Chapter 4

A State dependent M/M/1 single and batch service queue under the policy \((a, c, d)\)

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4.1 Introduction

Single as well as batch service queuing systems with poisson arrivals were considered by many researchers in the past (see Chaudhary and Templeton (1983), Neuts (1967), Cohan (1969) etc.). Single and batch service queueing systems were considered by Manoharan (1990), Baburaj and Manoharan (1997), Baburaj (1999), Baburaj and Manoharan (2004). Baburaj and Jayakumar (2005) considered an \((a,c,d)\) policy \(M/M/1\) queue with single and batch service.

Queueing systems with control limit policy has many concrete applications and also has the capability to narrate certain real life situations. Baburaj and Surendranath (2005) considered an \((a,c,d)\) policy single server queuing system where the server starts service when the queue size is at least \(c\) and serves a maximum of \(d\) units in a batch. If after a service completion epoch, the queue size is less than \(c\) but not less than a secondary limit \(a\), the server continues the service \((a \leq c \leq d)\). Here we assume that some set up cost is needed to initiate a batch service and when a batch service is started no set up cost is required. This type of models are very useful in production, manufacturing, telecommunication, transportation etc.

In this model we consider a state dependent \(M/M/1\) single and batch service queue under the policy \((a,c,d)\). The arrival process is assumed to be Poisson with parameter \(\lambda\). Here the customers are served by a single server either one at a time or in batches with different service rates depending on the number of units in the queue. If the number of units in the queue is at least \(c\), the server serves the units manually according to FCFS rule and the service time distribution is assumed to be exponential with parameter \(\mu_1\). This type of service is called Type I service. If \(n \geq c\) the server serves a maximum of \(d\) units in a batch and the service time distribution
is assumed to be exponential with parameter $\mu_2$. This type of service is called Type II service. If after a Type II service completion epoch, when the number of units in the queue is less than $c$ but not less than a secondary limit $a$ ($a \leq c \leq d$), the server serves them altogether in a batch and service time distribution is assumed to be exponential with parameter $\mu_3$. This type of service is called Type III service. After a batch service completion epoch, if the queue size is less than $a$, the server begins manual service. The analysis of this model is discussed in section 2. From section 3, we get the L.T of the transient probabilities. Section 4 explains the steady state probabilities of the model. Expected queue length is explained in section 5. Sections 6, 7, 8 and 9 respectively explain the busy period distribution in Type I service, batch service, Type II service and Type III service. Section 10 explains the optimal control limits.

4.2 Analysis of the Model

Let $Y(t)$ and $X(t)$ respectively denote the state of the server and the queue size at time $t$. The variable $Y(t)$ assumes the values 0, 1, 2 or 3 according as the server is idle, busy with a manual service, busy with type II service or busy with a type III service. Then the two-dimensional stochastic process $\{Y(t), X(t) \geq 0\}$ forms a Markov process with state space

$$S = S_1 \cup S_2 \cup S_3 \cup S_4$$

where

$$S_1 = \{(0,0)\},$$

$$S_2 = \{(1,n), n = 0, 1, 2, \ldots, c-1\}$$
\[ S_3 = \{(2, n), n = 0, 1, 2, \ldots\}, \]

and \[ S_4 = \{(3, n), n = 0, 1, 2, \ldots\} \]

Let \[ P(i, n, t) = P\{Y(t) = i, X(t) = n\} \]

Following are the transitions that can be occurred during \((t, t+h]\)

<table>
<thead>
<tr>
<th>Transitions during ((t, t+h])</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0) \rightarrow (0,0))</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1, n) \rightarrow (1, n+1)), (0 \leq n \leq c-2)</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1, c-1) \rightarrow (2, 0))</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1, n) \rightarrow (1, n-1)), (1 \leq n \leq c-1)</td>
<td>(\mu_1 h + O(h))</td>
</tr>
<tr>
<td>((1, 0) \rightarrow (0, 0))</td>
<td>(\mu_1 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (2, n+1)), (n = 0, 1, 2, \ldots)</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((2, 0) \rightarrow (0, 0))</td>
<td>(\mu_2 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (1, n-1)), (1 \leq n \leq a-1)</td>
<td>(\mu_2 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (3, 0)), (a \leq n \leq c-1)</td>
<td>(\mu_2 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (2, 0)), (c \leq n \leq d)</td>
<td>(\mu_2 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (2, n-d)), (n &gt; d)</td>
<td>(\mu_2 h + O(h))</td>
</tr>
<tr>
<td>((3, n) \rightarrow (3, n-1)), (n = 0, 1, 2, \ldots)</td>
<td>(\lambda h + O(h))</td>
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</tr>
<tr>
<td>((3, n) \rightarrow (2, n-d)), (n &gt; d)</td>
<td>(\mu_3 h + O(h))</td>
</tr>
</tbody>
</table>
Hence the difference differential equations governing the transitions are,

\[ P'(0, 0, t) = -\lambda P(0, 0, t) + \mu_1 P(1, 0, t) + \mu_2 P(2, 0, t) + \mu_3 P(3, 0, t) \]  
(4.2.1)

\[ P'(1, 0, t) = -(\lambda + \mu_1) P(1, 0, t) + \lambda P(0, 0, t) + \mu_1 P(1, 1, t) + \mu_2 P(2, 1, t) + \mu_3 P(3, 1, t) \]  
(4.2.2)

\[ P'(1, n, t) = -(\lambda + \mu_1) P(1, n, t) + \lambda P(1, n - 1, t) + \mu_1 P(1, n + 1, t) + \mu_2 P(2, n + 1, t) + \mu_3 P(3, n + 1, t), \quad 1 \leq n \leq a - 1 \]  
(4.2.3)

\[ P'(1, 0, t) = -(\lambda + \mu_1) P(1, n, t) + \lambda P(1, n - 1, t) + \mu_1 P(1, n + 1, t) + \mu_2 P(2, n + 1, t) + \mu_3 P(3, n + 1, t) \]  
(4.2.4)

\[ P'(1, c - 1, t) = -(\lambda + \mu_1) P(1, c - 1, t) + \lambda P(1, c - 2, t) \]  
(4.2.5)

\[ P'(2, 0, t) = -(\lambda + \mu_2) P(2, 0, t) + \lambda P(1, c - 1, t) + \mu_2 \sum_{n=c}^{d} P(2, n, t) + \mu_3 \sum_{n=c}^{d} P(3, n, t) \]  
(4.2.6)

\[ P'(2, n, t) = -(\lambda + \mu_2) P(2, n, t) + \lambda P(2, n - 1, t) + \mu_2 P(2, n + d, t) + \mu_3 P(3, n + d, t), \quad n = 1, 2, 3, \ldots \]  
(4.2.7)

\[ P'(3, 0, t) = -(\lambda + \mu_3) P(3, 0, t) + \mu_2 \sum_{n=a}^{c-1} P(2, n, t) + \mu_3 \sum_{n=a}^{c-1} P(3, n, t) \]  
(4.2.8)

\[ P'(3, n, t) = -(\lambda + \mu_3) P(3, n, t) + \lambda P(3, n - 1, t), \quad n = 1, 2, 3, \ldots \]  
(4.2.9)
4.3 Method of Solution

