was conceived also by Cameron [17] in connection to the study of \(S\)-closed spaces.

**Theorem (4.2.1):** Every regular open set is regular semiopen.

**Proof:** Obvious. //
**Remark (4.2.2):** The converse of Theorem (4.2.1) may be false. For, in the space $X$ of Example (2.2.2), the set $\{a, c\}$ is regular semiopen but it is not regular open.

**Theorem (4.2.2):** Every regular semiopen set is semiopen.

**Proof:** This follows because every regular open set is open. 

**Remark (4.2.3):** The converse of Theorem (4.2.2) may be false. For, in the space $X$ of Example (2.2.1), the set $\{a, b\}$ is semiopen but it is not regular semiopen.

**Remark (4.2.4):** The concepts of regular semiopen and open are independent. For, in the space $X$ of Example (2.2.2) the set $\{a, c\}$ is regular semiopen but it is not open whereas the set $\{a, b\}$ is open which is not regular semiopen.

**Diagram (4.2.1):** Thus we arrive at the following diagram of implications:

```
Regular open <-> Regular semiopen <=> Semiopen
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open
**REMARK (4.2.5):** Union of two regular semiopen sets need not be regular semiopen. For, in the space $X$ of Example (2.2.2), the sets $\{a\}$ and $\{b\}$ are regular semiopen but their union $\{a, b\}$ is not regular semiopen.

**REMARK (4.2.6):** Intersection of two regular semiopen sets need not be regular semiopen. For, in the space $X$ of Example (2.2.2) the sets $\{a, c\}$ and $\{b, c\}$ are regular semiopen but their intersection $\{c\}$ is not regular semiopen.

**THEOREM (4.2.3):** If $A$ is regular semiopen in a space $X$ then $\text{Cl}(X-A)$ is regular closed.

**PROOF:** Let $A$ be regular semiopen in $X$. Then there exists a regular open set $B$ such that $B \subset A \subset \text{Cl} B$. And so, $X-B = \text{Cl}(X-B) \supset \text{Cl}(X-A) \supset \text{Cl}(X-C \text{Cl} B) = \text{Cl} \text{Int}(X-B) = X-B$, because $X-B$ is regular closed. Thus, $\text{Cl}(X-A) = X-B$. Hence $\text{Cl}(X-A)$ is regular closed. //

**THEOREM (4.2.4):** If $A$ is regular semiopen in a space $X$ then $\text{Int} A$ is regular open.

**PROOF:** Let $A$ be regular semiopen in $X$. Then there is a regular open set $B$ such that $B \subset A \subset \text{Cl} B$. And so, $B = \text{Int} B \subset \text{Int} A \subset \text{Int} \text{Cl} B = B$. Thus, $B = \text{Int} A$. //
THEOREM (4.2.5): If A and B are regular semiopen in a space X then Int(A ∩ B) is regular open.

PROOF: If A and B are regular semiopen sets then by Theorem (4.2.4), Int A and Int B are regular open. Hence, Int(A ∩ B) = Int A ∩ Int B, which is regular open. //

THEOREM (4.2.6): If A is regular semiopen in a space X and A ⊆ B ⊆ Cl A, then B is regular semiopen.

PROOF: Let A be regular semiopen and A ⊆ B ⊆ Cl A. Then there is a regular open set O such that O ⊆ A ⊆ Cl O. And so, we obtain, O ⊆ A ⊆ B ⊆ Cl A = Cl O. This shows that B is regular semiopen. //

THEOREM (4.2.7): If O is open and A regular semiopen in a space X then O ∩ A is semiopen.

PROOF: If A is regular semiopen there is a regular open set B such that B ⊆ A ⊆ Cl B. If O is open we get O ∩ B ⊆ O ∩ A ⊆ Cl B ∩ O ⊆ Cl(O ∩ B). Since O ∩ B is open it follows that O ∩ A is semiopen. //
**Remark (4.2.7):** Intersection of an open set and regular semiopen set may not be regular semiopen. For, in the space $X$ of Example (2.2.2) the set $\{a,b\}$ is open, $X$ is regular semiopen but their intersection $\{a,b\}$ is not regular semiopen.

However, we have,

**Theorem (4.2.8):** If $O$ is regular open and $A$ is regular semiopen in a space $X$ then $O \cap A$ is regular semiopen.

**Proof:** As in Theorem (4.2.7) and the fact that intersection of two regular open sets is regular open.//

The following theorem gives a decomposition of regular openness of sets in view of Remark (4.2.4).

