CHAPTER (O)

INTRODUCTION

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0.1 TOPOLOGY:

The birth of set-theoretical topology can be traced back to the researches of G. Cantor on the theory of point sets situated in n-dimensional Euclidean space. The development of this theory did not go beyond the subsets of Euclidean spaces for several decades, although the definitions introduced and the demonstration methods had a more general validity. This fact was recognized by M. Fréchet when in 1906 he introduced the idea of Metric space and the still more general idea of topological space [11]. However, whilst the idea of metric space was soon recognized as a very useful tool, the attempt of M. Fréchet to give a system of axioms defining topological spaces as well as the efforts of F. Riesz [49], remained only attempts and it was F. Hausdorff [17] who succeeded in giving a satisfactory form to the definition of this notion, which was developed further by the Moscow topologists who introduced its generally adopted definitive form. From that time on one can speak
of General Topology, an axiomatic theory of topological spaces.

In mathematics topology was formerly considered to be the science of "situation, position, place" which is the literal meaning of a Greek word from which the word "Topology" is derived. An alternative name was "analysis situts".

The subject arose as a branch of geometry, but, in the course of its development topology has outgrown its geometrical origin and has involved in many other branches of mathematics. Nowadays it firmly stands alongside analysis and algebra as one of the most fundamental branch of mathematics. There are many domains in the broad field of topology.

Series of systems of axioms were formulated, some equivalent to other, in order to define the notion of topological space. In these systems of axioms we use as basic terms-open sets, closed sets, the closure operator, cluster points, limit points, neighbourhoods of a point and so on. A basic term used in a system of axioms is the primitive undefined term underlying therein to develop the notion of topology.
axiomatically. One could talk about a particular
structure according to each basic term and about
the equivalence of these different structures.
Strictly speaking, one could take a basic term
as a starting point of axiomatic development of
topology and to define other basic concepts in
terms of it.

It seems that most of the topologists
agree with a point that the topology starting
with the concept of open sets as the basic
concept leads to elegant results. Moreover,
source of the axioms that is to formulate the
collection of open sets in order to define a
topology on an abstract non-empty set $X$ is some
of the properties of collection of all open sub-
sets of a metric space.

**DEFINITION 0.1.1**: A topological space is a
pair $(X, T)$, where $X$ is non-empty set and $T$ is a
family of subsets of $X$ satisfying the following
axioms:

(i) $\emptyset \in T$ and $X \in T$,
(ii) $T$ is closed under arbitrary unions,
(iii) $T$ is closed under finite intersections.
The family $T$ is called a topology on $X$ and the members of $T$ are called the open sets of the space $(X, T)$.

0.2 MAPPINGS

A mapping is some thing to say a linkage between the mathematical systems under consideration. Many kinds of mappings occur in a great variety of situations. Now a days its meaning is much broader and deeper than its elementary one in classical analysis of real numbers.

A mapping $f: X \to Y$, $X$ and $Y$ non empty sets, is a rule $f$ which assigns to each $x \in X$ a single fully determined element $f(x)$ in $Y$.

A mapping is significant in the context of topology provided it be able to speak of a linkage of topologies under consideration. Of course, one may think of it in terms of open sets which are at hand to describe a topology completely.

As an immediate linkage one may naturally arrive at the following two conditions which are useful for the present consideration.

0.2.1 : $f(V)$ is open in $Y$ for each open $V$ in $X$.
0.2.2 : $f^{-1}(V)$ is open in $X$ for each open $V$ in $Y$. 
With condition 0.2.1 the mapping $f$ carries open sets over to open sets and is called an open mapping. With condition 0.2.2 the mapping $f$ pulls open sets back to open sets and is called a continuous mapping.

A homeomorphism is a mapping which is continuous, open, one to one and onto. Two topological spaces $X$ and $Y$ are said to be homeomorphic if there exists a homeomorphism of $X$ onto $Y$. The two homeomorphic spaces, therefore, differ only in the nature of points and, can, from the point of view of topology, be considered essentially identical.

Topology is a subject concerning with a study of topological properties. The topological properties are the topological invariants, i.e., to say, a property which is preserved under a homeomorphism is called a topological property.

