CHAPTER V

WEAK K-CONTINUITY

5.0 INTRODUCTION:

The concept of irresolute mapping due to Crossley and Hildebrand [1] is independent of that of the continuity. In [4], the concept of K-continuity has been introduced in such a way that it is stronger than both the concepts irresoluteness and continuity. Again, the weak continuity due to Levine [11] appeared as a generalization of continuity in the sense that the class of continuous mappings is properly included in the class of weakly continuous mappings. A mapping $f : X \rightarrow Y$ is defined to be weakly continuous if for any $x \in X$ and open set $V$ containing $f(x)$, there exists an open set $U$ such that $x \in U$ and $f(U) \subseteq \text{cl} V$. If 'openness' is replaced by 'semi-openness' for $U$, then the mapping so obtained is called weakly semi-continuous and has been studied by Kar and Bhattacharya [8].

A new class of mappings what are called weakly-K-continuous mappings, is the interest of
the present study. This concept may be seen arrived at in two ways. In one way, one may have an idea of generalizing the concept of K-continuity just as weak-continuity generalizes the continuity, and in another way, one may replace open set V by semi-open set in the aforesaid definition of weak continuity.

The introduced concept of weak K-continuity is found to be stronger than each of the concepts-weak — semi-continuity, weak continuity, and weak Θ-irresoluteness, but is weaker than the K-continuity. However, each of the concepts-irresoluteness, Θ-irresoluteness, weak irresoluteness, continuity, Θ-continuity, semi-continuity and Θ-semi continuity is independent of the concept of weak K-continuity. Further various characterizations of weak K-continuity have been obtained. Some other basic properties have also been examined.

5.1 TERMINOLOGY:

Throughout the present study, a space X always means a topological space. The closure, interior, semi-closure and semi-interior of a
subset $A$ are denoted by $\text{cl}(A)$, $\text{int}(A)$, $\text{scl}(A)$ and $\text{sint}(A)$ respectively. A subset $A$ is said to be semi-open [10], if there exists an open set $U$, such that $U \subseteq A \subseteq \text{cl}U$. The complement of a semi-open set is called a semi-closed set. The intersection of all semi-closed sets containing $A$ is called semi-closure of $A$. In a space $X$, for any subset $A$, $\text{scl}_g(A)$, and $\text{sint}_g(A)$ are defined as follows:

$$\text{scl}_g(A) = \{ x \in X : A \bigcap \text{cl} U \neq \emptyset, \text{ for every semi-open set } U \text{ containing } x \}.$$  

$$\text{sint}_g(A) = \{ x \in X : \text{cl} U \subseteq A, \text{ for some semi-open set } U \text{ containing } x \}.$$  

5.2 WEAKLY K-CONTINUOUS MAPPINGS:

**DEFINITION 5.2.1**: A mapping $f : X \rightarrow Y$ is said to be weakly $K$-continuous at $x \in X$, if for each semi-open set $V$, with $f(x) \in V$, there exists an open set $U$ with $x \in U$ such that $f(U) \subseteq \text{cl}V$. If $f$ is weakly $K$-continuous at every $x \in X$, then $f$ is said to be weakly $K$-continuous on $X$.

**DEFINITION 5.2.2** [2]: A mapping $f : X \rightarrow Y$ is said to be $\Theta_k$-continuous at $x \in X$, if for each
semi-open set \( V \) with \( f(x) \in V \), there exists an open set \( U \) with \( x \in U \), such that \( f(\text{cl } U) \subseteq \text{cl } V \).

**THEOREM 5.2.1**: Every \( \Theta_k \)-continuous mapping is weakly \( K \)-continuous.

**PROOF**: Obvious.

But the converse is not true, in general, as is shown by the example given below.

**EXAMPLE 5.2.1**: Let \( X = (a, b, c, d) \) with topology \( T = (X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\} \)

and \( Y = \{p, q, r, s\} \), with topology \( T' = (Y, \emptyset, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}) \).

