Chapter 3

An \((a, c, d)\) policy Bulk Service Queue with Accessible Batches under the Customer’s Choice

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3.1 Introduction

Batch service queueing models have potential applications in many areas, for example, in the loading and unloading of cargos at a seaport, in traffic signal systems, in communication systems (each message consisting of different packets), in transportation systems (buses, guided tours, medical evaluation systems, etc), in manufacturing systems (electrical, electronic and mechanical industries making products such as cars, computers, machines, etc., where each job is considered as the accumulation of different tasks), in a computer network where jobs are processed in batches. In typical batch service queueing models, once the service is started, arriving unit cannot enter the service station though enough space is available to accommodate it. But in many practical situations the arriving units will be considered for service with current batch in service with some limitation, for example cinema hall, elevator, etc. It may be noted that the accessible batch service has more economic utilizations in providing better service to the queue. For example, in many shuttle transportation systems, we observe units being transported according to accessibility rule with some limitation.

Queueing systems with accessible and non accessible batches were considered by many researchers in the past (see for example: Kleinrock (1975), Chiamsiri and leonard (1981), Sivasamy (1990), Gross and Harris (1999), Baburaj and Manoharan (1999), Baburaj (2000), Goswami and Samanta (2009) etc.). Baburaj and Surendranath (2006) considered an \((a,c,d)\) policy \(M/M/1\) bulk service queue with accessible and non accessible batches. In the model considered by Baburaj and Surendranath (2005), the server starts service only when the queue size is atleast \(c\) and continue to serve even when the system size is less than \(c\) but not less than a secondary limit \(a\)
(a ≤ c), after a service completion epoch, such a rule is called Modified General Bulk Service Rule (Modified GBSR). Baburaj and Surendranath (2006) further assumed that the server allows the late entries to join a batch in course of ongoing service till the batch size is less than the accessible limit b (a ≤ c ≤ b ≤ d).

Here a serious problem arises concerning the service of the accessible batches, where a customer arrives when the service of an accessible batch is going on. In certain cases, the arriving unit may not be interested in joining the ongoing service batch, which commenced the service before it’s arrival, since it can not get full service from that service batch. So the arriving unit has a chance either to join the ongoing service (accessible batch) or to wait in the queue till the service of the next batch begins.

Here we consider an (a, c, d) policy bulk service queue with accessible batches under the Customer’s Choice. In this model the customers arrive according to a poisson process with parameter λ. Here the server allows late arrivals to join the ongoing service till the batch size is less than b (accessible batch) and the arriving unit has a choice either to enter the accessible batch or to wait in the queue till the service of the next batch begins. Here we assume that the arriving unit join the accessible batch with probability p or with probability 1 − p he waits in the queue till the service of the next batch begins. In this model the server continue to serve even when the queue size n is less than c but not less than a secondary limit a (a ≤ c ≤ b ≤ d), after a service completion epoch. The service time distribution is assumed to be exponential with parameter µ. The rest of this paper is organized as follows: Section 2 provides the description of the model. Section 3 presents the analysis of the model. The transient and steady state behavior of the model is studied.
in sections 4. Explicit expressions for the steady state distribution, expected queue length and expected busy period are given in sections 5, 6 and 7. In section 8 the method of determining the optimal control limits $a$ and $c$ is discussed and obtained expressions for the optimal control limits $a$ and $c$.

### 3.2 Description of the Model

- Let $p(0,n)$, $(0 \leq n \leq c - 1)$ denote the probability that the server is idle and there are $n$ units waiting in the queue.

- $P(1,n)$, $(a \leq n \leq b - 1)$, denote the probability that the server is busy with an accessible batch and there are $n$ units in the queue.

- $P(2,n)$, $n=0,1,2,\ldots$ denote the probability that the server is busy with a non accessible batch and there are $n$ units in the queue (excluding those in service).

- Here the arriving unit may find the system in any of the following cases.

1. $(0,n)$, $(0 \leq n \leq c - 2)$ in this case the system is ideal.

2. $(0,c-1)$, in this case the state of transition will be $(1,0)$.

3. $(1,n)$, $(a \leq n \leq b - 2)$ in this case with probability $p$ the state of transition will be $(1,n+1)$ and with probability $1 - p$ the state of transition will be $(2,1)$.

4. $(1,b-1)$, in this case with probability $p$ the system enters to the new state $(2,0)$ and with probability $1 - p$ the new state of transition will be $(2,1)$.