In fact, a topological property identifies a class $[T]$ of topological spaces such that a topological space $Y \in [T]$ if it is homeomorphic to some topological space $X \in [T]$.

Undoubtedly, mappings are useful in various ways. As mentioned earlier continuous
mappings are in order to pull the open sets back
to the open sets while open mappings are in order
to carry the open sets over to the open sets.
Connected mappings appeared as a carrier of
connectedness property. The distinguishability
among the points of the space may be attained by
the mappings. In fact, in a complete Hausdorff
space, distinct points are separated by means of
real valued continuous mappings. A normal space
ensures the existence of certain real valued
continuous mappings and in turn, it is also
characterized by these mappings. A Tychonoff
space is found to be embedded in a "cube", i.e., it
is a carbon copy (homeomorphic) of a subspace of
a "cube".

In his 1962 paper [1] Alexandroff
suggested that many topological entities could be
best understood by studying their relations, via
"nice" maps, to other "nice" spaces. As pointed
out by Arhangel'ski! in the introduction to his
survey paper [2], solutions to problems of this
sort were obtained over a half-century ago by
Alexandroff and Hausdorff. More recently, the
Alexandroff's proposal has stimulated a renewed
interest in the classification of spaces via
mappings. As a result, various mappings having
of some specific nature relevant some topological entities were grown up and studied.

Now, our interest is to give an idea of some "nice" mappings centred around the continuous mapping and to observe them classified laying emphasis on the way with which they may be seen derived from a continuous mapping. We begin with continuity and some of its equivalent forms which facilitates our present consideration.

**THEOREM 0.2.1**: For a mapping \( f : X \rightarrow Y \) to be continuous on \( X \) the following conditions are equivalent:

(a) for each \( x \in X \) and each open set \( V \) containing \( f(x) \), there exists an open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

(b) \( f^{-1}(V) \) is open for each open \( V \) in \( Y \).

(c) \( f(\text{cl}(A)) \subseteq \text{cl}(f(A)) \) for each \( A \subseteq X \).

To concieve a variant of continuous mapping the inherent idea may be seen classified as follows.

(i) Varying the condition of continuity itself.

(ii) Replacing 'open set' concept itself.
(iii) Open sets together with some 'nice' property.

(iv) With some specific nature.

0.3 **Varying the Condition of Continuity Itself:**

We know that in a topological space for any set $E$, $E \subseteq \text{cl}(E)$. This simple fact, on one hand, with Theorem 0.2.1(a) gives rise to the condition $f(U) \subseteq \text{cl}V$, a weaker form of continuity, known as weak continuity due to Levine [29], and on the other hand, with Theorem 0.2.1(c) to $f(\text{cl}(A)) \subseteq f(A)$, a formulation of strong continuity of Levine [28]. Some of the mappings which may be seen yielded on this line, are being mentioned here.

**Definition 0.3.1 [10]:** A mapping $f: X \rightarrow Y$ is said to be $\theta$-continuous at $x \in X$ if for each neighbourhood $M$ of $f(x)$, there exists a neighbourhood $N$ of $x$ such that $f(\text{cl}M) \subseteq \text{cl}N$.

**Definition 0.3.2 [50]:** A mapping $f: X \rightarrow Y$ is said to be almost continuous (in the sense of Singal and Singal) at $x \in X$, if for every neighbourhood $M$ of $f(x)$ there is a neighbourhood $N$ of $x$, such that $f(N) \subseteq \text{int cl}M$. 
**DEFINITION 0.3.3 [18]** : A mapping \( f: X \to Y \) is said to be almost continuous (in the sense of Husain) at \( x \in X \) if for every neighbourhood \( V \) of \( f(x) \), \( \text{cl}(f^{-1}(V)) \) is a neighbourhood of \( x \).

**DEFINITION 0.3.4 [12]** : A mapping \( f: X \to Y \) is said to be almost continuous (in the sense of Frolik) if for every open subset \( V \) of \( Y \), \( f^{-1}(V) \subseteq \text{cl} \text{int}(f^{-1}(V)) \).

The definition of almost continuity in the sense of Stallings is considered to be given later. All these four types of almost continuities are independent of each other. Also each of these almost continuities is weaker than the continuity in the sense that they are implied by the continuity. In general, continuity is not implied by any of these almost continuities.