**Theorem (4.2.9):** A set is regular open if and only if it is open and regular semiopen.

**Proof:** Necessity: Since every regular open set is open and it is regular semiopen by Theorem (4.2.1).

Sufficiency: Let $A$ be both regular semiopen and open. Then by Theorem (4.2.4), $\text{Int } A$ is regular open. Since $A$ is open, $A = \text{Int } A$. Hence, $A$ is regular open. //
**Theorem (4.2.10):** If $A$ is regular semiopen in a space $X$ and $B$ is regular semiopen in a space $Y$ then $A \times B$ is a regular semiopen in the product space $X \times Y$. Let $A \subseteq E$.

**Proof:** Let $A$ be regular semiopen in $X$ and $B$ be regular semiopen in $Y$. Then there exist a regular open set $U$ in $X$ and a regular open set $V$ in $Y$ such that $U \subseteq A \subseteq \text{Cl} U$ and $V \subseteq B \subseteq \text{Cl} V$. And so, $U \times V \subseteq A \times B \subseteq \text{Cl} U \times \text{Cl} V = \text{Cl}(U \times V)$. Moreover, $U \times V$ is a regular open in $X \times Y$ by Lemma (4.1.1). Hence, $A \times B$ is regular semiopen in $X \times Y$. 

**Theorem (4.2.11):** Let $Y$ be a dense subspace of a space $X$ and $A \subseteq Y$. If $A$ is regular open in $X$ then it is regular open in $Y$.

**Proof:** By Lemma (4.1A) we have, $\text{Int}_Y \text{Cl}_Y A = \text{Int Cl} A \cap Y = A \cap Y$, if $A$ is regular open in $X$. Since $A \subseteq Y$, this reduces to $\text{Int}_Y \text{Cl}_Y A = A$. That is, $A$ is regular open in $Y$. 

**Theorem (4.2.12):** Let $Y$ be a dense subspace of a topological space $X$ and $A \subseteq Y$. If $A$ is regular semiopen in $X$ then it is regular semiopen in $Y$. 

\textbf{PROOF.} Let $O$ be regular open in $X$ such that $O \subseteq A \subseteq \text{Cl } O$. And so, $O \cap Y \subseteq A \cap Y \subseteq \text{Cl } O \cap Y$. Evidently, $O \subseteq Y$ because $A \subseteq Y$ by hypothesis. Therefore, $O \subseteq A \subseteq \text{Cl } O$ (closure of $O$ in $Y$). But by Theorem (4.2.11) $O$ is regular open in $Y$. Consequently, $A$ is regular semiopen in $Y$. //

We now proceed to expose the regular semiopen sets to some axioms of separation.

\textbf{DEFINITION (4.2.2):} A topological space $X$ is said to be regularly semi $T_0$ if for $x, y \in X$ and $x \neq y$, there exists a regular semiopen set $U$ containing $x$ but not $y$ or containing $y$ but not $x$.

\textbf{REMARK (4.2.3):} The axioms of $T_0$ and regularly semi $T_0$ are independent. For, the space $X$ of Example (2.3.5) is $T_0$ but it is not regularly semi $T_0$. On the other hand, the space $X$ of Example (2.3.6) is regularly semi $T_0$ but it is not $T_0$.

\textbf{DEFINITION (4.2.4):} A topological space $X$ is said to be semi $T_0$ (resp. semi $T_1$) if for $x,y \in X$ and $x \neq y$, there exists a semiopen set $U$ such that $x \in U$, $y \notin U$, or (resp. and) a semiopen set $V$ such that $x \notin V$, $y \in V$ [76].
**Theorem (4.2.13):** Every regularly semi $T_0$ space is semi $T_0$.

**Proof:** This follows because of Theorem (4.2.2).

**Remark (4.2.9):** The converse of Theorem (4.2.13) may be false. For, the space of Example (2.3.5) is semi $T_0$ but it is not regularly semi $T_0$.

**Remark (4.2.10):** The property of being regularly semi $T_0$ is not even closed hereditary. For, $\{c, d\}$ is a closed subset of the regularly semi $T_0$ space $X$ of Example (2.3.6), but it is not a regularly semi $T_0$ subspace of $X$.