A mapping \( f : X \to Y \) be defined as

\[
\begin{align*}
  f(a) &= p, & f(b) &= q, & f(c) &= r, & f(d) &= p.
\end{align*}
\]

Here, the mapping \( f \) is weakly-\( K \)-continuous but not, \( \Theta_k \)-continuous.

**DEFINITION 5.2.3 [8]**: A mapping \( f : X \to Y \) is said to be weakly semi-continuous at \( x \in X \), if for each open set \( V \) with \( f(x) \in V \), there exists a
semi-open set $U$ with $x \in U$, such that $f(U) \subseteq \text{cl}V$.

**Theorem 5.2.2**: Every weak K-continuous mapping is weakly semi-continuous.

**Proof**: Obvious.

But the converse is not true, in general, as is shown by the following example.

**Example 5.2.2**: Let $X = \{a, b, c\}$, with topology $\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$.

A mapping $f : X \rightarrow X$ be defined as:

$f(a) = f(b) = a, \quad f(c) = c$.

Here, the mapping $f$ is weakly semi-continuous but not weakly K-continuous.

**Definition 5.2.4** [7]: A mapping $f : X \rightarrow Y$ is said to be weakly $\Theta$-irresolute at $x \in X$, if for each semi-open set $V$ containing $f(x)$, there exits a semi-open set $U$ with $x \in U$, such that $f(U) \subseteq \text{cl}V$.

**Theorem 5.2.3**: Every weakly K-continuous mapping is weakly $\Theta$-irresolute.
PROOF : Obvious.

But the converse is not true in general as is shown by the example given below.

EXAMPLE 5.2.3 : Let \( X = \{a, b, c\} \) with topology \( T = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \).

A mapping \( f : X \rightarrow X \) be defined as -
\[
f(a) = a, \quad f(b) = b, \quad f(c) = c.
\]

Here, the mapping \( f \) is weakly \( \theta \)-irresolute but not weakly \( K \)-continuous.

DEFINITION 5.2.5 [9] : A mapping \( f : X \rightarrow Y \) is said to be \( \theta \)-irresolute if, for each point \( x \in X \), and for each semi-open set \( V \) containing \( f(x) \), there exists a semi-open set \( U \) with \( x \in U \), such that \( f(\text{cl}U) \subseteq \text{cl}V \).

DEFINITION 5.2.6 [3] : A mapping \( f : X \rightarrow Y \) is said to be \( \theta \)-semi continuous at \( x \in X \), if for each open set \( V \) with \( f(x) \in V \), there exists a semi-open set \( U \) with \( x \in U \), such that \( f(\text{cl}U) \subseteq \text{cl}V \).

DEFINITION 5.2.7 [6] : A mapping \( f : X \rightarrow Y \) is said to be \( \theta \)-continuous at \( x \in X \) if for each
open set \( V \) with \( f(x) \in V \), there exists an open set \( U \) with \( x \in U \), such that \( f(\text{cl}U) \subseteq \text{cl}V \).

**DEFINITION 5.2.6**: A mapping \( f : X \rightarrow Y \) is said to be irresolute if for each semi-open \( V \) in \( Y \), \( f^{-1}(V) \) is semi-open in \( X \) \([1]\), \( f \) is said to be semi-continuous if for each open set \( V \) in \( Y \), \( f^{-1}(V) \) is semi-open in \( X \) \([10]\). Finally, \( f \) is said to be \( K \)-continuous \([4]\) if for each semi-open set \( V \), \( f^{-1}(V) \) is open in \( X \).

**DEFINITION 5.2.9** \([5]\) : A mapping \( f : X \rightarrow Y \) is said to be weakly irresolute at \( x \in X \), if for each semi-open set \( V \) with \( f(x) \in V \), there exists a semi-open set \( U \) with \( x \in U \), such that \( f(U) \subseteq \text{scl} V \).

**DEFINITION 5.2.10** \([13]\) : A mapping \( f : X \rightarrow Y \) is said to be semi-weakly continuous at \( x \in X \), if for each open set \( V \) with \( f(x) \in V \), there exists a semi-open set \( U \) with \( x \in U \), such that \( f(U) \subseteq \text{scl} V \).