In [32] Long and Carnahan are interested in comparing almost continuities of Husain [18], Stallings [52] and Singal and Singal [50]. Example 1 of [33] shows that an almost continuous mapping in the sense of Husain need not be almost continuous in the sense of Singal and Singal and Example 2.1 of [50] shows that an almost
continuous mapping in the sense of Singal and Singal need not be almost continuous in the sense of Husain. Consequently, the mappings, given by Definition 0.3.2 and 0.3.3 are completely independent of each other. Likewise, Examples 1 and 2 of [33] are to show that the almost continuity in the sense of Husain is independent of the almost continuity in the sense of Stallings. Further, Example 2 of [33] shows that almost continuity in the sense of Stallings need not be almost continuity in the sense of Singal and Singal, while Mathur and Deb [41] observed the independency of both these almost continuities.

Frolík was interested in introducing almost continuous mappings in his sense in connection with the study of Baire spaces. Arya and Deb [3] extended a study on the almost continuous mappings in the sense of Frolík. This almost continuity is found to be independent of almost continuity in the sense of Singal and Singal and almost continuity in the sense of Stallings [52]. The notion of quasi-continuous mappings is due to Kempisty [23]. Kempisty considered quasi-continuous real valued mappings
of several real variables to extend some results of Hahn and Baire on real valued mappings of several real variables which are continuous in each variable separately. Marcus [37] proved that quasi-continuous mappings are equivalent to Bledsoe's neighbourly functions [5] and that if $f$ is a derivative function which is continuous almost everywhere, then it is quasi-continuous. Neighbourly functions have also been studied by Martin [38] and Thielman [54]. Martin considered quasi-continuous mappings for arbitrary topological spaces and proved that some of the results of Kempisty hold for more general spaces. Quasi-continuous mappings have also been studied by Neugebauer [42] and Niastad [43]. Levine [30] in his own concept of semi-open sets to introduce semi-continuous mappings. Arya and Deb [3] observed that almost continuity in the sense of Frolik, quasi-continuity and semi-continuity are equivalent to each other.

The $\theta$-continuous mappings were introduced by Fomin [10]. These were found useful in the study of Hausdorff non-regular spaces. Also, $\theta$-continuous mappings turn out to be the natural tool for studying almost compact spaces of Alexandroff and Urysohn, since $\theta$-continuous image
of an almost compact space is almost compact. The θ-continuous mappings were later studied by Iliadis [19] and Iliadis and Fomin [20]. Fomin studied these mappings while obtaining some results about H-closed extensions of topological spaces and Iliadis used these in connection with his study of absolutes. These θ-continuous mappings have also been talked of in [47], [48] and [50]. A θ-continuous mapping may be seen [50], failed to be almost continuous mapping in the sense of Singal and Singal. Arya and Deb [4] studied θ-continuous mappings extensively and showed that almost continuity in the sense of Singal and Singal [50] implies the θ-continuity which implies the weak continuity. Moreover, θ-continuity is independent of Stallings almost continuity. They [4] also developed the notion of θ-continuous homotopy.

0.4 REPLACING 'OPEN SET' CONCEPT ITSELF:

Various types of sets appeared not only to generalize but also to strengthen the concept of open set. These sets are frequently used in place of open sets to introduce various mappings.

It is mentioned earlier that a continuous mapping is to pull the open sets back to the open
sets. In this formulation of continuity the idea of open sets occurs at two places. This may be replaced by some other one only at one place or at both the places.

More precisely a replacement of open set by a weaker concept at the range place only gives rise to a stronger form of the continuity, while at the domain place a weaker form. Analogously, one may think of the situation regarding the replacement by a stronger form. A replacement at both the places is not necessarily comparable with the original one.

In a topological space a set $A$ is known to be regular open, pre-open, $\kappa$-set and semi-open if $A = \text{Int} \ cl(A)$, $A \subseteq \text{Int} \ cl(A)$, $A \subseteq \text{Int} \ cl \ Int(A)$ and $A \subseteq cl \ Int(A)$, respectively.