However, we have,

**Theorem (4.2.14):** Every regular open dense subspace $Y$ of regularly semi $T_0$ space $X$ is regularly semi $T_0$.

**Proof:** Let $x, y \in Y$ and $x \neq y$. Since $X$ is regularly semi $T_0$, there exists a regular semiopen set $A$ in $X$ such that, suppose $x \in A$ and $y \notin A$. By Theorem (4.2.8), $A \cap Y$ is regular semiopen in $X$ because $Y$ is regular open in $X$ by hypothesis. Since $Y$ is dense in $X$ and $A \cap Y \subseteq Y$, by Theorem (4.2.12) we obtain that $A \cap Y$ is regular semiopen in $Y$. Evidently, $x \in A \cap Y$ and $y \notin A \cap Y$. Hence, $Y$ is regularly semi $T_0$ subspace of $X$. //
**Theorem (4.2.15):** If $X$ and $Y$ be two regularly semi $T_0$ spaces then so is the product space $X \times Y$.

**Proof:** Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, such that $(x_1, y_1) \neq (x_2, y_2)$. Suppose that $x_1 \neq x_2$. Since $x_1, x_2 \in X$ and $X$ is regularly semi $T_0$, there is a regular semiopen set $A$ in $X$ containing (without any loss in the generality) say $x_1$ but not $x_2$. But, $Y$ is regular semiopen in itself. Therefore by Theorem (4.2.10) $A \times Y$ is regular semiopen in the product space $X \times Y$. It is clear that $(x_1, y_1) \in A \times Y$ but $(x_2, y_2) \notin A \times Y$. Consequently, the product space $X \times Y$ is regularly semi $T_0$. 

**Definition (4.2.3):** A topological space $X$ is said to be regularly semi $T_1$ if for $x, y \in X$ and $x \neq y$, there exist regular semiopen sets $U$ and $V$ such that $x \in U$, $y \notin U$ and $x \notin y$, $y \in V$.

**Remark (4.2.11):** The axioms of regularly semi $T_1$ and $T_1$ are independent. For, the space $X$ of Example (2.3.6) is regularly semi $T_1$ but it is not $T_1$. On the other hand, the space $X$ of Example (2.1.3) is $T_1$ but it is not regularly semi $T_1$ because the only regular semiopen sets are $\emptyset$ and $X$. 

**Theorem (4.2.16):** Every regularly semi $T_1$ space is semi $T_1$.

**Proof:** Follows from Theorem (4.2.2).

**Remark (4.2.12):** A semi $T_1$ space may fail to be regularly semi $T_1$. For, the space $X$ of Example (2.1.3) is semi $T_1$ but it is not regularly semi $T_1$.

**Theorem (4.2.17):** Every regularly semi $T_1$ space is regularly semi $T_0$.

**Proof:** Follows from Definitions (4.2.2) and (4.2.3).

**Remark (4.2.13):** The axiom of regularly semi $T_1$ is not even closed hereditary. For, $\{b,c,d\}$ is a closed subset of the regularly semi $T_1$ space $X$ of Example (2.3.6), but it is not a regularly semi $T_1$ subspace of $X$.

However, we have,

**Theorem (4.2.18):** Every regular open dense subspace of regularly semi $T_1$ space is regularly semi $T_1$. 
THEOREM (4.2.19): Product of any two regularly semi $T_1$ spaces is regularly semi $T_1$.

The proofs of Theorems (4.2.18) and (4.2.19) are analogous to those of Theorems (4.2.14) and (4.2.15) respectively.

DEFINITION (4.2.4): A topological space $X$ is said to be regularly semi $T_2$ if for $x, y \in X$ and $x \neq y$ there exist regular semi open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

DEFINITION (4.2.B): A topological space $X$ is said to be semi $T_2$ if for $x, y \in X$ and $x \neq y$, there exist semiclosed sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. [76].

REMARK (4.2.14): The space of Example (2.3.6) is regularly semi $T_2$ but it is not $T_2$.

THEOREM (4.2.20): Every regularly semi $T_2$ space is regularly semi $T_1$.

PROOF: Follows from Definitions (4.2.4) and (4.2.3).

THEOREM (4.2.21): Every regularly semi $T_2$ space is semi $T_2$. 