In example 5.2.1 the mapping \( f \) is weakly \( K \)-continuous but not irresolute, \( \theta \)-irresolute, weakly-irresolute, \( \theta \)-continuous.
0-semi continuous and semi-weakly continuous. Also, in example 5.2.3, the mapping \( f \) is irresolute, \( \theta \)-irresolute, weakly-irresolute, \( \theta \)-continuous, \( \theta \)-semi continuous and semi weakly continuous but not weakly \( K \)-continuous. Therefore, the weak \( K \)-continuity is independent of each of the concepts-irresoluteness, \( \theta \)-irresoluteness, weak-irresoluteness, \( \theta \)-continuity, \( \theta \)-semi continuity and semi-weak continuity. Further, by example 5.2.1 and 5.2.2 one may conclude that weak \( K \)-continuity is independent of continuity and semi-continuity.

**DEFINITION 5.2.11** [11] : A mapping \( f : X \rightarrow Y \) is said to be weakly continuous at \( x \in X \), if for each open set \( V \) with \( f(x) \in V \), there exists an open set \( U \), with \( x \in U \), such that \( f(U) \subseteq \text{cl}V \).

**THEOREM 5.2.4** : Every weakly \( K \)-continuous mapping is weakly continuous.

**PROOF** : Obvious. But the converse is not true, in general, as is in the example 5.2.3.
THEOREM 5.2.5: If the mapping \( f : X \rightarrow Y \) be a weakly \( K \)-continuous and the space \( X \) is regular, then the mapping \( f \) is \( \theta_K \)-continuous.

PROOF: Let \( x \in X \) and \( W \) be any semi-open set containing \( f(x) \). Then by weak \( K \)-continuity there exists an open set \( V \) such that \( x \in V \), and \( f(V) \subseteq \text{cl}W \). Now, in the regular space \( X \) we have \( x \in U \subseteq \text{cl}U \subseteq V \), for some open set \( U \). Therefore, \( f(\text{cl}U) \subseteq \text{cl}W \). Consequently \( f \) is \( \theta_K \)-continuous.

DEFINITION 5.2.12 [15]: A space \( X \) is said to be \( K \)-regular if for any \( x \in X \) and semi-open set \( V \) with \( x \in V \), there exists an open set \( U \) such that \( x \in U \subseteq \text{cl}U \subseteq V \).

THEOREM 5.2.6: If the mapping \( f : X \rightarrow Y \) be a weakly \( \theta \)-irresolute mapping and space \( X \) is \( K \)-regular then \( f \) is weakly \( K \)-continuous.

PROOF: Let \( x \in X \) and \( W \) be any semi-open set containing \( f(x) \). Then by Weak-\( \theta \)-irresoluteness there exists a semi-open set \( V \) such that \( x \in V \),
\( f(V) \subseteq \text{cl}W \). Now, in K-regular space \( X \) we have \( x \in U \subseteq \text{cl}U \subseteq V \), for some open set \( U \). Therefore, \( f(U) \subseteq f(\text{cl}U) \subseteq \text{cl}W \). Consequently, \( f \) is a weakly K-continuous mapping.

Every K-regular space is regular. Consequently we have the following result.

**Theorem 5.2.7**: For a mapping \( f : X \to Y \), where \( X \) is K-regular, the following are equivalent:

1. \( f \) is \( \Theta_k \)-continuous.
2. \( f \) is weakly K-continuous.
3. \( f \) is weakly \( \Theta \)-irresolute.

**Theorem 5.2.8**: If \( f : X \to Y \) be a weakly continuous mapping and space \( Y \) is K-regular, then the mapping \( f \) is K-continuous; and hence, \( f \) is weakly K-continuous mapping.