Let $P^*(i, n, s)$ denote the Laplace transform of $P(i, n, t)$ and assume that $P(0, 0, 0) = 1$. Then the following are the laplace transform of the transient probabilities corresponding to system of equations (4.2.1) to (4.2.9)

\[(s + \lambda)P^*(0, 0, s) - 1 = \mu_1 P^*(1, 0, s) + \mu_2 P^*(2, 0, s) + \mu_3 P^*(3, 0, s) \tag{4.3.1}\]

\[(s + \lambda + \mu_1)P^*(1, 0, s) = \lambda P^*(0, 0, s) + \mu_1 P^*(1, 1, s) + \mu_2 P^*(2, 1, s) + \mu_3 P^*(3, 1, s) \tag{4.3.2}\]

\[(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s) + \mu_2 P^*(2, n+1, s) + \mu_3 P^*(3, n+1, s), \quad 1 \leq n \leq a-1 \tag{4.3.3}\]

\[(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s), \quad a \leq n \leq c-2 \tag{4.3.4}\]

\[(s + \lambda + \mu_1)P^*(1, c-1, s) = \lambda P^*(1, c-2, s) \tag{4.3.5}\]

\[(s + \lambda + \mu_2)P^*(2, 0, s) = \lambda P^*(1, c-1, s) + \mu_2 \sum_{n=c}^{d} P^*(2, n, s) + \mu_3 \sum_{n=c}^{d} P^*(3, n, s) \tag{4.3.6}\]

\[(s + \lambda + \mu_2)P^*(2, n, s) = \lambda P^*(2, n-1, s) + \mu_2 P^*(2, n+d, s) + \mu_3 P^*(3, n+d, s), \quad n \geq 1 \tag{4.3.7}\]

\[(s + \lambda + \mu_3)P^*(3, 0, s) = \mu_2 \sum_{n=a}^{c-1} P^*(2, n, s) + \mu_3 \sum_{n=a}^{c-1} P^*(3, n, s) \tag{4.3.8}\]

\[(s + \lambda + \mu_3)P^*(3, n, s) = \lambda P^*(3, n-1, s) + \mu_3 P^*(3, n+1, s), \quad n \geq 1 \tag{4.3.9}\]

From (4.3.9) we get $P^*(3, n, s) = e_3^n P^*(3, 0, s), \quad n \geq 1$
Invoking Rouche’s theorem and solving (4.3.7) as a difference equation in $P^*(2, n, s)$, we get

$$P^*(2, n, s) = P^*(2, 0, s)R^n - \frac{e_9e_3^{n+d}P^*(3, 0, s)}{K(e_3)}, \quad n \geq 1,$$

where

$$K(z) = \mu_2z^{d+1} - (s + \lambda + \mu_2)z + \lambda$$

and $R$ is the unique positive real root less than unity of the equation $K(z) = 0$

From (4.3.8) $P^*(3, 0, s) = P^*(2, 0, s).A_1$

Solving (4.3.4) as a difference equation in $P^*(1, n, s)$ we get

$$P^*(1, n, s) = A_3R^a_1p^*(1, 0, s), \quad a \leq n \leq c - 2$$

where $R_1$ is the unique positive real root less than unity of the equation

$$\mu_1z^2 - (s + \lambda + \mu_1)z + \lambda = 0$$

From (4.3.3) $P^*(1, n, s) = \left[ R_1^n - \left\{ \frac{e_5R_1^{n+1}}{K_1(R)} + \frac{e_8e_3^{n+1}A_1}{K_1(e_3)} \right\} e_1e_2R_1^{c+a-3}A_2 \right] P^*(1, 0, s), \quad 1 \leq n \leq a - 1$

where

$$K(z) = \mu_1z^2 - (s + \lambda + \mu_1)z + \lambda$$

From (4.3.5) $P^*(1, c - 1, s) = e_1A_3P^*(1, 0, s)R_1^{c-2}$

From (4.3.6) $P^*(2, 0, s) = e_1e_2A_2R_1^{c+a-3}P^*(1, 0, s)$

From (4.3.1) $P^*(0, 0, s) = e_{14} + [e_{12} + (e_{11} + e_{13}A_1)e_1e_2R_1^{c+a-3}A_2] P^*(1, 0, s)$

and

$$e_1 = \frac{\lambda}{s + \lambda + \mu_1}, \quad e_2 = \frac{\lambda}{s + \lambda + \mu_2}, \quad e_3 = \frac{\lambda}{s + \lambda + \mu_3}$$

$$e_4 = \frac{\mu_1}{s + \lambda + \mu_1}, \quad e_5 = \frac{\mu_2}{s + \lambda + \mu_1}, \quad e_6 = \frac{\mu_2}{s + \lambda + \mu_2}$$
Hence the Laplace transform of transient probabilities are

\[
e_7 = \frac{\mu_2}{s + \lambda + \mu_3} \quad e_8 = \frac{\mu_3}{s + \lambda + \mu_1} \quad e_9 = \frac{\mu_3}{s + \lambda + \mu_2}
\]

\[
e_{10} = \frac{\mu_3}{s + \lambda + \mu_4} \quad e_{11} = \frac{\lambda}{s + \lambda + \mu_1} \quad e_{12} = \frac{\mu_2}{s + \lambda}
\]

\[
e_{13} = \frac{\mu_3}{s + \lambda} \quad e_{14} = \frac{1}{s + \lambda}
\]

\[
A_1 = e_7 \frac{(R^n - R^c)}{1 - R} \left[ 1 + \left( \frac{e_7 e_9 e_{3,1}}{K(e_3)} - e_{10} \right) \left( \frac{e_3 - e_{3,1}}{1 - e_3} \right) \right]^{-1}
\]

\[
A_2 = 1 + \frac{e_1 e_2 e_5 R_{1,1}^{c+a-2}}{K_1(R)} + \frac{e_1 e_2 e_8 e_3 A_1}{K_1(e_3)} + \frac{e_6 e_9 e_3 A_1(e_3 - e_3^{d+1})}{K(e_3)(1 - e_3)} - \frac{e_9 (R^c - R^{d+1})}{1 - R} - \frac{e_9 A_1 (e_3 - e_3^{d+1})}{1 - e_3} \quad (1 - e_3)
\]

\[
A_3 = R_{1,1}^{a-1} - \frac{e_5 R^n}{K_1(R)} + \frac{e_8 e_{3,1} A_1}{K_1(e_3)} e_1 e_2 R_{1,1}^{c+a-3} A_2
\]

Hence the Laplace transform of transient probabilities are

\[
P^*(0,0,s) = P^*(1,0,s) e_{14} + \left[ e_{11} + (e_{12} + e_{13} A_1) e_1 e_2 R_{1,1}^{c+a-3} A_2 \right] \quad (4.3.10)
\]

\[
P^*(1,n,s) = P^*(1,0,s) \left[ R^n - \left\{ \frac{e_5 R^{n+1}}{K_1(R)} + \frac{e_8 e_{3,1} A_1}{K_1(e_3)} \right\} e_1 e_2 R_{1,1}^{c+a-3} A_2 \right] , \quad 1 \leq n \leq a - 1 \quad (4.3.11)
\]

\[
P^*(1,n,s) = P^*(1,0,s) A_3 R^n_1 A_1 , \quad a \leq n \leq c - 2 \quad (4.3.12)
\]

\[
P^*(1,c-1,s) = P^*(1,0,s) e_1 A_3 R_{1,1}^{c-2} \quad (4.3.13)
\]

\[
P^*(2,n,s) = P^*(2,0,s) R^n - \frac{e_9 e_3^{n+d} P^*(3,0,s)}{K(e_3)} , \quad n \geq 1 \quad (4.3.14)
\]

\[
P^*(3,0,s) = P^*(2,0,s) A_1 \quad (4.3.15)
\]

\[
P^*(3,n,s) = P^*(2,0,s) e_3 A_1 \quad (4.3.16)
\]