5. $(2,n)$, $n=0,1,2,\ldots$ the state of transition will be $(2,n+1)$. 
3.3 Analysis of the Model

Here the state space of the system is

\[ S = S_1 \cup S_2 \cup S_3, \]

where

\[ S_1 = \{(0, n), n = 0, 1, ..., c - 1\}, \]

\[ S_2 = \{(1, n), n = a, a + 1, a + 2, ..., b - 1\} \]

and

\[ S_3 = \{(2, n), n = 0, 1, 2, ...\} \]

Following are the transitions that can be occurred during \((t, t+h]\) with the indicated probabilities

<table>
<thead>
<tr>
<th>Transitions during ((t, t+h])</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,n) \rightarrow (0,n+1), ) (1 \leq n \leq c -2 )</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((0,c-1) \rightarrow (1,0))</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (1,n+1), ) (a \leq n \leq b -2 )</td>
<td>(p\lambda h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (0,0), ) (a \leq n \leq b -1 )</td>
<td>(\mu h + O(h))</td>
</tr>
<tr>
<td>((1,b-1) \rightarrow (2,0))</td>
<td>(p\lambda h + O(h),)</td>
</tr>
<tr>
<td>((1,n) \rightarrow (2,1), ) (a \leq n \leq b -1 )</td>
<td>((1-p)\lambda h + O(h))</td>
</tr>
<tr>
<td>((2,n) \rightarrow (2,n+1), ) (n \geq 0 )</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((2,n) \rightarrow (0,n), ) (0 \leq n \leq a -1 )</td>
<td>(\mu h + O(h))</td>
</tr>
<tr>
<td>((2,n) \rightarrow (1,n), ) (a \leq n \leq b -1 )</td>
<td>(\mu h + O(h))</td>
</tr>
<tr>
<td>((2,n) \rightarrow (2,0), ) (b \leq n \leq d )</td>
<td>(\mu h + O(h))</td>
</tr>
<tr>
<td>((2,n) \rightarrow (2,n-d), ) (n &gt; d )</td>
<td>(\mu h + O(h))</td>
</tr>
</tbody>
</table>
Hence the forward equations governing the transitions are

\[ P'(0,0,t) = -\lambda P(0,0,t) + \mu \sum_{n=a}^{b-1} P(1,n,t) + \mu P(2,0,t) \quad (3.3.1) \]

\[ P'(0,n,t) = -\lambda P(0,n,t) + \lambda P(0,n-1,t) + \mu P(2,n,t), \quad 1 \leq n \leq a-1 \]

\[ P'(0,n,t) = -\lambda P(0,n,t) + \lambda P(0,n-1,t), \quad a \leq n \leq c-1 \quad (3.3.3) \]

\[ P'(1,a,t) = -(\lambda + \mu) P(1,a,t) + \mu P(2,a,t) \quad (3.3.4) \]

\[ P'(1,n,t) = -(\lambda + \mu) P(1,n,t) + p\lambda P(1,n-1,t) + \mu P(2,n,t), \quad a + 1 \leq n \leq c-1 \quad (3.3.5) \]

\[ P'(1,c,t) = -(\lambda + \mu) P(1,c,t) + \mu P(2,c,t) + \lambda P(0,c-1,t) \]

\[ +\mu P(2,c,t) \]

\[ P'(1,n,t) = -(\lambda + \mu) P(1,n,t) + p\lambda P(1,c-1,t) + \lambda P(0,c-1,t) \]

\[ +\mu P(2,c,t) \]

\[ c + 1 \leq n \leq b-1 \quad (3.3.6) \]

\[ P'(2,0,t) = -(\lambda + \mu) P(2,0,t) + \mu \sum_{n=b}^{d} P(2,n,t) + p\lambda P(1,b-1,t) \quad (3.3.8) \]

\[ P'(2,1,t) = -(\lambda + \mu) P(2,1,t) + \lambda P(2,0,t) + (1-p)\lambda \sum_{n=a}^{b-1} P(1,n,t) \]

\[ +\mu P(2,1+d,t) \]

\[ +\mu P(2,1+d,t) \]

\[ n \geq 2 \quad (3.3.9) \]

\[ P'(2,n,t) = -(\lambda + \mu) P(2,n,t) + \lambda P(2,n-1,t) + \mu P(2,n+d,t) \]

\[ n \geq 2 \quad (3.3.10) \]
3.4 Method of Solution

Let \( P^*(i, n, s) \) denote the Laplace transform of \( P(i,n,t) \). Here we assume that \( P(0,0,0) = 1 \).