PROOF: Follows from Theorem (4.2.2).//

REMARK (4.2.15): Example (2.3.6) shows that the property of being regularly semi $T_2$ is not even closed hereditary. But, we have,

**Theorem (4.2.22):** Every regular open dense subspace of a regularly semi $T_2$ space is regularly semi $T_2$.

**Proof:** Analogous to Theorem (4.2.14).//

**Theorem (4.2.23):** If $X$ and $Y$ be regularly semi $T_2$ spaces then so is the product space $X \times Y$.

**Proof:** Let $(a, b), (c, d) \in X \times Y$ and $(a, b) \neq (c, d)$. Suppose that $a \neq c$, $b \neq d$. There exist disjoint regular semiopen sets, $U, V$ in $X$ such that $a \in U$, $c \in V$. Similarly, let $G, H$ are disjoint regular semiopen sets in $Y$ such that $b \in G$, $d \in H$. By Theorem (4.2.10), $U \times G$, $V \times H$ are regular semiopen in $X \times Y$ containing $(a, b)$ and $(c, d)$ respectively. Moreover, $(U \times G) \cap (V \times H) = (U \cap V) \times (G \cap H) = \emptyset$.

Consequently, the product space $X \times Y$ is regularly semi $T_2$.//

**Remark (4.2.16):** The space $X$ of Example (2.1.3) is semi $T_1$ but it is not regularly semi $T_0$. 
**Remark (4.2.17):** The space $X$ of Example (2.3.6) is regularly semi $T_2$ but it is not regularly $T_0$.

**Remark (4.2.4):** Semi $T_2$ $\Rightarrow$ Semi $T_1$ $\Rightarrow$ Semi $T_0 [76]$.  

**Diagram (4.2.2):** Let us abbreviate 'Regularly' and 'Semi' by 'R' and 'S' respectively. Then, the study in this section leads to the following implications:

\[
\begin{align*}
R T_0 & \Rightarrow R S T_0 \Rightarrow S T_0 \\
& \uparrow \quad \uparrow \quad \uparrow \\
R T_1 & \Rightarrow R S T_1 \Rightarrow S T_1 \\
& \uparrow \quad \uparrow \quad \uparrow \\
R T_2 & \Rightarrow R S T_2 \Rightarrow S T_2
\end{align*}
\]

**Note:** In Diagram (4.2.2), the problems of the existence of $R T_0$ space which is not $R S T_1$ & $R T_1$ space which is not $R S T_2$ & $R S T_0$ but not $S T_1$ & $R S T_1$ but not $S T_2$ and $S T_2$ but not $R S T_0$, remain open.
4.3. SOME NEW MAPPINGS

The classes of open mappings and closed mappings occupy a central place in the study of topological spaces. Various weaker forms of these mappings have been introduced in the literature from time to time. One such weaker form of open mappings had been given by Biswas [11] in 1969, as follows:

**DEFINITION (4.3.A):** A mapping \( f : X \rightarrow Y \) is said to be semi open if for any open set \( U \) in \( X \), \( f(U) \) is a semi open set in \( Y \) [11].

Noiri [100] obtained several characterization of these mappings. In analogy to this concept in the year 1973 he [102] has also given a generalization of closed mappings in the form of semiclosed mappings as follows:

**DEFINITION (4.3.B):** A mapping \( f : X \rightarrow Y \) is said to be semiclosed if the image \( f(F) \) of each closed set \( F \) in \( X \) is semiclosed in \( Y \) [102].

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This section puts forth a new class of mappings called \( g \)-semiclosed mappings which contain the class of \( g \)-semiclosed mappings and a new class of mappings called \( g \)-semiopen mappings which contain the class of semiopen mappings and presents their study.

**DEFINITION (4.3.1):** A mapping \( f : X \to Y \) is said to be generalized semiclosed (or \( g \)-semiclosed) if the image \( f(F) \) for each regular closed set \( F \) in \( X \) is semiclosed in \( Y \).

**DEFINITION (4.3.2):** A mapping \( f : X \to Y \) is said to be generalized semiopen (or, \( g \)-semiopen) if for each regular open set \( U \) in \( X \), \( f(U) \) is semiopen in \( Y \).

**DEFINITION (4.3.3):** A mapping \( f : X \to Y \) is said to be \( \beta \)-continuous if the inverse image of every regular closed (resp. regular open) subset of \( Y \) is regular closed (resp. regular open) in \( X \).

**REMARK (4.3.1):** The concepts of continuity and \( \beta \)-continuity are independent. For,

**EXAMPLE (4.3.1):** Let \( X = \{a, b, c\} \) be equipped with the topology \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \) and \( Y \) be the space of Example(2.1.2). Define \( f : X \to Y \) by \( f(a) = f(b) = y \) and \( f(c) = x \). Then \( f \) is \( \beta \)-continuous but it is not continuous.
EXAMPLE (4.3.2): Let \( X = \{a, b, c\} \) be equipped with the topology \( \tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \) and \( Y \) be the space of Example (2.3.3). Define \( f : X \to Y \) by \( f(a) = x, f(b) = y \) and \( f(c) = z \). Then \( f \) is continuous but it is not \( \beta \)-continuous.