\[
P^*(2,0,s) = P^*(1,0,s) e_1 e_2 A_2 R_{1,1}^{c-3} \quad (4.3.17)
\]

and \( P^*(1,0,s) \) can be obtained by using the normalizing condition

\[
\sum_i \sum_n P^*(i,n,s) = \frac{1}{s} \quad \text{as}
\]

\[
P^*(1,0,s) = \left( \frac{1}{s} - e_{14} \right) A_4 \quad (4.3.18)
\]
where
\[ A_4 = \left[ e_{11} + 1 + \frac{R_1(1 - R_1^{a-1})}{1 - R_1} - \left\{ e_5 R_1^2 (1 - R_1^{a-1}) \frac{e_8 e_3^2 (1 - e_3^{a-1})}{K_1(R)(1 - R)} + \frac{e_8 e_3^2 (1 - e_3^{a-1})}{K_1(e_3)(1 - e_3)} \right\} e_1 e_2 R_1^{c+a-3} A_2 + \left\{ 1 + e_{12} + e_{13} A_1 + \frac{R}{1 - R} - \frac{e_9 e_3^{d+1} A_1}{K(e_3)(1 - e_3)} + A_1 + \frac{A_1 e_3}{1 - e_3} \right\} e_1 e_2 R_1^{c+a-3} A_2 + A_3 \left( \frac{R_1^a - R_1^{c-1}}{1 - R_1} + e_1 A_3 R_1^{c-2} \right) \right]^{-1} \]

### 4.4 Steady State Probabilities

The steady state probabilities of the system states can be obtained by using final value theorem on Laplace transforms as

\[ P(i, n) = \lim_{t \to \infty} P(i, n, t) = \lim_{s \to 0} s P^*(i, n, s) \]

Hence from (3.10) to (3.18), the steady state probabilities can be obtained as

\[ P(0, 0) = P(1, 0) \left[ \theta_1 + (\theta_2 + \theta_3 B_1) \theta_4 \theta_5 r_1^{c+a-3} B_2 \right] \]

(4.4.1)

\[ P(1, n) = P(1, 0) \left[ r_1^n - \left\{ \frac{\theta_8 r_1^{n+1}}{K_1(r)} + \frac{\theta_{11} E_1^{n+1}}{K_1(\theta_6)} \right\} \theta_4 \theta_5 r_1^{c+a-3} B_2 \right]; \]

\[ 1 \leq n \leq a - 1 \]

(4.4.2)

\[ P(1, n) = P(1, 0) B_3 r_1^n; \quad a \leq n \leq c - 2 \]

(4.4.3)

\[ P(1, c - 1) = P(1, 0) \theta_4 B_3 r_1^{c-2} \]

(4.4.4)

\[ P(2, 0) = P(1, 0) \theta_4 \theta_5 r_1 c + a - 3 B_2 \]

(4.4.5)

\[ P(2, n) = P(1, 0) \theta_4 \theta_5 r_1^{c+a-3} B_2 \left[ r_1^n - \frac{\theta_{12} e_3^{a+d} B_1}{K'(\theta_6)} \right]; \quad n \geq 1 \]

(4.4.6)

\[ P(3, 0) = P(1, 0) \theta_4 \theta_5 r_1^{c+a-3} B_1 B_2 \]

(4.4.7)

\[ P(3, n) = P(2, 0) \theta_5^n B_1 \theta_4 \theta_5 r_1^{c+a-3}; \quad n \geq 1 \]

(4.4.8)
\[ P(1,0) = \left[ 1 - \left\{ \frac{\theta_8 r^2 (1 - r^{a-1})}{K_1'(r)(1-r)} + \frac{\theta_{11} \theta_6^2 (1 - \theta_6^{c-1})}{K_1'(\theta_6)(1-\theta_6)} \right\} \theta_4 \theta_5 r_1^{c+a-3} B_2 + \theta_1 + \right. \\
\left. \left\{ 1 + \theta_2 + \theta_3 B_1 + \frac{r}{1-r} - \frac{\theta_{12} \theta_6^{d+1} B_1}{K'(\theta_6)(1-\theta_6)} + B_1 + \frac{B_1 \theta_6}{1-\theta_6} \right\} \theta_4 \theta_5 r_1^{c+a-3} B_2 + B_3 \left( \frac{r_1^a - r_1^{c-1}}{1-r_1} \right) + \theta_4 B_3 r_1^{c-2} + \frac{r_1 (1 - r_1^{a-1})}{1-r_1} \right]^{-1} \] (4.4.9)

where

\[ \theta_1 = \frac{\mu_1}{\lambda}, \quad \theta_2 = \frac{\mu_2}{\lambda}, \quad \theta_3 = \frac{\mu_3}{\lambda}, \quad \theta_4 = \frac{\lambda}{\lambda + \mu_1} \]

\[ \theta_5 = \frac{\lambda}{\lambda + \mu_2}, \quad \theta_6 = \frac{\lambda}{\lambda + \mu_3}, \quad \theta_7 = \frac{\mu_1}{\lambda + \mu_1} \]

\[ \theta_8 = \frac{\mu_2}{\lambda + \mu_1}, \quad \theta_9 = \frac{\mu_2}{\lambda + \mu_2} \]

\[ \theta_{10} = \frac{\mu_2}{\lambda + \mu_3}, \quad \theta_{11} = \frac{\mu_3}{\lambda + \mu_1}, \quad \theta_{12} = \frac{\mu_3}{\lambda + \mu_2}, \quad \theta_{13} = \frac{\mu_3}{\lambda + \mu_3} \]

\[ B_1 = \theta_{10} \left( \frac{r_1^a - r_1^{c}}{1-r} \right) \left[ 1 + \left( \frac{\theta_{10} \theta_6^d \theta_6^a}{K'(\theta_6)} - \theta_{13} \right) \left( \frac{\theta_6^a - \theta_6^c}{1-\theta_6} \right) \right]^{-1} \]

\[ B_2 = 1 + \frac{\theta_4 \theta_5 \theta_8 r_1^{c+a-2}}{K_1'(r)} + \frac{\theta_4 \theta_6 \theta_11 \theta_6^d B_1}{K_1'(\theta_6)} + \frac{\theta_9 \theta_{12} \theta_6^d B_1 (\theta_6^a - \theta_6^{d+1})}{K'(\theta_6)(1-\theta_6)} - \frac{\theta_9 (r_1^c - r_1^{d+1})}{1-r} - \frac{\theta_{12} B_1 (\theta_6^c - \theta_6^{d+1})}{(1-\theta_6)} \]

\[ B_3 = r_1^{a-1} \left[ \frac{\theta_8 r_1^a}{K_1'(r)} + \frac{\theta_{11} \theta_6^a B_1}{K_1'(\theta_6)} \right] \theta_4 \theta_5 r_1^{c+a-3} B_2 \]

where \( r \) is the unique positive real root less than unity of the equation

\[ \mu_2 z^{d+1} - (\lambda + \mu_2) z + \lambda = 0 \quad \text{and} \quad r_1 = \frac{\lambda}{\mu_1}. \]

Here for the existence of steady state distribution we assume that \( \frac{\lambda}{\mu_1} < 1 \) and \( \frac{\lambda}{d \mu_2} < 1 \)
4.5 Expected Queue Length