Hence the Laplace transforms for the transient distribution of the system states are

\[
(s + \lambda)P^*(0,0,s) - 1 = \mu \sum_{n=a}^{b-1} P^*(1,n,s) + \mu P^*(2,0,s) \quad (3.4.1)
\]

\[
(s + \lambda)P^*(0,n,s) = \lambda P^*(0,n-1,s) + \mu P^*(2,n,s), \quad 1 \leq n \leq a - 1 (3.4.2)
\]

\[
(s + \lambda)P^*(0,n,s) = \lambda P^*(0,n-1,s) \quad , \quad a \leq n \leq c - 1 (3.4.3)
\]

\[
(s + \lambda + \mu)P^*(1,a,s) = \mu P^*(2,a,s) \quad (3.4.4)
\]

\[
(s + \lambda + \mu)P^*(1,n,s) = p\lambda P^*(1,n-1,s) + \mu P^*(2,n,s), \quad a + 1 \leq n \leq c - 1 (3.4.5)
\]

\[
(s + \lambda + \mu)P^*(1,c,s) = p\lambda P^*(1,c-1,s) + \lambda P^*(0,c-1,s) + \mu P^*(2,c,s) \quad (3.4.6)
\]

\[
(s + \lambda + \mu)P^*(1,n,s) = p\lambda P^*(1,n-1,s) + \mu P^*(2,n,s) \quad c + 1 \leq n \leq b - 1 (3.4.7)
\]

\[
(s + \lambda + \mu)P^*(2,0,s) = \mu \sum_{n=b}^{d} P^*(2,n,s) + p\lambda P^*(1,b-1,s) \quad (3.4.8)
\]

\[
(s + \lambda + \mu)P^*(2,1,s) = \lambda P^*(2,0,s) + (1-p)\lambda \sum_{n=a}^{b-1} P^*(1,n,s) + \mu P^*(2,1+d,s) \quad (3.4.9)
\]

\[
(s + \lambda + \mu)P^*(2,n,s) = \lambda P^*(2,n-1,s) + \mu P^*(2,n+d,s) \quad n \geq 2 (3.4.10)
\]

Invoking Rouche’s theorem and solving (3.4.10) as a difference equation in \( P^*(2,n,s) \),
we get

\[
P^*(2,n,s) = P^*(2,1,s),R^n, \quad n \geq 2
\]
where \( R \approx R(s) \) is the unique positive real root less than unity of the equation
\[
\mu z^{d+1} - (s + \lambda + \mu)z + \lambda = 0.
\]

From (3.4.4) and (3.4.5)
\[
P^*(1, a, s) = e_2 \cdot P^*(2, 1, s) \cdot R^a
\]
\[
P^*(1, n, s) = e_2 R^a P^*(2, 1, s) \left\{ \frac{(pe_1)^{n-a+1} - R^{n-a+1}}{pe_1 - R} \right\} , \quad a+1 \leq n \leq c-1
\]

from (3.4.7) to (3.4.9).
\[
P^*(1, n, s) = (pe_1)^{n-c} P^*(1, c, s) + e_2 P^*(2, 1, s) R^{c+1} \left\{ \frac{(pe_1)^{n-c} - R^{n-c}}{pe_1 - R} \right\} , \quad c+1 \leq n \leq b-1
\]
\[
P^*(2, 0, s) = P^*(1, c, s) D_5
\]
\[
P^*(2, 1, s) = P^*(1, c, s) D_4
\]

From (3.4.1) to (3.4.3)
\[
P^*(0, 0, s) = e_6 + P^*(1, c, s) D_6
\]
\[
P^*(0, n, s) = e_1^n e_6 + P^*(1, c, s) \left\{ e_4 D_6 + e_5 D_4 + e_5 D_4 R^2 \left( \frac{1 - R^{a-1}}{1 - R} \right) \right\} , \quad 1 \leq n \leq a-1
\]
\[
P^*(0, n, s) = e_4^n e_6 + P^*(1, c, s) e_4^{n-a+1} \left\{ e_4 D_6 + e_5 D_4 + e_5 D_4 R^2 \left( \frac{1 - R^{a-2}}{1 - R} \right) \right\} , \quad a \leq n \leq c-1
\]

where \( e_1 = \frac{\lambda}{s + \lambda + \mu} \), \( e_2 = \frac{\mu}{s + \lambda + \mu} \), \( e_3 = \frac{1}{s + \lambda + \mu} \)
\( e_4 = \frac{\lambda}{s + \lambda} \), \( e_5 = \frac{\mu}{s + \lambda} \), \( e_6 = \frac{1}{s + \lambda} \)