**Proof**: Let any \( x \in X \) and \( V \) be any semi-open set in \( Y \) containing \( f(x) \). Since the space \( Y \) is K-regular, there exists an open set \( U \) such that \( f(x) \in U \subseteq \text{cl}U \subseteq V \). Further, since \( f \) is weakly...
continuous, for the open set \( U \), there exists some open set \( W \) in \( X \) such that \( x \in W \) and \( f(W) \subseteq \text{cl}U \), and hence, \( f(W) \subseteq \text{cl}U \subseteq V \). Consequently, \( f \) is \( K \)-continuous.

**Theorem 5.2.9**: If a mapping \( f : X \to Y \) be \( \Theta \)-continuous and space \( Y \) is \( K \)-regular, then the mapping \( f \) is \( K \)-continuous, and hence, \( f \) is weakly \( K \)-continuous.

**Proof**: Let for any \( x \in X \) and \( V \) be any semi-open set in \( Y \) containing \( f(x) \). Since the space \( Y \) is \( K \)-regular there exists an open set \( U \) such that \( f(x) \in U \subseteq \text{cl}U \subseteq V \). Further, since \( f \) is \( \Theta \)-continuous, for open set \( U \) there exists some open set \( W \) in \( X \) such that \( x \in W \) and \( f(\text{cl}W) \subseteq \text{cl}U \), and hence, \( f(\text{cl}W) \subseteq \text{cl}U \subseteq V \). Consequently, \( f \) is \( K \)-continuous.

**Theorem 5.2.10**: If a mapping \( f : X \to Y \) is irresolute and space \( X \) is extremally disconnected, then \( f \) is a weakly \( K \)-continuous mapping.
**Proof**: In [2] an irresolute mapping $f$ is defined on, an extremally disconnected space is found to be $\Theta_k$-continuous. Again, $\Theta_k$-continuity implies weak $K$-continuity.

The implication relations discussed earlier may be concluded as follows.

\[
\Theta_k\text{-C} \implies W.K\text{-C} \implies W-\Theta\text{-irr} \iff W.\text{ irr} \\
\downarrow \quad \downarrow \\
W.C. \implies W.S.C. \iff S.W.C.
\]

In the diagram we have abbreviated as follows:

$\Theta_k$-C = $\Theta_k$-continuous, W.K-C = Weak K-continuous, W-\Theta-irr = Weak $\Theta$-irresolute, W.S.C. = Weak semi continuous, W. irr = Weak-irresolute, S.W.C. = Semi weak continuous.

Recall that the weak $K$-continuity has been shown to be independent of each of the concepts - irresoluteness, $\Theta$-irresoluteness, weak irresoluteness, $\Theta$-continuity, $\Theta$-semi continuity, semi-weak continuity and semi-continuity.
THEOREM 5.2.11: A mapping \( f : X \rightarrow Y \) is weak K-continuous if and only if for every semi-open set \( V \) in \( Y \), \( f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}V)) \).

PROOF: Let \( V \) be any semi-open set in \( Y \) and \( x \in f^{-1}(V) \). Since \( f \) is weak K-continuity, there exists an open set \( U \) in \( X \) such that \( x \in U \subseteq f^{-1}(\text{cl}V) \). Therefore, we have \( x \in U \subseteq f^{-1}(\text{cl}V) \) and hence \( x \in \text{int}(f^{-1}(\text{cl}V)) \). This proves that \( f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}V)) \). Conversely, let \( x \in X \) and \( V \) be any semi-open set in \( Y \) with \( f(x) \in V \). Then \( x \in f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}V)) \). Put \( U = \text{int}(f^{-1}(\text{cl}V)) \). Then open set \( U \) is such that \( x \in U \subseteq f^{-1}(\text{cl}V) \). Therefore, \( f(U) \subseteq \text{cl}V \).

THEOREM 5.2.12: For any mapping \( f : X \rightarrow Y \), the following conditions are equivalent.

(a) \( f \) is weakly K-continuous.

(b) \( f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}V)) \), for semi-open set \( V \) in \( Y \).

(c) \( f(\text{cl}A) \subseteq \text{sclg}(f(A)) \), for each \( A \subseteq X \).