The expected queue length is given by

\[ L_q = \sum_{n=1}^{c-1} nP(1,n) + \sum_{n=1}^\infty nP(2,n) + \sum_{n=1}^\infty nP(3,n) \]

\[ = r_1(1-r_1)^{-2}(1-r_1^{a-1}) - \left[ \frac{\theta_3 r^2}{K'(r)}(1-r)^{-2}[1-r^{a-1}] + \frac{\theta_{11}\theta_6 B_1}{K'(\theta_6)}\theta_6(1-\theta_6)^{-2}[1-\theta_6^{a-1}] \right] \]

\[ \theta_4\theta_3 r_1^{c+a-3}B_2 + B_3 \left[ a(1-r_1)^{-1}(r_a - r_1^{c-1}) + r_1^{a+1}(1-r_1)^{-2} \right] \]

\[ [1-r_1^{c-a-2}((c-a-1)(1-r_1) + r_1)] + (c-1)B_3\theta_4 r_1^{c-2} \]

\[ + r(1-r)^{-2} + \frac{\theta_{12}\theta_6^2 B_1}{K'(\theta_6)}\theta_6(1-\theta_6)^{-2} + B_1\theta_6(1-\theta_6)^{-2} \left[ \theta_4\theta_3 r_1^{c+a-3}B_2P(1,0) \right] \]

4.6 Busy Period Distribution in Type I service

In this model, the busy period of the server in Type I service begins with the arrival of a unit in the system and lasts till the queue size is less than \( c \) for the first time. In this case the server serves the units manually with a rate \( \mu_1 \).

The distribution of the busy period of the server \( B_1 \) can be obtained by considering the states \( (0,0) \), \( (2,n) \), \( n = 0, 1, 2, \ldots \) and \( (3,n) \), \( n = 0, 1, 2, \ldots \) are absorbing. Also assume \( P(1,0,0) = 1 \)

Let \( f_{0,0}(t) = P(t \leq B_1 < t + dt, Y(t + dt) = 0, X(t + dt) = 0) \)

\( f_{2,n}(t) = P(t \leq B_1 < t + dt, Y(t + dt) = 2, X(t + dt) = n \), \( n = 0, 1, 2\ldots \) and \( f_{3,n}(t) = P(t \leq B_1 < t + dt, Y(t + dt) = 3, X(t + dt) = n \), \( n = 0, 1, 2\ldots \) Then \( f_{0,0}(t) = \frac{d}{dt}P(0,0,t) \)

\( f_{2,n}(t) = \frac{d}{dt}P(2,n,t) \), \( n = 0, 1, 2\ldots \)

and \( f_{3,n}(t) = \frac{d}{dt}P(3,n,t) \), \( n = 0, 1, 2\ldots \)

Let the Laplace transform of \( f_{0,0}(t) \), \( f_{2,n}(t) \) and \( f_{3,n}(t) \) are,
\[ f_{0,0}^*(s) = sP^*(0, 0, s) \]
\[ f_{2,n}(s) = sP^*(2, n, s) , \quad n = 0, 1, 2... \]
\[ f_{3,n}^*(s) = sP^*(3, n, s) , \quad n = 0, 1, 2... \]

Hence the Laplace transform of the busy period distribution in Type I service is

\[
b_1^*(s) = f_{0,0}^*(s) + \sum_{n=0}^{\infty} f_{2,n}^*(s) + \sum_{n=0}^{\infty} f_{3,n}^*(s) = sf_{0,0}^*(s) + \sum_{n=0}^{\infty} sP^*(2, n, s) + \sum_{n=0}^{\infty} sP^*(3, n, s)\]

The Laplace transform of the transient probabilities of the system are given by

\[
sP^*(0, 0, s) = \mu_1 P^*(1, 0, s) \quad (4.6.1) \]
\[
(s + \lambda + \mu_1)P^*(1, 0, s) - 1 = \mu_1 P^*(1, 1, s) \quad (4.6.2) \]
\[
(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n - 1, s) + \mu_1 P^*(1, n + 1, s), \quad 1 \leq n \leq c - 2 \quad (4.6.3) \]
\[
(s + \lambda + \mu_1)P^*(1, c - 1, s) = \lambda P^*(1, c - 2, s) \quad (4.6.4) \]
\[
sP^*(2, 0, s) = \lambda P^*(1, c - 1, s) \quad (4.6.5) \]

Solving (4.6.3) as a difference equation in \( P^*(1, n, s) \), we get

\[ P^*(1, n, s) = P^*(1, 0, s) R_1^n , \quad 1 \leq n \leq c - 2 \]

where \( R_1 \) is the unique positive real root less than unity of the equation

\[ \mu_1 z^2 - (s + \lambda + \mu_1) z + \lambda = 0. \]

From (4.6.1)

\[ P^*(0, 0, s) = P^*(1, 0, s) \frac{\mu_1}{s} \]
From (4.6.4) to (4.6.5)

\[ P^*(1, c-1, s) = P^*(1, 0, s) e_1 R_1^{c-2} \]
\[ P^*(2, 0, s) = P^*(1, 0, s) \frac{\lambda}{s} e_1 R_1^{c-2} \]

and \( P^*(1, 0, s) \) can be obtained by using the normalizing condition

\[ \sum_i \sum_n P^*(i, n, s) = \frac{1}{s}, \quad \text{as} \]
\[ P^*(1, 0, s) = \left\{ \mu_1 + \frac{s R_1 (1 - R_1^{c-2})}{1 - R_1} + s e_1 R_1^{c-2} + \lambda e_1 R_1^{c-2} \right\}^{-1} \]

Hence the Laplace transform of the busy period distribution is given by

\[ \tilde{B}_1 = \left. -\frac{d}{ds} b_1^*(s) \right|_{s=0} = \frac{\lambda(1 + r_1) - 2 r_1^c}{(1 - r_1)(\lambda + \mu_1 - r_1^c)} \] (4.6.6)

where \( r_1 = \frac{\lambda}{\mu_1} \).

### 4.7 Busy Period Distribution in Batch service

In this model, the busy period of the server in batch service begins when there are at least \( c \) units in the queue and lasts till the queue size is less than \( a \) for the first time after the service completion epoch of a Type II service.

The distribution of the busy period of the server \( B_2 \) can be obtained by considering the states \((0,0)\) and \((1, n)\), \( n = 0, 1, 2, \ldots, a-1 \). Also assume \( \text{P}(2,0,0)=1 \)

Let \( f_{0,0}(t) = P(t \leq B_2 < t + dt, \ Y(t + dt) = 0, \ X(t + dt) = 0) \)
\( f_{1,n}(t) = P(t \leq B_2 < t + dt, \ Y(t + dt) = 1, \ X(t + dt) = n, \ n = 0, 1, 2 \ldots a-1) \)
Then \[ f_{0,0}(t) = \frac{d}{dt} P(0, 0, t) \]
\[ f_{1,n}(t) = \frac{d}{dt} P(1, n, t), \quad n = 0, 1, 2 \ldots a - 1 \]

Let the Laplace transform of \( f_{0,0}(t) \) and \( f_{1,n}(t) \) are,

\[ f^*_0(s) = sP^*(0, 0, s) \]
\[ f^*_1(s) = sP^*(1, n, s), n = 0, 1, 2 \ldots a - 1 \]

Hence the Laplace transform of the busy period distribution in batch service is

\[ b^*_2(s) = f^*_0(s) + \sum_{n=0}^{a-1} f^*_1(s) \]
\[ = sf^*_0(s) + \sum_{n=0}^{a-1} sP^*(1, n, s) \]