Hence the Laplace Transform of the transient probabilities are
\[
P^*(0, 0, s) = e_6 + D_6 P^*(1, c, s) \tag{3.4.11}
\]
\[
P^*(0, n, s) = e_4^n e_6 + P^*(1, c, s) \left\{ e_4 D_6 + e_5 D_4 + e_5 D_4 R^2 \left( \frac{1 - R^{a-1}}{1 - R} \right) \right\} , \quad 1 \leq n \leq a-1 \tag{3.4.12}
\]
\[
P^\ast(0, n, s) = e_4^n e_6 + P^\ast(1, c, s) e_4^{n-a+1} \left[ e_4 D_6 + e_5 D_4 + e_5 D_4 R^2 \left( \frac{1 - R^{a-2}}{1 - R} \right) \right]
\]
\[
a \leq n \leq c - 1 \quad (3.4.13)
\]
\[
P^\ast(1, a, s) = e_2 \cdot P^\ast(2, 1, s) \cdot R^a
\]
\[
(3.4.14)
\]
\[
P^\ast(1, n, s) = e_2 R^a P^\ast(2, 1, s) \left[ \frac{(pe_1)^{n-a+1} - R^{n-a+1}}{pe_1 - R} \right], a + 1 \leq n \leq c - 1 \quad (3.4.15)
\]
\[
P^\ast(1, n, s) = (pe_1)^{n-c} P^\ast(1, c, s) + e_2 P^\ast(2, 1, s) R^{c+1} \left[ \frac{(pe_1)^{n-c} - R^{n-c}}{pe_1 - R} \right]
\]
\[
c + 1 \leq n \leq b - 1 \quad (3.4.16)
\]
\[
P^\ast(2, 0, s) = P^\ast(1, c, s) D_5
\]
\[
(3.4.17)
\]
\[
P^\ast(2, 1, s) = P^\ast(1, c, s) D_4
\]
\[
(3.4.18)
\]
\[
P^\ast(2, n, s) = P^\ast(2, 1, s) R^n, \quad n \geq 2, \quad (3.4.19)
\]

and \(P^\ast(1, c, s)\) can be obtained by using the normalizing condition
\[
\sum_i \sum_n P^\ast(i, n, s) = \frac{1}{s}, \text{ as}
\]
\[
P^\ast(1, c, s) = D_7 D_8^{-1} \quad (3.4.20)
\]

where
\[
D_1 = e_2 \left[ e_2 R^{a(1-p)} e_1 \left[ (pe_1)^{b-c-1} - R^{b-c-1} \right] \right]
\]
\[
D_2 = e_1 D_1 + \frac{e_2 R^a (1-p)e_1}{pe_1 - R} \left[ (pe_1)^{b-c-1} - R^{b-c-1} \right] \left( \frac{1}{1 - R} \right) + \frac{1}{pe_1 - R} \left( \frac{1}{pe_1 - R} \right) \left( \frac{1}{1 - R} \right) + \frac{e_2 R^{c+1}}{pe_1 - R} \left( \frac{1}{pe_1 - R} \right) \left( \frac{1}{1 - R} \right)
\]
\[
D_3 = e_1 (pe_1)^{b-c} + (1-p)e_1 + (1-p)pe_1^2 \left( \frac{1}{1 - R} \right)
\]
\[
D_4 = \frac{D_3}{1 - D_2}
\]
\[ D_5 = D_1 D_4 + (pe_1)^{b-c} \]

\[ D_6 = e_5 D_4 \left[ e_2 R^a + \frac{e_2 R^a}{pe_1 - R} \left( (pe_1)^2 \left( \frac{1 - (pe_1)^{c-a-1}}{1 - pe_1} \right) - R^2 \frac{1 - R^{c-a-1}}{1 - R} \right) \right] + e_5 \left[ 1 + pe_1 \frac{1 - (pe_1)^{b-c-1}}{1 - pe_1} + D_5 \right] + \frac{e_2 R^{c+1}}{pe_1 - R} \left[ \frac{pe_1 1 - (pe_1)^{b-c-1}}{1 - pe_1} - R \frac{1 - R^{b-c-1}}{1 - R} \right] \]

\[ D_7 = D_6 + (e_4 D_6 + e_5 D_4) (a-1) + \frac{e_5 D_4 R^2}{1 - R} \left[ a - 1 - \frac{1 - R^{a-1}}{1 - R} \right] \]