A mapping under which $f^{-1}(V)$ is open for regular open set $V$ is the almost continuity discussed earlier in the sense of Singal and Singal. Also, the semi-continuity under which inverse image of open set is semi-open, is due to Levine [30] and it is equivalent to the almost continuity in the sense of Frolik. The concept of semi-open set is due to Levine and it is
equivalent to that of \( \beta \)-set of Niastad [43]. The idea of pre-open sets [39] was in existence earlier as "nearly open" sets in [46] where Pettis discussed the nearly continuity in his sense in connection with some topological questions related to open mapping and closed graph theorems. If the inverse of open set is \( \alpha \)-set [43] then the mapping so obtained is called \( \alpha \)-continuity and studied in [40] and [53]. If for every semi-open set \( V \), \( f^{-1}(V) \) is open, then the mapping \( f \) is called K-continuous [9].

The K-continuous mapping is also appeared in [51], wherein it is called S-continuous mapping. Analogous to a continuous mapping an irresolute mapping is formulated by Crossley and Hildebrand [7] to pull semi-open sets back to the semi-open sets. The concept of irresolute mapping is weaker than that of K-continuity, but stronger than the semi continuity, however, independent of the continuity. Later, these mappings have been studied considerably not only as a new mapping but also to make it convenient for being well-behaved to stand firmly upto certain extent with semi-open sets alongside as the continuous mappings with open sets.
In this direction, the work carried out by Crossley and Hildebrand is worth to mention. In [7], topological spaces \( X \) and \( Y \) are said to be semi-homeomorphic if there exists a mapping \( f : X \to Y \) such that \( f \) is surjective, injective and \( f^{-1} \) and \( f^{-1} \) both are irresolute. Such \( f \) is called a semi-homeomorphism. A property which is preserved under a semi-homeomorphism is called a semi-topological property [7]. If \( f : X \to Y \) is a homeomorphism, then it is a semi-homeomorphism but not conversely [7]. Consequently, a semi-topological property is a topological property but not conversely, in general. Semi-homeomorphic is an equivalent relation between topological spaces [7]. Further Crossley and Hildebrand [7] introduced an equivalence relation of semi-correspondence on the collection of topologies on a non-empty set. Two topologies \( T \) and \( T' \) on a non-empty set \( X \) are said to be semi-correspondent if they have the same semi-open sets. Also, an equivalence class under the semi-correspondence is called a semi-topological class. In fact, a semi-topological class of \( X \) is a collection of topological spaces which have \( X \) as their set of points and have the same semi-open sets.
0.5 Open sets together with some 'nice' property:

Here, we are concerned with a way which appears to be followed most commonly in the literature for getting a generalization of the continuity. The continuous mappings are characterized explicitly by means of open sets involved therein. In the present technique, simply a restriction, with some 'nice' property, is to be made to get ready only some of the open sets (of course, not all of the open sets) for having their inverse images as open. Compactness and connectedness type properties are frequently used for the aforesaid restriction.

This way of approach has been pursued in the introduction of various concepts, viz., $C^*$-continuity, $C^-$continuity, $H$-continuity, $N$-continuity, semi-connected mappings, $s$-continuity etc. Each of these concepts, except semi-connected mappings is implied by that of continuity.

A mapping $f : X \rightarrow Y$ is called weak semi connected (respectively semi connected) if for each closed and connected set $K \subseteq Y$, $f^{-1}(K)$ is closed (respectively closed and connected).

Several authors namely Lee [27], Jones [22] and
Long [31] have studied semi connected mappings. Weak semi connected mappings are also called s-continuous mappings. Kohli [24] gave several characterizations of an s-continuous mapping. The concepts of continuous and semi connected mappings are independent of each other and both imply s-continuity [24].