The Laplace transform of the transient probabilities of the system are given by

\[ sP^*(0, 0, s) = \mu_2P^*(2, 0, s) + \mu_3P^*(3, 0, s) \] (4.7.1)
\[ sP^*(1, 0, s) = \mu_2P^*(2, 1, s) + \mu_3P^*(3, 1, s) \] (4.7.2)
\[ sP^*(1, n, s) = \mu_2P^*(2, n + 1, s) + \mu_3P^*(3, n + 1, s), \]
\[ 1 \leq n \leq a - 1 \] (4.7.3)
\[ (s + \lambda + \mu_2)P^*(2, 0, s) - 1 = \mu_2 \sum_{n=c}^{d} P^*(2, n, s) + \mu_3 \sum_{n=c}^{d} P^*(3, n, s) \] (4.7.4)
\[ (s + \lambda + \mu_2)P^*(2, n, s) = \lambda P^*(2, n - 1, s) + \mu_2P^*(2, n + d, s) \]
\[ + \mu_3P^*(3, n + d, s), \quad n \geq 1 \] (4.7.5)
\[ (s + \lambda + \mu_3)P^*(3, 0, s) = \mu_2 \sum_{n=a}^{c-1} P^*(2, n, s) + \mu_3 \sum_{n=a}^{c-1} P^*(3, n, s) \] (4.7.6)
\[ (s + \lambda + \mu_3)P^*(3, n, s) = \lambda P^*(3, n - 1, s), \quad n \geq 1 \] (4.7.7)

From (4.7.7)

\[ P^*(3, n, s) = P^*(3, 0, s)e_3^n, \quad n \geq 1 \]
Invoking Rouche’s theorem and solving (7.7) as a difference equation in $P^*(2, n, s)$, we get

$$P^*(2, n, s) = P^*(2, 0, s) R^n - \frac{e_9 e_3^{n+d} P^*(3, 0, s)}{K(e_3)}, n \geq 1$$

where

$$K(z) = \mu_2 z^{d+1} - (s + \lambda + \mu_2) z + \lambda$$

and $R$ is the unique positive real root less than unity of the equation $K(z) = 0$

From (4.7.6)

$$P^*(3, 0, s) = P^*(2, 0, s) A_1$$

From (4.7.1) to (4.7.3)

$$P^*(0, 0, s) = P^*(2, 0, s) \left\{ \frac{\mu_2}{s} + \frac{\mu_3 A_1}{s} \right\}$$

$$P^*(1, n, s) = P^*(2, 0, s) \left\{ \frac{\mu_2}{s} \left( R^{n+1} - \frac{e_9 e_3^{n+d+1} A_1}{K(e_3)} \right) + \frac{\mu_3 e_3^{n+1} A_1}{s} \right\}, 0 \leq n \leq a - 1$$

and $P^*(2, 0, s)$ can be obtained by using the normalizing condition

$$\sum_i \sum_n P^*(i, n, s) = \frac{1}{s}$$

Hence the Laplace transform of the busy period distribution is given by

$$\hat{B}_2 = \left. \frac{-d}{ds} b_2^*(s) \right|_{s=0}$$

$$= \left( (1 - \theta_6) K(\theta_6) + D_1(1 - r)[k(\theta_6) - \theta_12 \theta_6^{d+1}] \right)$$

$$\times \left\{ \mu_3 D_1 K(\theta_6)(1 - r)(1 - \theta_6^{a+1}) + \mu_2 (1 - r^{a+1})(1 - \theta_6) K(\theta_6) - \mu_2 \theta_12 \theta_6^{d+1} D_1 (1 - \theta_6^a)(1 - r) \right\}^{-1}$$

(4.7.8)
where
\[ D_1 = \theta_{10} \frac{(r_a - r_c)}{1 - r} \left[ 1 + \left( \frac{\theta_{10} \theta_{12} \theta_6^d}{K(\theta_6)} - \theta_{13} \right) \left( \frac{\theta_6^e - \theta_6^d}{1 - \theta_6} \right) \right]^{-1} \]

### 4.8 Busy Period Distribution in Type II service

In this model, the busy period of the server in Type II service begins if there are at least \( c \) units in the queue and lasts till the queue size is less than \( c \) for the first time. The server renders Type II service with rate \( \mu_2 \).

The distribution of the busy period of the server \( B_3 \) can be obtained by considering the states \((0,0),(1,n), n=0,1,2,...a-1 \) and \((3,n), n=0,1,2,... \) are absorbing. Also assume \( P(2,0,0)=1 \)

Let \( f_{0,0}(t) = P(t \leq B_3 < t + dt, Y(t + dt) = 0, X(t + dt) = 0) \)

\( f_{1,n}(t) = P(t \leq B_3 < t + dt, Y(t + dt) = 1, X(t + dt) = n, n = 0,1,2...a - 1) \)
and \( f_{3,n}(t) = P(t \leq B_3 < t + dt, Y(t + dt) = 3, X(t + dt) = n, n = 0,1,2...) \)

Then \( f_{0,0}(t) = \frac{d}{dt} P(0,0,t) \)

\( f_{1,n}(t) = \frac{d}{dt} P(1,n,t), n = 0,1,2...a - 1 \)
and \( f_{3,n}(t) = \frac{d}{dt} P(3,n,t), n = 0,1,2... \)

Let the Laplace transform of \( f_{0,0}(t), f_{1,n}(t) \) and \( f_{3,n}(t) \) are,

\( f_{0,0}^*(s) = sP^*(0,0,s) \)
\( f_{1,n}^*(s) = sP^*(1,n,s), n = 0,1,2...a - 1 \)
\( f_{3,n}^*(s) = sP^*(3,n,s), n = 0,1,2... \)
Hence the Laplace transform of the busy period distribution in Type II service is

\[ b_1^*(s) = f_{0,0}^*(s) + \sum_{n=0}^{a-1} f_{1,n}^*(s) + \sum_{n=0}^{\infty} f_{3,n}^*(s) = sf_{0,0}^*(s) + \sum_{n=0}^{a-1} sP^*(1, n, s) + \sum_{n=0}^{\infty} sP^*(3, n, s) \]

The Laplace transform of the transient probabilities of the system are given by

\[
sP^*(0, 0, s) = \mu_2 P^*(2, 0, s) \tag{4.8.1}
\]
\[
sP^*(1, 0, s) = \mu_2 P^*(2, 1, s) \tag{4.8.2}
\]
\[
sP^*(1, n, s) = \mu_2 P^*(2, n + 1, s), 1 \leq n \leq a - 1 \tag{4.8.3}
\]
\[
(s + \lambda + \mu_2)P^*(2, 0, s) - 1 = \mu_2 \sum_{n=c}^{d} P^*(2, n, s) \tag{4.8.4}
\]
\[
(s + \lambda + \mu_2)P^*(2, n, s) = \lambda P^*(2, n - 1, s) + \mu_2 P^*(2, n + d, s), \quad n \geq 1 \tag{4.8.5}
\]
\[
sP^*(3, 0, s) = \mu_2 \sum_{n=a}^{c-1} P^*(2, n, s) \tag{4.8.6}
\]

Invoking Rouche’s theorem and solving (4.8.5) as a difference equation in \(P^*(2, n, s)\) we get

\[ P^*(2, n, s) = P^*(2, 0, s)R^n, \quad n \geq 1 \]

where \(R\) is the unique positive real root less than unity of the equation

\[ \mu_2 z^{d+1} - (s + \lambda + \mu_2)z + \lambda = 0 \]