\[ + \left[ e_4 D_6 + e_5 D_4 + \frac{e_5 D_4 R^2}{pe_1 - R} \left( \frac{1 - R^{a-2}}{1 - R} \right) \right] \left[ (pe_1)^2 \frac{1 - (pe_1)^{c-a-1}}{1 - pe_1} - R^2 \frac{1 - R^{c-a-1}}{1 - R} \right] \]

\[ + 1 + pe_1 \frac{1 - (pe_1)^{b-c-1}}{1 - pe_1} + e_2 D_4 R^{c+1} \left[ (pe_1)^2 \frac{1 - (pe_1)^{b-c-1}}{1 - pe_1} - R \frac{1 - R^{b-c-1}}{1 - R} \right] \]

\[ + D_5 + D_4 + D_4 \left( \frac{R^2}{1 - R^2} \right) \]

\[ D_8 = \frac{1}{s} e_6 \left( 1 + \frac{e_4 - e_5}{1 - e_4} \right) \]

### 3.5 Steady State Probabilities

The steady state probabilities of the system states can be obtained by using final value theorem on Laplace transforms as

\[ P(i, n) = \lim_{t \to \infty} P(i, n, t) = \lim_{s \to 0} s P^*(i, n, s) \]

Hence from (3.4.11) to (3.4.20), the steady state probabilities can be obtained as

\[ P(0, 0) = P(1, c) T_6 \quad (3.5.1) \]

\[ P(0, n) = P(1, c) \left[ T_6 + \theta_3 T_4 + \theta_3 T_4 r^2 \frac{1 - \gamma^{n-1}}{1 - r} \right], \quad 1 \leq n \leq a - 1 \quad (3.5.2) \]

\[ P(0, n) = P(1, c) \left[ T_6 + \theta_3 T_4 + \theta_3 T_4 r^2 \frac{1 - \gamma^{a-2}}{1 - r} \right], \quad a \leq n \leq c - 1 \quad (3.5.3) \]
\[ P(1, a) = P(1, c) \theta_2 r^a T_4 \]  
\[ P(1, n) = P(1, c) \theta_2 r^a T_4 \left[ \frac{(p\theta_1)^n - r^n}{p\theta_1 - r} \right], \quad a + 1 \leq n \leq c - 1 \]  
\[ P(1, n) = P(1, c) \left[ \theta_2 r^{n-c} + \theta_2 T_4 r^{c+1} \left( \frac{(p\theta_1)^n - r^n}{p\theta_1 - r} \right) \right], \quad c + 1 \leq n \leq b - 1 \]  
\[ P(2, 0) = T_5 P(1, c) \]  
\[ P(2, 1) = T_4 P(1, c) \]  
\[ P(2, n) = r^n T_4 P(1, c) \text{ and} \]  
\[ P(1, c) = T^{-1} \]

where

\[ \theta_1 = \frac{\lambda}{\lambda + \mu} , \quad \theta_2 = \frac{\mu}{\lambda + \mu} , \quad \theta_3 = \frac{\mu}{\lambda} \]

\[ T_1 = \theta_2 \left( \frac{r^b - r^{d+1}}{1 - r} + p\theta_1 r^{c+1} \left( \frac{p\theta_1)^b - r^{b-1}}{p\theta_1 - r} \right) \right) \]

\[ T_2 = T_1 \theta_1 + \theta_2 r^a (1-p)\theta_1 \left[ 1 + (p\theta_1)^2 \left( \frac{1 - (p\theta_1)^{c-a-1}}{1 - p\theta_1} - \frac{r^2 (1 - r^{c-a-1})}{1 - r} \right) \right] \]

\[ T_3 = \theta_1 (p\theta_1)^{b-c} + (1-p)\theta_1 \left[ 1 + p\theta_1 \left( \frac{1 - (p\theta_1)^{b-c-1}}{1 - p\theta_1} \right) \right] \]

\[ T_4 = \frac{T_3}{1 - T_2} \]

\[ T_5 = T_1 T_4 + (p\theta_1)^{b-c} \]