REMARK (4.3.2): It is clear that every semiclosed mapping is \( g \)-semiclosed. But, the converse is false, as shown by the following example:

EXAMPLE (4.3.3): Let \( X \) be the space of Example (2.2.1) and \( Y \) be the space of Example (2.1.2). Define \( f : X \to Y \) by \( f(a) = y, f(b) = x \) and \( f(c) = z \). Then the mapping \( f \) is \( g \)-semiclosed but it is not semiclosed.

REMARK (4.3.3): It is clear that every semiopen mapping is \( g \)-semiopen. But the converse may be false is asserted by Example (4.3.3).

REMARK (4.3.4): The concepts of \( g \)-semiopen and \( g \)-semiclosed mappings are independent. For,

EXAMPLE (4.3.4): Let \( X \) and \( Y \) be the spaces of Example (4.3.3). Define \( \varphi_1 : X \to Y \) by \( \varphi_1(a) = \varphi_1(b) = \varphi_1(c) = y \) and \( \varphi_2 : X \to Y \) by \( \varphi_2(a) = \varphi_2(b) = x \) and \( \varphi_2(c) = y \). Then the
mapping \( \varnothing_1 \) is g-semiclosed but it is not g-semiopen whereas the mapping \( \varnothing_2 \) is g-semiopen but it is not g-semiclosed.

**Remark (4.3.5):** The composition mapping of two g-semiclosed mappings is not always g-semiclosed. Consider the following example.

**Example (4.3.5):** Let \( X \) and \( Y \) be the spaces of Example (4.3.3). Let \( Z = \{u, v, w\} \) be equipped with the topology \( \mathcal{T}_3 = \{\varnothing, \{u\}, Z\} \). Define \( f : X \to Y \) by \( f(a) = f(b) = f(c) = y \) and \( h : Y \to Z \) by \( h(x) = v, h(y) = u, h(z) = w \). Then \( f \) and \( h \) are g-semiclosed mappings but \( hof \) is not g-semiclosed.

**Remark (4.3.6):** The composition of two g-semiopen mappings need not be g-semiopen. For,

**Example (4.3.6):** Let \( X, Y, Z \) be the spaces as in Example (4.3.5). Define \( f_1 : X \to Y \) by \( f_1(a) = f_1(b) = x, f_1(c) = y \) and \( f_2 : Y \to Z \) by \( f_2(x) = v, f_2(y) = w, f_2(z) = u \). Then the mappings \( f_1 \) and \( f_2 \) are g-semiopen but \( f_2 \circ f_1 \) is not g-semiopen.

**Lemma (4.3.4):** If \( f : X \to Y \) is an open and semi-continuous mapping then the inverse image \( f^{-1}(B) \) of each semiclosed (resp. semiopen) set \( B \) in \( Y \) is semiclosed (resp. semiopen) in \( X \). [102].
**Theorem (4.3.1):** Let \( f : X \rightarrow Y \) and \( h : Y \rightarrow Z \) be two mappings and let \( h \circ f : X \rightarrow Z \) is a \( g \)-semiclosed (resp. \( g \)-semiopen) mapping. Then:

1. If \( f \) is \( \beta \)-continuous and surjective then \( h \) is \( g \)-semiclosed (resp. \( g \)-semiopen).

2. If \( h \) is open, semicontinuous and injective then \( f \) is \( g \)-semiclosed (resp. \( g \)-semiopen).

**Proof (1):** Suppose \( H \) is a regular closed set in \( Y \). Then \( f^{-1}(H) \) is regular closed in \( X \), because \( f \) is \( \beta \)-continuous. Since \( h \circ f \) is \( g \)-semiclosed and \( f \) is surjective \( (h \circ f)(f^{-1}(H)) = h(f(f^{-1}(H))) = h(H) \) is semiclosed in \( Z \). This implies that \( h \) is a \( g \)-semiclosed mapping. The proof of the respectively part is similar.

(2). Suppose \( F \) is a regular closed set in \( X \). Then \( (h \circ f)(F) \) is semiclosed in \( Z \) because \( h \circ f \) is \( g \)-semiclosed. Since \( h \) is injective, we have \( h^{-1}(h \circ f)(F)) = f(F) \). It follows immediately from Lemma (4.3.A) that \( f(F) \) is a semiclosed set in \( Y \) because \( h \) is open and semicontinuous. This implies that \( f \) is \( g \)-semiclosed. The proof of the respectively part is similar. //
DEFINITION (4.3.6): A mapping \( f : X \rightarrow Y \) is said to be almost closed (resp. almost open) if the image \( f(A) \) of every regular closed (resp. regular open) set \( A \) in \( X \) is closed (resp. open) in \( Y \) [148].