The idea of c-continuous mapping was conceived by Gentry and Hovie, Ill [13] in an entirely unrelated setting. These mappings have a rather nice relationship to the classical theorem "Every one-to-one onto continuous mapping from a compact space onto a Hausdorff space is a homeomorphism" and have many basic properties of their own similar to properties possessed by continuous mappings. A mapping \( f : X \rightarrow Y \) is called c-continuous if for each closed and compact set \( K \subseteq Y \), \( f^{-1}(K) \) is closed. Further, using c-continuous homotopy for the equivalence relation instead of usual homotopy a new type of fundamental group is defined and called c-continuous fundamental group [14]. c-continuous fundamental group is found to be non-trivial and different from the usual fundamental group.
Using the idea of countably compactness Park [44] introduced a class of $C^n$-continuous mappings. A $C^n$-continuous mapping is defined to be a mapping under which inverse of countably compact closed sets are closed. Further, Long and Herrington [35] discussed various properties of these mappings together with $c$-continuous mappings. They observed that in a topological space $(Y, T)$ the collection of open sets having countably compact complements forms a base for a new topology $T^*$. Of course, $T^* \subseteq T$ and $(Y, T^*)$ is always a countably compact space. A mapping $f: X \to (Y, T)$ is $C^n$-continuous iff $f: X \to (Y, T^*)$ is continuous. These considerations of $C^n$-continuous mapping lead them in [35], to define a $C^n$-continuous fundamental group analogous to the $c$-continuous fundamental group of [14].

Long and Hamlett [34] used $H$-closed set to introduce $H$-continuity. A mapping $f: X \to Y$ is called $H$-continuous if for each closed, $H$-closed subset $B \subseteq Y$, $f^{-1}(B)$ is closed in $X$. Here, $B$ is to be an $H$-closed subset of $Y$ but not $H$-closed as a sub-space of $Y$. Carnahan [6] defined $N$-closed set which generalizes the notion of compact set. A subset $A$ of a space $X$ is said to be $N$-closed if for any open cover of $A$ (open
in the space X) there exists a finite sub-
collection \( U \) such that \( A \subseteq \bigcup U \) (int (cl(U)) : \( U \in U \)). The motivation for the definition of \( N \)-
continuous mappings comes from the concept of
\( c \)-continuous, \( C^\ast \)-continuous and \( H \)-continuous
mappings. In [36], Ma'ligan and Hanchinamani
introduced and studied \( N \)-continuous mappings. A
mapping \( f : X \rightarrow Y \) is called \( N \)-continuous if for
each closed, \( N \)-closed subset \( V \) of \( Y \), \( f^{-1}(V) \) is
closed in \( X \). Every mapping that is \( C^\ast \)-continuous
is also \( c \)-continuous but the converse need not
hold [44]. An \( H \)-continuous mapping is also
\( c \)-continuous but a \( c \)-continuous mapping need not
be \( H \)-continuous. Further, \( H \)-continuity is found
to be independent of \( C^\ast \)-continuity [34]. The
class of \( N \)-continuous mappings includes the
class of \( H \)-continuous mappings and is included in
the class of \( c \)-continuous mappings [36].

0.6 WITH SOME SPECIFIC NATURE:

Our interest turns up, here, to the
mappings which on first sight seem to be of
having no interaction with continuous mappings
yet they have.

Some problems of topology may be solved
only through the study of non-continuous
mappings. Connectivity maps are a species of such mappings, their importance lies in Hamilton's theorem: A connectivity map of an n-cell into itself has a fixed point [16]. In order to extend Hamilton's theorem and method of proof therein, Stallings [52] introduced the almost continuous and Polyhedrally almost continuous mappings, some sorts of non-continuous mappings. A mapping $f: X \rightarrow Y$ may be well understood by its graph $G(f) = \{(x, f(x)) : x \in X\}$. The mapping $f: X \rightarrow Y$ is said to be a connectivity map [16] if for any connected subset $C$ of $X$, $G(f/C)$ is connected. A mapping $f: X \rightarrow Y$ is called almost continuous [52] if for any open set $N \subseteq X \times Y$ with $G(f) \subseteq N$, there exists a continuous mapping $g: X \rightarrow Y$ such that $G(g) \subseteq N$. The almost continuity is extended to the idea of polyhedral almost continuity in the situation of polyhedron [52].