\[ T_6 = T_1 \theta_3 \left[ \theta_2 r^a \left( \frac{p\theta_1)^2 \left( \frac{1 - (p\theta_1)^{c-a-1}}{1 - p\theta_1} - \frac{r^2 (1 - r^{c-a-1})}{1 - r} \right) \right) + \frac{\theta_2 r^{c+1}}{p\theta_1 - r} \left( \frac{p\theta_1 \left( \frac{1 - (p\theta_1)^{b-c-1}}{1 - p\theta_1} - r \frac{1 - r^{b-c-1}}{1 - r} \right) + 1 + p\theta_1 \left( \frac{1 - (p\theta_1)^{b-c-1}}{1 - p\theta_1} \right)}{1 - p\theta_1} + T_5 \right) \]
\[ T = aT_6 + (a-1)\theta_3T_4 + \frac{\theta_2T_4r^2}{1-r} \left( \frac{a(a-1)}{2} - \frac{1-r^a}{1-r} \right) \\
+ (a-1) \left[ T_6 + \theta_3T_4r^2 \frac{1-r^a-2}{1-r} + r^a\theta_2T_4 + 1 + p\theta_1 \left( \frac{1-(p\theta_1)^{b-c-1}}{1-p\theta_1} \right) \right] \\
+ \frac{\theta_2r^aT_4}{p\theta_1 - r} \left[ (p\theta_1)^2 \frac{1-(p\theta_1)^{c-a-1}}{1-p\theta_1} - r^2 \frac{1-r^{c-a-1}}{1-r} \right] \\
+ \frac{\theta_2T_4r^{c+1}}{p\theta_1 - r} \left[ \frac{(p\theta_1)(1-(p\theta_1)^{b-c-1})}{1-p\theta_1} - r \left( \frac{1-r^{b-c-1}}{1-r} \right) \right] + T_5 + T_4 + \frac{T_4r^2}{1-r} \]

where \( r \) is the unique positive real root less than unity of the equation \( \mu z^{d+1} - (\lambda + \mu)z + \lambda = 0 \). Here for the existence of steady state distribution we assume that \( \frac{\lambda}{d\mu} < 1 \) and \( \frac{\lambda}{\mu} = \frac{r - r^{d+1}}{1-r} \).

### 3.6 Expected Queue Length

The expected number of units in the queue is given by,

\[
L_q = \sum_{n=1}^{c-1} nP(0,n) + \sum_{n=1}^{\infty} nP(2,n) \\
= P(1,c) \left[ (T_6 + \theta_3T_4) \frac{a(a-1)}{2} + D_6 + \theta_3D_4 + T_4 \right. \\
+ \left. \frac{\theta_2T_4r^2}{1-r} \left( \frac{a(a-1)}{2} + (1-r)^2(1-r^a) - ar^{a-1}(1-r)^{-1} \right) \right. \\
+ \left. \theta_3D_4r^2 \frac{1-r^{a-2}}{1-r} \left[ \frac{(c-a)(c+a-1)}{2} \right] + T_4r((1-r)^{-2} - 1) \right] \tag{3.6.1}
\]

### 3.7 Busy Period Distribution

Let the random variable B denote the Busy period of the server. Here the busy period commences with the start of a batch service and lasts till the system size is \( n, n = 0, 1, 2, ...a - 1 \) after a service completion epoch.
Let $Y(t)$ denote the state of the server and $X(t)$ denote the queue size at time $t$. Here $Y(t)$ can assume values 0, 1 or 2 according as the server is idle, busy with an accessible batch or busy with a non accessible batch.

The distribution of the busy period $B$ can be obtained as follows:

Let $f_n(t) = P(t \leq B < t + dt, Y(t + dt) = n, n = 0, 1, 2, \cdots, a - 1)$

Let $f_n^*(s)$ be the Laplace transform of $f_n(t)$

Then $f_n(t) = \frac{d}{dt}P(0, n, t), \quad n = 0, 1, 2, \cdots, a - 1$

$f_n^*(s) = sP^*(0, n, s), \quad n = 0, 1, 2, \cdots, a - 1$

Hence the Laplace transform of the busy period distribution is

$$b^*(s) = \sum_{n=0}^{a-1} f_n^*(s) \quad (3.7.1)$$

The Laplace transform of the distribution of the system states avoiding the states $(0,n), n=0,1, \ldots a-1$ are

$$sP^*(0,0,s) = \mu \sum_{n=a}^{b-1} P^*(1,n,s) + \mu P^*(2,0,s) \quad (3.7.2)$$

$$sP^*(0,n,s) = \mu P^*(2,n,s), \quad 1 \leq n \leq a - 1 \quad (3.7.3)$$

$$(s + \lambda + \mu)P^*(1,a,s) = \mu P^*(2,a,s) \quad (3.7.4)$$

$$(s + \lambda + \mu)P^*(1,n,s) = p\lambda P^*(1,n-1,s) + \mu P^*(2,n,s), \quad a + 1 \leq n \leq c - 1 \quad (3.7.5)$$