(d) \( \text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{sclgB}) \), for each \( B \subseteq Y \).

(e) \( f^{-1}(\text{sintgB}) \subseteq \text{int}(f^{-1}(B)) \), for each \( B \subseteq Y \).
PROOF: (a) \(\iff\) (b) by theorem 5.2.11.

(b) \(\iff\) (c): Let \(A\) be any subset of \(X\) and \(x \in \text{cl}A\). Take any semi-open set \(V\) such that \(f(x) \in \text{cl}V\). Then by theorem 5.2.11, \(f^{-1}(V) \subseteq \text{int}(f^{-1}(\text{cl}V))\).

Therefore, there exists some open set \(W\) such that \(x \in W\) and \(W \subseteq f^{-1}(\text{cl}V)\). Again, \(W \cap A \neq \emptyset\) as \(x \in \text{cl}A\). Therefore, \(\text{cl}V \cap f(A) \neq \emptyset\); and hence, \(f(x) \in \text{scl}g(f(A))\). Consequently, \(f(\text{cl}A) \subseteq \text{scl}g f(A)\).

(c) \(\iff\) (d): For any set \(B\) in \(Y\), suppose \(x \in \text{cl}f^{-1}(B)\). Then, \(f(x) \in f(\text{cl}f^{-1}(B))\) and by (c), \(f(x) \in f(\text{cl}(f^{-1}(B))) \subseteq \text{scl}g(f(f^{-1}(B))), \text{i.e., } x \in f^{-1}[\text{scl}g(f^{-1}(B))]\) implies that \(x \in f^{-1}(\text{scl}gB)\). Consequently, \(\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{scl}gB)\).

(d) \(\iff\) (e): Let \(B\) be any subset of \(Y\) and \(x \in X\) be such that \(f(x) \in \text{sint}gB\). Then there exists some semi-open set \(V\) such that \(f(x) \in \text{cl}V \subseteq B\). Hence, \(f(x) \in \text{scl}g(Y-B)\). Therefore, by (d), \(\text{cl}(f^{-1}(Y-B)) \subseteq f^{-1}(\text{scl}g(Y-B))\). Hence, \(x \in \text{cl}(f^{-1}(Y-B))\) i.e. \(x \in \text{cl}(X-f^{-1}(B))\). Therefore, \(x \in \text{int}(f^{-1}(B))\). Therefore, \(f^{-1}(\text{sint}gB) \subseteq \text{int}(f^{-1}(B))\).
(e) \implies (a): Let \( x \in X \) and \( V \) be any semi-open set such that \( f(x) \in V \). Then by (e), we have \( f^{-1}(\text{sintgV}) \subseteq \text{int}(f^{-1}(V)) \). Recall that for any semi-open set \( V \), \( V \subseteq \text{sintg clV} \). Thus we have \( f^{-1}(V) \subseteq \text{int} f^{-1}(\text{clV}) \). Therefore, there exists an open set \( U \) such that \( x \in U \subseteq f^{-1}(\text{clV}) \) and hence, \( f(U) \subseteq \text{clV} \).

5.3 SOME OTHER RESULTS:

DEFINITION 5.3.1 [12]: A topological space \( Y \) is said to be semi- \( T'_{2} \) if for each two distinct points \( x, y \in Y \), there exists semi-open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \), and \( \text{clU} \cap \text{clV} = \emptyset \).

THEOREM 5.3.1: If a mapping \( f : X \to Y \) is weakly \( K \)-continuous, injective and space \( Y \) is semi- \( T'_{2} \), then the space \( X \) is \( T_{2} \).