From (4.8.6)

\[ P^*(3, 0, s) = P^*(2, 0, s) \frac{(R^a - R^c)}{1 - R} \]
From (4.8.1) to (4.8.3)

\[ P^*(0, 0, s) = P^*(2, 0, s) \frac{\mu_2}{s} \]

\[ P^*(1, n, s) = P^*(2, 0, s) R^{n+1} \frac{\mu_2}{s}, \quad 0 \leq n \leq a - 1 \]

and \( P^*(2, 0, s) \) can be obtained by using the normalizing condition

\[ \sum_i \sum_n P^*(i, n, s) = \frac{1}{s} \text{ as} \]

\[ P^*(2, 0, s) = \left\{ \mu_2(1 + R^a) + \frac{\mu_2(R - R^c)}{1 - R} + s + \frac{sR}{1 - R} \right\}^{-1} \]

Hence the Laplace transform of the busy period distribution is given by

\[ \bar{B}_3 = -\frac{d}{ds} b^*(s) \bigg|_{s=0} \]

\[ = \left\{ \mu_2[(1 - r^c) + r^a(1 - r)] \right\}^{-1} \quad (4.8.7) \]

### 4.9 Busy Period Distribution in Type III service

In this model, the server begins Type III service with rate \( \mu_2 \) only if queue size becomes less than \( c \) and not less than a secondary limit \( a \), after a Type II service completion epoch. Hence the busy period of the server in Type III service begins when the number of units in the queue is less than \( c \) but not less than the limit \( a \) (i.e., \( a \leq n \leq c - 1 \)), after a Type II service completion epoch and lasts till the queue size becomes less than \( a \) or above the level \( c \) after a Type II service completion epoch. In this case the server serves the units altogether in a batch with rate \( \mu_3 \).

The distribution of the busy period of the server \( B_4 \) can be obtained by considering the states \((0,0),(1,n),n=0,1,2,...a-1\) and \((2,n),n=0,1,2,...\) are absorbing. Also assume \( P(3,0,0)=1 \)

Let \( f_{0,0}(t) = P(t \leq B_1 < t + dt, Y(t + dt) = 0, X(t + dt) = 0) \)
\[ f_{1,n}(t) = P(t \leq B_1 < t + dt, \ Y(t + dt) = 1, X(t + dt) = n, n = 0, 1, 2...a - 1) \]

and \[ f_{2,n}(t) = P(t \leq B_1 < t + dt, \ Y(t + dt) = 2, X(t + dt) = n, n = 0, 1, 2...) \]

Then \[ f_{0,0}(t) = \frac{d}{dt}P(0,0,t) \]

\[ f_{1,n}(t) = \frac{d}{dt}P(1,n,t), n = 0, 1, 2...a - 1 \]

and \[ f_{2,n}(t) = \frac{d}{dt}P(2,n,t), n = 0, 1, 2... \]

Let the Laplace transform of \( f_{0,0}(t), f_{1,n}(t) \) and \( f_{2,n}(t) \) are,

\[ f^*_{0,0}(s) = sP^*(0,0,s) \]

\[ f^*_{1,n}(s) = sP^*(1,n,s), n = 0, 1, 2...a - 1 \]

\[ f^*_{2,n}(s) = sP^*(2,n,s), n = 0, 1, 2...a - 1 \]

Hence the Laplace transform of the busy period distribution in Type III service is

\[
\begin{align*}
    b^*_4(s) &= f^*_{0,0}(s) + \sum_{n=0}^{a-1} f^*_{1,n}(s) + \sum_{n=0}^\infty f^*_{2,n}(s) \\
    &= sf^*_{0,0}(s) + \sum_{n=0}^{a-1} sP^*(1,n,s) + \sum_{n=0}^\infty sP^*(2,n,s)
\end{align*}
\]

The Laplace transform of the transient probabilities of the system are given by

\[
\begin{align*}
    sP^*(0,0,s) &= \mu_3P^*(3,0,s) \quad (4.9.1) \\
    sP^*(1,0,s) &= \mu_3P^*(3,1,s) \quad (4.9.2) \\
    sP^*(1,n,s) &= \mu_3P^*(3,n+1,s), \quad 1 \leq n \leq a - 1 \quad (4.9.3) \\
    sP^*(2,0,s) &= \mu_3 \sum_{n=c}^{d} P^*(3,n,s) \quad (4.9.4) \\
    sP^*(2,n,s) &= \mu_3P^*(3,n+d,s), \quad n \geq 1 \quad (4.9.5) \\
    (s + \lambda + \mu_3)P^*(3,0,s) - 1 &= \mu_3 \sum_{n=a}^{c-1} P^*(3,n,s) \quad (4.9.6) \\
    (s + \lambda + \mu_3)P^*(3,n,s) &= \lambda P^*(3,n-1,s), \quad n \geq 1 \quad (4.9.7)
\end{align*}
\]

From (4.9.7) \( P^*(3,n,s) = P^*(3,0,s)e_3^n, \quad n \geq 1 \)
From (4.9.5) \( P^*(2, n, s) = P^*(3, 0, s) \frac{\mu_3}{s} e_3^{n+1} \), \( n \geq 1 \).

From (4.9.4) \( P^*(2, 0, s) = P^*(3, 0, s) \frac{\mu_3}{s} \left[ e_3^{c} - \frac{e_3^{d+1}}{1 - e_3} \right] \).

From (4.9.1) to (4.9.3)

\[
P^*(0, 0, s) = P^*(3, 0, s) \frac{\mu_3}{s}
\]

\[
P^*(1, n, s) = P^*(3, 0, s) \frac{\mu_3}{s} e_3^{n+1}, \quad 0 \leq n \leq a - 1
\]

and \( P^*(3, 0, s) \) can be obtained by using the normalizing condition

\[
\sum_i \sum_n P^*(i, n, s) = \frac{1}{s}
\]

as

\[
P^*(3, 0, s) = \left\{ \frac{\mu_3}{1 - e_3} [1 + e_3^{c} - e_3^{d+1}] + \frac{s}{1 - e_3} \right\}^{-1}
\]

Hence the Laplace transform of the busy period distribution is given by

\[
\tilde{B}_4 = \left. -\frac{d}{ds} b_4^*(s) \right|_{s=0} = \mu_3 \left\{ 1 + \theta_6^c - \theta_6^{a+1} \right\}^{-1}
\]

(4.9.8)

In this model the server is idle when he is in the state (0,0). The expected length of an idle period \( E(I) \) is given by

\[
\bar{I} = \frac{1}{\lambda}
\]

Hence the expected length of a Busy cycle \( E(c) \) is given by

\[
\bar{c} = \bar{B} + \bar{I}
\]

where

\[
\bar{B} = \bar{B}_1 + \bar{B}_3 + \bar{B}_4
\]
4.10 Optimal Policy

The design of an optimal policy for a queueing system has received a lot of attention, as shown by the survey conducted by Tadj and Choudhury (2005). This is known in queueing theory as the optimal control of the system. The aim is to find the best values that the decision maker would implement in order to minimize the total expected cost per unit of time.

Let $C_h$: The holding cost per unit time in the system.

$C_a$: Start up cost per unit time for the preparatory work of the server before starting the service.

$C_0$: Set up cost per busy cycle.

$C_{s1}$: Cost per unit time for keeping the server in Type I service.

$C_{s3}$: Cost per unit time for keeping the server in Type II service.

$C_{s4}$: Cost per unit time for keeping the server in Type III service. In our model, the start up cost $C_a$ is charged only for starting the service. There is no start up cost for the subsequent service batches in a busy period.