$$(s + \lambda + \mu)P^*(1,c,s) - 1 = p\lambda P^*(1,c-1,s) + \mu P^*(2,c,s) \quad (3.7.6)$$

$$(s + \lambda + \mu)P^*(1,n,s) = p\lambda P^*(1,n-1,s) + \mu P^*(2,n,s) \quad c + 1 \leq n \leq b - 1 \quad (3.7.7)$$
\[(s + \lambda + \mu)P^*(2, 0, s) = \mu \sum_{n=b}^{d} P^*(2, n, t) + p\lambda P^*(1, b - 1, s) \quad (3.7.8)\]

\[(s + \lambda + \mu)P^*(2, 1, s) = \lambda P^*(2, 0, s) + (1 - p)\lambda \sum_{n=a}^{b-1} P^*(1, n, s) + \mu P^*(2, 1 + d, s) \quad (3.7.9)\]

\[(s + \lambda + \mu)P^*(2, n, s) = \lambda P^*(2, n - 1, s) + \mu P^*(2, n + d, s) \quad n \geq 2 \quad (3.7.10)\]

Invoking Rouche’s theorem and solving (3.7.10) as a difference equation in \(P^*(1, n, s)\), we get

\[P^*(2, n, s) = P^*(2, 1, s) \cdot R^n, \quad n \geq 2, \quad (3.7.11)\]

where \(R \approx R(s)\) is the unique positive real root less than unity of the equation

\[\mu z^{d+1} - (s + \lambda + \mu)z + \lambda = 0.\]

From (3.7.4) to (3.7.7) we get

\[P^*(1, a, s) = P^*(2, 1, s) e_2 R^a, \quad (3.7.12)\]

\[P^*(1, n, s) = P^*(2, 1, s) e_2 R^a \left[ \frac{(pe_1)^{n-a+1} - R^{n-a+1}}{pe_1 - R} \right], \quad a + 1 \leq n \leq c - 1 \quad (3.7.13)\]

\[P^*(1, c, s) = P^*(2, 1, s) e_2 R^a \left[ \frac{(pe_1)^{c-a+1} - R^{c-a+1}}{pe_1 - R} \right] + e_3 \quad (3.7.14)\]

\[P^*(1, n, s) = P^*(2, 1, s) e_2 R^a \left[ \frac{(pe_1)^{n-a+1} - R^{n-a+1}}{pe_1 - R} \right] + e_3 (pe_1)^{n-c} \quad c + 1 \leq n \leq b - 1 \quad (3.7.15)\]

From (3.7.8)

\[P^*(2, 0, s) = B_1 P^*(2, 1, s) + e_3 (pe_1)^{n-c} \quad (3.7.16)\]
From (3.7.2) and (3.7.3)

\[ P^*(0,0,s) = P^*(2,1,s) \left[ \frac{\mu}{s} \left( e_2 R^a + \frac{e_2 R^a}{p e_1 - R} B_2 + B_1 \right) \right] + \frac{\mu}{s} e_3 \frac{1 - (p e_1)^{b-c+1}}{1 - p e_1} \]

\[ P^*(0,1,s) = \frac{\mu}{s} P^*(2,1,s) \quad (3.7.18) \]

\[ P^*(0,n,s) = \frac{\mu}{s} R^n P^*(2,1,s) , \quad 2 \leq n \leq a - 1 \quad (3.7.19) \]

Using the normalizing condition

\[ \sum_i \sum_j P^*(i,n,s) = \frac{1}{s} \]

we get

\[ P^*(2,1,s) = \frac{1 - (\mu + s)e_3 \frac{1 - (p e_1)^{b-c+1}}{1 - p e_1}}{B_3(\mu + s) - \frac{\mu R^a}{1 - R}} \quad (3.7.20) \]

where

\[ B_3 = 1 + B_1 + e_2 R^a + e_2 R^a \frac{B_2}{p e_1 - R} + \frac{R^2}{1 - R} \]

\[ B_1 = e_2 \left( \frac{R^b - R^{d+1}}{1 - R} \right) + e_2 R^a \frac{p e_1 ((p e_1)^{b-a} - R^{b-a})}{p e_1 - R} \]

\[ B_2 = (p e_1)^2 \frac{(1 - (p e_1)^{b-a+1})}{1 - p e_1} - R^2 \frac{(1 - R^{b-a+1})}{1 - R} \]