PROOF: Let \( x_{1} \) and \( x_{2} \) be two distinct points of \( X \). Since \( f \) is injective and space \( Y \) is semi- \( T'_{2} \), there exists semi-open \( V_{i}(i = 1, 2) \) such that \( f(x_{1}) \in V_{1} \) and \( \text{clV}_{1} \cap \text{clV}_{2} = \emptyset \). Since \( f \) is weakly \( K \)-continuous there exists open set \( U_{i}(i = 1, 2) \) containing \( x_{1} \) such that \( f(U_{i}) \subseteq \text{clV}_{i} \). Therefore, \( U_{1} \cap U_{2} = \emptyset \) and hence space \( X \) is \( T_{2} \).
**Theorem 5.3.2**: Let a mapping $f : X \rightarrow Y$ be weakly $K$-continuous and $A$ be any subset of $Y$. Then the restriction $f/A = g$ is weakly $K$-continuous.

**Proof**: Let for $x \in A$, $V$ be any semi-open set containing $f(x)$. Since $f$ is weakly $K$-continuous there exists an open set $U$ in $X$ such that $x \in U$, and $U \subseteq f^{-1}(\overline{V})$. Then $U \cap A$ is open relative to $A$ such that $x \in U \cap A \subseteq f^{-1}(\overline{V}) \cap A = g^{-1}(\overline{V})$. It follows that $f/A$ is weakly $K$-continuous.

Following example is to show that the composition of weakly $K$-continuous mappings is not weakly $K$-continuous.

**Example 5.3.1**: Let $X = \{a,b,c,d\}$ be equipped with topology

$$T = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}\}$$

and $Y = \{p, q, r, s\}$ with topology,

$$T' = \{Y, \emptyset, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}\}.$$  

Mappings $f : X \rightarrow Y$ and $g : Y \rightarrow Y$ be defined as -
f(a) = p, f(b) = q, f(c) = r, f(d) = p
and g(p) = r, g(q) = s, g(r) = q, g(s) = q.

Then the mappings f and g are weakly K-continuous but g ∘ f is not weakly K-continuous.

**Theorem 5.3.3:** If a mapping f : X → Y is continuous and mapping g : Y → Z is weakly K-continuous then the mapping (g ∘ f) is weakly K-continuous.

**Proof:** Let x ∈ X and W be any semi-open set in Z such that (g ∘ f)(x) ∈ W. Since g is weakly K-continuous there exists an open set V in Y such that f(x) ∈ V, g(V) ⊆ clW. Again, since f is continuous, there exists an open set U of X such that x ∈ U and f(U) ⊆ V. Thus for any semi-open set W with g(f(x)) ∈ W, there exists an open set U such that x ∈ U and g(f(U)) ⊆ clW. It follows that the mapping (g ∘ f) is weakly K-continuous.

**Theorem 5.3.4:** If the mapping f : X → Y is weakly K-continuous and mapping g : Y → Z is 0-irresolute then the mapping g ∘ f is weakly K-continuous.
PROOF: For any $x \in X$, let any semi-open set $W$ in $Z$ be such that $g(f(x)) \in W$. Then $g$ being $\Theta$-irresolute there exists a semi-open set $V$, such that $f(x) \in V$, and $g(\text{cl}V) \subseteq \text{cl}W$. Again, $f$ is weakly $K$-continuous. Then there exists some open set $U$ such that $x \in U$ and $f(U) \subseteq \text{cl}V$. Therefore, the open set $U$ containing $x$ is such that $g(f(U)) \subseteq \text{cl}W$. Hence, the mapping $g \circ f$ is weakly $K$-continuous.

THEOREM 5.3.5: If mappings $f : X \to Y$ and $g : Y \to Z$ are weakly $K$-continuous and space $Y$ is regular then the composition $g \circ f$ is weakly $K$-continuous.

PROOF: Since the space $Y$ is regular and the mapping $g$ is weakly $K$-continuous, by theorem 5.2.5, $g$ is $\Theta_K$-continuous. Further, $\Theta_K$-continuity implies $\Theta$-irresoluteness. Therefore, by theorem 5.3.4 $g \circ f$ is weakly $K$-continuous.