The linear cost function constructed for determining the optimal control limits $c$ and $a$ is

$$TC1(a, c) = C_h L_q + C_{s1} \frac{B_1}{C} + C_{s3} \frac{B_3}{C} + C_{s4} \frac{B_4}{C} + C_0 \frac{1}{C} + C_a \frac{T}{C}$$
4.11 Particular Cases

Case 1: When $\mu_3 = \mu_2$

From equation (4.4.1) to (4.4.9), the steady state probabilities are obtained as follows

Here $\theta_2 = \theta_3$, $\theta_5 = \theta_6$, $\theta_9 = \theta_{10}$, and $\theta_{12} = \theta_{13}$

\[ P(0, 0) = B_4 P(2, 0) \]  
(4.11.1)

\[ P(1, 0) = B_2 P(2, 0) \]  
(4.11.2)

\[ P(1, n) = \left[ B_2 r_1^n - \frac{\theta_8 r^{n+1}}{K_1'(r)} - \frac{\theta_{11} \theta_5^{n+1} B_1}{K_1'(\theta_5)} \right] P(2, 0); \]

\[ 1 \leq n \leq a - 1 \]  
(4.11.3)

\[ P(1, n) = B_3 r_1^n P(2, 0); \quad a \leq n \leq c - 2 \]  
(4.11.4)

\[ P(1, c - 1) = \theta_4 B_3 r_1^{c-2} P(2, 0) \]  
(4.11.5)

\[ P(2, 0) = B_5 \]  
(4.11.6)

\[ P(2, n) = r^n P(2, 0) - \frac{\theta_2 \theta_5^{n+d} B_1}{K'(\theta_5)} P(2, 0); \quad n \geq 1 \]  
(4.11.7)

\[ P(3, 0) = B_1 P(2, 0) \]  
(4.11.8)

\[ P(3, n) = \theta_5^n B_1 P(2, 0); \quad n \geq 1 \]  
(4.11.9)

\[ B_1 = \theta_9 \left( \frac{r^a - r^c}{1 - r} \right) \left[ 1 + \left( \frac{\theta_9 \theta_{12} \theta_5^d}{K'(\theta_5)} - \theta_{12} \left( \frac{\theta_5^a - \theta_5^c}{1 - \theta_5} \right) \right]^{-1} \]

\[ B_2 = \frac{\theta_4 \theta_2^2 B_1 - \frac{\theta_7 \theta_5}{K_1'(r)} - \frac{\theta_2 \theta_{11} \theta_5^2 B_1}{K_1'(\theta_5)} + \theta_8 r - \frac{\theta_8 \theta_{12} \theta_5^{d+1} B_1}{K'(\theta_5)} + \theta_9 \theta_5 B_1}{1 - \theta_4 \theta_1 - \theta_7 r_1} \]

\[ B_3 = B_2 r_1^{a-1} - \frac{\theta_8 r^a}{K_1'(r)} - \frac{\theta_{11} \theta_5^2 B_1}{K_1'(\theta_5)} \]

\[ B_4 = \theta_1 B_2 + \theta_2 + \theta_2 B_1 \]
\[ B_5 = \left[ B_4 + B_2 + B_2 r_1 \frac{(1 - r_1^{a-1})}{1 - r_1} - \frac{\theta sr^2(1 - r_1^{a-1})}{K_1'(r)(1 - r)} - \frac{\theta_{11}\theta_5^2 B_1(1 - \theta_5^{a-1})}{K_1'(\theta_5)(1 - \theta_5)} \right]^{-1} + B_3 \left( \frac{r_1^a - r_1^{c-1}}{1 - r_1} \right) + \theta_4 B_3 r_1^{c-2} + 1 + \frac{r}{1 - r} - \frac{\theta_{12}\theta_5^{d+1} B_1}{K_1'(\theta_5)(1 - \theta_5)} + \frac{B_1 \theta_5}{1 - \theta_5} \]

where \( r \) is the unique positive real root less than unity of the equation \( \mu_2 z^{d+1} - (\lambda + \mu_2)z + \lambda = 0 \) and \( r_1 = \frac{\lambda}{\mu_1} \).

From equation (4.6.6), the expected busy period of the server in Type I service is,

\[ \bar{B}_1 = \frac{\lambda(1 + r_1) - 2r_1^{c}}{(1 - r_1)(\lambda + \mu_1 - r_1^{c})} \]  \hspace{1cm} (4.11.10)

From (4.7.8), the busy period of the server in batch service is,

\[ \bar{B}_2 = \left( (1 - \theta_5)K(\theta_5) + D_1(1 - r)[k(\theta_5) - \theta_{12}\theta_5^{d+1}] \right) \times \left\{ \mu_3 D_1 K(\theta_5)(1 - r)(1 - \theta_5^{a+1}) + \mu_2(1 - r^{a+1})(1 - \theta_5) K(\theta_5) - \mu_2\theta_{12}\theta_5^{d+1} D_1(1 - \theta_5^{a})(1 - r) \right\}^{-1} \]  \hspace{1cm} (4.11.11)

where

\[ D_1 = \theta_9 \frac{(r^a - r^{c})}{1 - r} \left[ 1 + \left( \frac{\theta_9\theta_{12}\theta_5^d}{K(\theta_9)} - \theta_{12} \right) \left( \frac{\theta_5^a - \theta_5^{c}}{1 - \theta_5} \right) \right]^{-1} \]

From (4.9.7), the busy period of the server in Type II service is,

\[ \bar{B}_3 = \{ \mu_2[(1 - r^{c}) + r^a(1 - r)] \}^{-1} \]  \hspace{1cm} (4.11.12)

From (4.9.8), the busy period of the server in Type III service is,

\[ \bar{B}_4 = \mu_3 \{ 1 + \theta_5^c - \theta_5^{a+1} \}^{-1} \]  \hspace{1cm} (4.11.13)

which agrees with the results of 'An \((a,c,d)\) policy M/M/1 queue with single and batch service' considered by C.Baburaj and P.P.Jayakumar (2005).
Case 2: When $\mu_3 = \mu_2$, $c=a$

Then from (4.4.1) to (4.4.9) the steady state probabilities are obtained as follows.

Here $\theta_2 = \theta_3$, $\theta_5 = \theta_6$, $\theta_9 = \theta_{10}$, and $\theta_{12} = \theta_{13}$

\[
P(0,0) = B_4 P(2,0) \quad (4.11.14)
\]

\[
P(1,0) = B_2 P(2,0) \quad (4.11.15)
\]

\[
P(1,n) = B_3 r_1^n P(2,0) \quad ; \quad 1 \leq n \leq c - 2 \quad (4.11.16)
\]

\[
P(1,c-1) = \theta_4 B_3 r_1^{c-2} P(2,0) \quad (4.11.17)
\]

\[
P(2,0) = B_5 \quad (4.11.18)
\]

\[
P(2,n) = r^n P(2,0) ; \quad n \geq 1 \quad (4.11.19)
\]

\[
B_2 = \theta_8 r \frac{\theta_7 \theta_8 r^2}{K'_1(r)}
\]

\[
B_3 = B_2 r_1^{a-1} - \frac{\theta_8 r^a}{K'_1(r)}
\]

\[
B_4 = \theta_1 B_2 + \theta_2
\]

\[
B_5 = \left[ B_4 + B_2 + B_3 \frac{(1 - r_1^{c-2})}{1 - r_1} + \theta_4 B_3 r_1^{c-2} + 1 + \frac{r}{1 - r} \right]^{-1}
\]

where $r$ is the unique positive real root less than unity of the

equation $\mu_2 z^{d+1} - (\lambda + \mu_2)z + \lambda = 0$ and $r_1 = \frac{\lambda}{\mu_1}$

From equation (4.6.6), the expected busy period of the server in Type I service is,

\[
\bar{B}_1 = \frac{\lambda (1 + r_1) - 2r_1^a}{(1 - r_1)(\lambda + \mu_1 - r_1^a)} \quad (4.11.20)
\]

From (4.7.8), the busy period of the server in batch service is,

\[
\bar{B}_2 = \left\{ \mu_2 (1 - r^{a+1}) \right\}^{-1}
\]
From (4.8.7), the busy period of the server in Type II service is,
\[ \bar{B}_3 = \{\mu_2(1-r)\}^{-1} \quad (4.11.21) \]

From (4.9.8), the busy period of the server in Type III service is,
\[ \bar{B}_4 = \mu_3 \{1 + \theta_5^c - \theta_5^{c+1}\}^{-1} \quad (4.11.22) \]

which agrees with the results of 'A single and batch service queue with a control on batch size' considered by M.Manoharan and C.Baburaj (1999).