Some mappings are explicitly defined as a carrier of some "nice" properties. A fundamental property of continuous mappings is that the image of a connected set is a connected set. This property was taken by Pervin and Levine [45] as the definition of connected mappings. Since the requirement for a set to be connected are quite strong, the connected mappings, as expected are
found to have interesting topological properties of their own. They examined various conditions that may be placed upon the mapping or upon the spaces in order to be able to conclude that a connected mapping is continuous. Kwak [26] was interested in studying the mapping which appeared as the carrier of the property "connectedness between sets" of a space. Intuitively a topological space having the property of connectedness may be thought a space yielded as a single piece in a topological sense. A motivating idea of a space to be connected between a pair of its subsets is that this pair is to be considered as a single piece retaining a topological sense, i.e. the pair has a certain interaction with which sets of the pair are not distinguishable to a certain degree in their own right. Infact, a topological space is said to be connected between subsets A and B if there is no closed-open set F such that $A \subseteq F$ and $F \cap B = \emptyset$ [44]. Kwak [26] called a mapping $f : X \rightarrow Y$ to be a "set-connected" mapping if $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to relative topology $f(X)$ whenever $X$ is connected between $A$ and $B$. He found that $f$ is a set-connected mapping iff $f^{-1}(F)$ is closed-open in $X$ for every closed-open subset $F$ of $f(X)$. In [8],
using the concept of semi-open sets, the concept of s-connectedness between sets and its carrier set s-connected mappings, were introduced. A mapping \( f : X \rightarrow Y \) is said to be set s-connected iff \( f^{-1}(F) \) is semi-closed and semi-open in \( X \) for each semi-closed semi-open in \( f(X) \). A set connected mapping is independent of set-s-connected mapping.

Irudayanathan [21] enabled to define a mapping on a space \( X \) to be nearly continuous in terms of its certain relation with a continuous mapping on \( X \). Two mappings \( f : X \rightarrow Y \) and \( g : X \rightarrow Y \) are said to be \( \alpha \)-near for an open covering \( \alpha \) of \( Y \) if for every \( x \in X \), \( f(x) \) and \( g(x) \) belong to the same set \( U \in \alpha \). A mapping \( f : X \rightarrow Y \) is said to be nearly continuous if for every open covering \( \alpha \) of \( Y \) there exists a continuous mapping \( g : X \rightarrow Y \) which is \( \alpha \)-near to \( f \). Every continuous mapping is nearly continuous in the sense of [21] but not conversely, in general. Further, examples given in [4], [41] and [50], are to show that near continuity is independent of each of the almost continuity in the sense of Frolik [12], the almost continuity in the sense of Stallings [52], the almost continuity in the sense of Singal and Singal [50] and \( \theta \)-continuity [10].
One of the "nice" properties of continuous mappings is that the image of a compact set is compact. In order to achieve more deeper and fruitful results up to a certain extent, some extra conditions are frequently imposed on a general topological space. Amongst these restrictions the separation axioms and covering conditions are basically most common. A motivating idea of separation axioms is to make the points and sets of a space topologically distinguishable. The separation axioms enable us to state with precision that a topological space has a rich enough supply of open sets to serve the purpose. The supply of open sets possessed by a topological space is intimately linked to the supply of continuous functions on it. In 1982 Gentry and Hoyle, [15] introduced a hierarchy of non-continuous mappings which are called $T_i$-continuous mapping ($i=1, 2, 3$). These mappings are framed with the interaction of certain covering conditions and are found to have a close relation with separation axioms and continuous mappings. It is found that continuity $\Rightarrow T_1$-continuity $\Rightarrow T_2$-continuity $\Rightarrow T_3$-continuity. A $T_i$-continuous mapping $f$ of a space $X$ to a $T_i$-space is continuous where $i = 1, 2, 3$. They proved that these weaker concepts share also
some of the important properties of continuous mappings. On one hand, the image of a compact space under a surjective $T_1$-continuous mapping is compact and on the other hand, the image of a connected space under a surjective $T_3$-continuous mapping is connected.

CONCLUDING REMARKS:

We have attempted only to give an idea of a few important mappings which may be considered centered around the continuous mapping, laying emphasis more on the way with which they may be seen derived from a continuous mapping. We have tried also to present the mappings classified on the basis of the nature of their introduction. We have tried to make the task exhaustive but it is, by no means, claimed to be complete. In the above considerations we have not included any mention of the mappings which appear significantly centered around the open mapping.
REFERENCES