THEOREM 5.3.6: Let $f : X \to Y$ be a mapping and $g : X \to X \times Y$ be its graph mapping given by $g(x) = (x, f(x))$ for every $x \in X$. If $g$ is weakly $K$-continuous then $f$ is weakly $K$-continuous.
PROOF : Let \( x \in X \) and \( V \) be any semi-open set in \( Y \) containing \( f(x) \). Then \( X \times V \) is semi-open set containing \( g(x) \). Since \( g \) is weakly \( K \)-continuous there exists an open set \( U \) in \( X \) containing \( x \) such that \( g(U) \subseteq \text{cl}(X \times V) = X \times \text{cl}V \). Also, \( g \) being the graph mapping of \( f \) we have \( f(U) \subseteq \text{cl}V \). Therefore, \( f \) is weakly \( K \)-continuous.

THEOREM 5.3.7 : Let \( X \) be any space, \( Y \) be a semi-\( T'2 \) space and mappings \( f, g : X \rightarrow Y \) be weakly \( K \)-continuous. Then

(i) \( A = \{x \in X : f(x) = g(x)\} \) is a closed set in \( X \).

(ii) If a dense subset \( D \) of \( X \) is such that \( f = g \) on \( D \), then \( f = g \) on \( X \).

PROOF : (i) For any \( x \in X - A \), \( f(x) \neq g(x) \).
Since the space \( Y \) is semi-\( T'2 \), there exist semi-open sets \( U \) and \( V \) such that \( f(x) \in U \), \( g(x) \in V \) and \( \text{cl}U \cap \text{cl}V = \emptyset \). Further, \( f \) and \( g \) are weakly \( K \)-continuous mappings. Therefore, there exist open sets \( W \) and \( W' \) such that \( x \in W \), \( x \in W' \), \( f(W) \subseteq \text{cl}U \) and \( f(W') \subseteq \text{cl}V \). Thus \( x \in W \cap W' \) is an open set such that \( (W \cap W') \cap A = \emptyset \). If it is not, then for some \( p \in (W \cap W') \cap A \), we have \( f(p) = g(p) \); and hence, \( \text{cl}U \cap \text{cl}V \neq \emptyset \), a contradiction.
(ii) We have $D \subseteq \{x : f(x) = g(x)\}$ and so $X = \text{cl}D \subseteq \{x : f(x) = g(x)\}$.

**THEOREM 5.3.8**: Let $f : X \rightarrow Y$ be continuous, $g : X \rightarrow Y$ be weakly $K$-continuous and space $Y$ be $T_2$. Then, $A = \{x \in X : f(x) = g(x)\}$ is a closed set.

**PROOF**: For $x \in Y - A$, $f(x) \neq g(x)$ and in $T_2$-space $Y$, there exists open sets $U$ and $V$ such that $f(x) \in U$, $g(x) \in V$ and $U \cap \text{cl}V = \emptyset$. The rest proof of the theorem is similar as that of theorem 5.3.7.

**THEOREM 5.3.9**: If the mapping $f : X \rightarrow Y$ is weakly $K$-continuous and space $Y$ is semi-$T'_2$, then the set $A = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in $X \times X$.

**PROOF**: Let $(x_1, x_2) \in X \times \{X - A\}$. Then $f(x_1) \neq f(x_2)$. Since the space $Y$ is semi-$T'_2$, there exists semi-open set $V_i$ containing $f(x_i)$, where $i = 1, 2$, such that $\text{cl}(V_1) \cap \text{cl}(V_2) = \emptyset$. Since the mapping $f$ is weakly $K$-continuous, there exists open set $U_i$ containing $x_i$ such that $f(U_i) \subseteq \text{cl}V_i$. Let $U = U_1 \times U_2$. Then $(x_1, x_2) \in U$.
and U is open in X × X such that A ∩ U = ∅. Therefore, \((x_1, x_2) \not\in \text{cl}A\), and hence, A is closed in X × X.