**Case 3:** When \( \mu_1 = 0, \mu_2 = \mu_3 = \mu \)

From (4.4.1) to (4.4.9), the steady state probabilities are obtained as follows
\[
\begin{align*}
P(0,0) &= P(2,0) \frac{\mu}{\lambda} \\
P(0,n) &= P(2,0) \frac{\mu}{\lambda} \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}, \quad 1 \leq n \leq a - 1 \quad (4.11.23) \\
P(0,n) &= P(2,0) \frac{\mu}{\lambda} \left\{ \frac{1 - r^a}{1 - r} \right\}, \quad a \leq n \leq c - 1 \quad (4.11.24) \\
P(2,n) &= P(2,0) r^n, \quad n \geq 1 \quad (4.11.25)
\end{align*}
\]

and
\[
\begin{align*}
P(2,0) &= \left\{ \frac{\mu}{\lambda(1-r)} \left[ a - \frac{1 - r^a}{1 - r} + (1 - r^a)(c - a) \right] + \frac{1}{1 - r} \right\}^{-1} \quad (4.11.26)
\end{align*}
\]

where \( r \) is the unique positive real root less than unity of the equation
\[ \mu z^{d+1} - (\lambda + \mu) z + \lambda = 0. \]

The busy period of the server is,
\[ \bar{B} = \{\mu(1-r^a)\}^{-1} \quad (4.11.27) \]

These agrees with the corresponding results of the \( M/M(a,c,d)/1 \) model considered by Baburaj and Surendranath (2005).
Case 4: When $\mu_1 = 0, \mu_2 = \mu_3 = \mu, a=c$

From (4.4.1) to (4.4.9), the steady state probabilities are obtained as follows

\[
P(0, 0) = P(2, 0) \frac{\mu}{\lambda} \tag{4.11.29}
\]
\[
P(0, n) = P(2, 0) \frac{\mu}{\lambda} \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}, \quad 1 \leq n \leq c - 1 \tag{4.11.30}
\]
\[
P(2, n) = P(2, 0)r^n, \quad n \geq 1 \tag{4.11.31}
\]

and

\[
P(2, 0) = \left\{ \frac{\mu}{\lambda(1 - r)} \left[ a - \frac{1 - r^a}{1 - r} \right] + \frac{1}{1 - r} \right\}^{-1}, \tag{4.11.32}
\]

where \( r \) is the unique positive real root less than unity of the equation

\[
\mu z^{a+1} - (\lambda + \mu) z + \lambda = 0.
\]

The busy period of the server is,

\[
\bar{B} = \left\{ \mu(1 - r^a) \right\}^{-1} \tag{4.11.33}
\]

These agrees with the corresponding results of the standard $M/M(a, b)/1$ model.
4.12 Numerical Illustration

The table giving the values of Steady state probabilities for the values of $\lambda=1.6$, $\mu_1=3.5$, $\mu_2=2.5$, $\mu_3=1.7$ $a = 3$, $c = 7$ and $d = 30$ are $P(0,0) = 0.539224$ and

<table>
<thead>
<tr>
<th>n</th>
<th>$P(1,n)$</th>
<th>$P(2,n)$</th>
<th>$P(3,n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.46408E-01</td>
<td>1.25779E-04</td>
<td>1.02E-05</td>
</tr>
<tr>
<td>1</td>
<td>1.12647E-01</td>
<td>4.90846E-05</td>
<td>4.93935E-06</td>
</tr>
<tr>
<td>2</td>
<td>5.14956E-02</td>
<td>1.91549E-05</td>
<td>2.39484E-06</td>
</tr>
<tr>
<td>3</td>
<td>4.91960E-03</td>
<td>7.47510E-06</td>
<td>1.16113E-06</td>
</tr>
<tr>
<td>4</td>
<td>2.24896E-03</td>
<td>2.91711E-06</td>
<td>5.62974E-07</td>
</tr>
<tr>
<td>5</td>
<td>1.02810E-03</td>
<td>1.13838E-06</td>
<td>2.72957E-07</td>
</tr>
<tr>
<td>6</td>
<td>3.22540E-04</td>
<td>4.44247E-07</td>
<td>1.32343E-07</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1.73365E-07</td>
<td>6.41662E-08</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>6.76545E-08</td>
<td>3.11109E-08</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>2.64018E-08</td>
<td>1.50841E-08</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>1.03031E-08</td>
<td>7.31348E-09</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>4.02073E-09</td>
<td>3.54593E-09</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
The graph of the expected queue length computed using the equation (4.5.1) plotted for $\lambda = 1.6$, $\mu_1 = 3.5$, $\mu_2 = 2.5$, $\mu_3 = 1.7$, $d = 35$ and different values $a$.

**Figure 4.12.1**

**Remarks**: The Expected Queue length increases up to $a = 5$ and then it decreases.
The table giving the values of the expected cost function for the values of $\lambda=0.8$, $\mu_1=1.2$, $\mu_2=2.5$, $\mu_3=1.7$, $d = 25$, $C_h = 10$, $C_0 = 40$, $C_{s1} = 200$, $C_{s3} = 300$, $C_{s4} = 500$, $C_a = 60$.

Table 4.12.2

<table>
<thead>
<tr>
<th>$c \backslash a$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>284.6937</td>
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<td>282.5488</td>
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</tr>
<tr>
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<td>284.7379</td>
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<td>282.5092</td>
<td>283.2046</td>
<td>284.1477</td>
</tr>
<tr>
<td>13</td>
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<td>282.4888</td>
<td>283.1618</td>
<td>284.0887</td>
</tr>
<tr>
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<td>284.0513</td>
</tr>
<tr>
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<td>284.8414</td>
<td>282.6711</td>
<td>282.4754</td>
<td>283.1207</td>
<td>284.0276</td>
</tr>
<tr>
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<td>284.8635</td>
<td>282.6811</td>
<td>282.4746</td>
<td>283.1116</td>
<td>284.0126</td>
</tr>
<tr>
<td>17</td>
<td>284.8805</td>
<td>282.6893</td>
<td>282.4753</td>
<td>283.1064</td>
<td>284.0032</td>
</tr>
<tr>
<td>18</td>
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<td>282.6960</td>
<td>282.4765</td>
<td>283.1034</td>
<td>283.9972</td>
</tr>
<tr>
<td>19</td>
<td>284.9029</td>
<td>282.7011</td>
<td>282.4778</td>
<td>283.1018</td>
<td>283.9935</td>
</tr>
<tr>
<td>20</td>
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<td>282.7051</td>
<td>282.4790</td>
<td>283.1010</td>
<td>283.9912</td>
</tr>
<tr>
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<td>284.9151</td>
<td>282.7080</td>
<td>282.4800</td>
<td>283.1006</td>
<td>283.9898</td>
</tr>
</tbody>
</table>

Remarks: The $TC!(a, c)$ is minimum for $a = 4$ and $c = 16$