REMARK (4.3.7): Evidently, every almost closed mapping is \( g \)-semiclosed. But the converse need not be true. For, the mapping \( f_1 : X \rightarrow Y \) defined in Example (4.3.4) is \( g \)-semiclosed but it is not almost closed.

REMARK (4.3.8): Every almost open mapping is \( g \)-semiopen. But the converse may be false. For, the mapping \( f_1 : X \rightarrow Y \) defined in Example (4.3.6) is \( g \)-semiopen but it is not almost open.

THEOREM (4.3.2): Let \( f : X \rightarrow Y \) and \( h : Y \rightarrow Z \) be two mappings. If \( f \) is almost closed (resp. almost open) and \( h \) is semiclosed (resp. semiopen) then the mapping \( h \circ f : X \rightarrow Z \) is \( g \)-semiclosed (resp. \( g \)-semiopen).

PROOF: Let \( F \) be regular closed in \( X \). Then \( f(F) \) is closed in \( Y \) because \( f \) is almost closed. Therefore, \( h(f(F)) \) is semiclosed in \( Z \) because \( h \) is semiclosed. Since \( (h \circ f)(F) = h(f(F)) \), it follows that the mapping \( h \circ f \) is \( g \)-semiclosed. The proof of the respectively part is similar. //
THEOREM (4.3.3): A surjective mapping \( f: X \rightarrow Y \) is g-semiclosed if and only if for each subset \( B \) in \( Y \) and each regular open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists a semiopen set \( V \) in \( Y \) containing \( B \) such that \( f^{-1}(V) \subseteq U \).

PROOF: Necessity: Suppose \( B \) is an arbitrary subset of \( Y \) and \( U \) is any regular open set containing \( f^{-1}(B) \). We put \( V = Y - f(X - U) \). Then \( V \) is semiopen in \( Y \). Since \( f^{-1}(B) \subseteq U \), it follows from a straightforward calculation that \( B \subseteq V \). Moreover, we have, \( f^{-1}(V) = X - f^{-1}(f(X - U)) \subseteq U \).

Sufficiency: Suppose \( F \) is any regular closed set in \( X \). Let \( y \) be any point in \( Y - f(F) \). Then \( f^{-1}(y) \subseteq X - f^{-1}(f(F)) \subseteq X - F \) and \( X - F \) is regular open in \( X \). Hence, by the hypothesis, there exists a semiopen set \( V_y \) containing \( y \) such that \( f^{-1}(V_y) \subseteq X - F \). This implies that \( y \in V_y \subseteq Y - f(F) \). We obtain that \( Y - f(F) = \bigcup \{ V_y \mid y \in Y - f(F) \} \) is semi open in \( Y \). Therefore, \( f(F) \) is semiclosed. This shows that \( f \) is g-semiclosed. //

REMARK (4.3.9): The concepts of almost closed and semiclosed mappings are independent. For, the mapping \( f: X \rightarrow Y \) defined in Example (4.3.3) is almost closed but it is not semi closed. On the other hand, the mapping \( f_1: X \rightarrow Y \) considered in Example (4.3.4) is semiclosed but it is not almost closed.
**Remark (4.3.10):** The notions of almost open and semiopen mappings are also independent. For, the mapping \( f: X \rightarrow Y \) of Example (4.3.3) is almost open but it is not semiopen. On the other hand, the mapping \( f_1: X \rightarrow Y \) defined in Example (4.3.6) is semiopen but it is not almost open.

**Remark (4.3.11):** It should also be noted that the mapping \( f_2: X \rightarrow Y \) considered in Example (4.3.4) is semiclosed but it is not semiopen. On the contrary the mapping \( f_2: X \rightarrow Y \) in Example (4.3.6) is semiopen but it is not semiclosed.

**Remark (4.3.A):** Every open (resp. closed) mapping is almost open (resp. almost closed) but not conversely [148].

**Diagram:** Thus we arrive at the following diagram of implications amongst the mappings appeared in this section.

```
almost open
  /     \
open  -----> semiopen  -----> g-semiopen
     \
  /     \
closed  -----> semiclosed  -----> g-semiclosed
      \
almost closed
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