**DEFINITION 5.3.2:** A subset W of X × Y is said to be \(\theta\)-semi closed with respect to Y if for each \((x, y) \not\in W\), there exist open set U and semi-open V such that \(x \in U\), \(y \in V\) and \((U \times \text{cl}V) \cap W = \emptyset\). Further, a mapping \(f : X \rightarrow Y\) has a \(\theta\)-semi closed graph with respect to Y if the graph \(G(f)\) is \(\theta\)-semi closed with respect to Y.

**REMARK:** It is clear that the concept of \(\theta\)-semi closed graph with respect to Y is weaker than that of strongly closed graph.

**THEOREM 5.3.10:** If the mapping \(f : X \rightarrow Y\) is weakly \(K\)-continuous and space Y is semi-\(T'\_2\), then the graph \(G(f)\) is \(\theta\)-semi closed with respect to Y.

**Proof:** Let \((x, y) \not\in G(f)\). Since the space Y is semi-\(T'\_2\) there exist semi-open sets V and W such that \(f(x) \in W\) and \(y \in V\) and clV \(\cap\) clW = \(\emptyset\). Since the mapping f is weakly \(K\)-continuous, there exists an open set U with \(x \in U\) such that \(f(U) \subseteq\) clW. Therefore, \(f(U) \cap \text{cl}V = \emptyset\), and hence,
\[ U \times \text{cl}V \setminus G(f) = \emptyset. \] Consequently, \( G(f) \) is \( \theta \)-semi closed graph with respect to \( Y \).

**Theorem 5.3.11**: If a mapping \( f : X \rightarrow Y \) is injective, weakly \( K \)-continuous and with \( \theta \)-semi closed graph with respect to \( Y \) then the space \( X \) is \( T_2 \).

**Proof**: Let \( x_1, x_2 \) be any two distinct points of \( X \). Then \( f(x_1) \neq f(x_2) \), and so \( (x_1, f(x_2)) \notin G(f) \). Therefore, there exist open set \( U \) containing \( x_1 \) and semi-open set \( V \) containing \( f(x_2) \) such that \( f(U) \setminus \text{cl}V = \emptyset \). Since \( f \) is weakly \( K \)-continuous, there exists an open set \( W \) such that \( x_2 \in W \) and \( f(x_2) \in f(W) \subseteq \text{cl}V \subseteq Y - f(U) \). But, \( x_1 \in U, x_2 \in W \subseteq f^{-1}(\text{cl}V) \subseteq X - U \). So that space \( X \) is \( T_2 \).

**Definition 5.3.3** [14]: A topological space \( X \) is said to be \( S \)-closed if for each semi-open cover \( (V_i) \) of \( X \), there is a finite subfamily \( (V_1, V_2, \ldots, V_n) \) such that \( X = \bigcup_{i=1}^{n} \text{cl}(V_i) \).

**Theorem 5.3.12**: Image of a compact space under a weakly \( K \)-continuous surjective mapping is \( S \)-closed.
**Proof**: Let \( f \) be a weakly \( K \)-continuous mapping of a compact space \( X \) onto a space \( Y \). Let \( \{V_i\} \) be any semi-open cover of \( Y \). Then, for each semi-open set \( V_i \), we have, by theorem 5.2.11, \( f^{-1}(V_i) \subseteq \text{int} f^{-1}(\text{cl}V_i) \), as \( f \) is weakly \( K \)-continuous.

Therefore, the family \( \{\text{int}(f^{-1}(\text{cl}V_i))\} \) forms an open cover of \( X \). Since space \( X \) is compact, there exists a finite subfamily \( \{\text{int} f^{-1}(\text{cl}V_i) : i = 1, 2, \ldots, n\} \) such that \( X = \bigcup_{i=1}^{n} (\text{int}(f^{-1}(\text{cl}V_i))) \), or \( X = \bigcup_{i=1}^{n} f^{-1}(\text{cl}V_i) \). Therefore, it follows that \( f(X) = f\left(\bigcup_{i=1}^{n} (f^{-1}(\text{cl}V_i))\right) = \bigcup_{i=1}^{n} (\text{cl}V_i) = Y \). Therefore, \( Y \) is \( S \)-closed.
REFERENCES