Hence the Laplace transform of the busy period distribution is given by,

\[ b^*(s) = \sum_{n=0}^{a-1} s P^*(0,n,s) \]

\[ = \frac{\mu B_3 - \mu R^a}{1 - R} + \frac{\mu R^a}{1 - R} Se_3 \frac{1 - (p e_1)^{b-c+1}}{1 - p e_1} \]

\[ = \frac{B_3(\mu + S) - \mu R^a}{B_3(\mu + S) - \mu R^a} \quad (3.7.21) \]
The expected busy period of the server is given by

\[ E(B) = B \]

\[ = -\frac{d}{ds} b^*(s)|_{s=0} \]

\[ = \left\{B(1-r)(\lambda + \mu)^{b-c+1}(\lambda + \mu - p\lambda) - \mu r^a \left[ (\lambda + \mu)^{b-c+1} - (p\lambda)^{b-c+1} \right] \right\} \]

\[ \left\{ \mu(\lambda + \mu)^{b-c+1}(\lambda + \mu - p\lambda) [B(1-r) - r^a] \right\}^{-1} \quad (3.7.22) \]

where

\[ B = 1 + \left( \frac{\mu}{\lambda + \mu} \right) \left( \frac{r^b - 1}{1-r} \right) \]

\[ + \frac{\mu}{\lambda + \mu} r^a + \frac{\mu}{\lambda + \mu} r^a \left[ \frac{(p\lambda)^{b-a+1} - (p\lambda)^{b-a}}{(\lambda + \mu)^{b-a+2}} \left( \frac{\lambda + \mu - p\lambda}{\lambda + \mu - p\lambda} \right) \right] \]

\[ - r^2 \left( 1 - r^{-b-a+1} \right) + \frac{r^2}{1-r} \]

In this model the server is idle when he is in the state \((0, n); n = 0, 1, 2, \cdots\)

The expected length of an idle period \(E(I)\) is given by

\[ I = \frac{\sum_{n=0}^{c-1} (c-n) P(0, n)}{\lambda \sum_{n=1}^{c-1} P(0, n)} \]

\[ = \frac{1}{\lambda} \left( \frac{I_1}{I_2} \right) \quad (3.7.23) \]

where

\[ I_1 = P(1, c) \left[ (T_6 + \theta_3 T_4)^{a-1} \left( \frac{a(a-1)}{2} + \frac{\theta_3 T_4 r^2}{1-r} \left( \frac{a(a-1)}{2} + (1-r)^{-2}(1-r^a) \right) \right) \right. \]

\[ - a r^{a-1} (1-r)^{-1} + \left( T_6 + \theta_3 T_4 + \theta_3 T_4 r^2 \left( \frac{1-r^{a-2}}{1-r} \right) \right) \left( \frac{(c-a)(c-a+1)}{2} \right) \]

\[ I_2 = P(1, c) \left[ T_6 + (a-1)(T_6 + \theta_3 T_4) + \frac{\theta_3 T_4 r^2}{1-r} \left( \frac{a-1}{1-r} \right) \right. \]

\[ + (c-a) \left( T_6 + \theta_3 T_4 + \frac{\theta_3 T_4 r^2}{1-r} (1-r^{a-2}) \right) \]
Hence the expected length of a Busy cycle $E(C)$ is given by

$$\bar{C} = \bar{B} + \bar{I}$$

### 3.8 Optimal Policy

The design of an optimal policy for a queueing system has received a lot of attention, as shown by the survey conducted by Tadj and Choudhury (2005). This is known in queueing theory as the optimal control of the system. The aim is to find the best values that the decision maker would implement in order to minimize the total expected cost per unit of time.

Let $C_h$: The holding cost per unit time in the system.

$C_a$: Start up cost per unit time for the preparatory work of the server before starting the service.

$C_s$: Set up cost per busy cycle.

$C_0$: Cost per unit time for keeping the server on and in operation. In our model, the start up cost $C_a$ is charged only for starting the service. There is no start up cost for the subsequent service batches in a busy period.

The linear cost function constructed for determining the optimal control limits $c$ and $a$ is

$$TC1(a, c) = C_h L_q + C_0 \frac{\bar{B}}{C} + C_s \frac{1}{C} + C_a \frac{\bar{I}}{C}$$
### 3.9 Particular Cases

**Case 1:** When \( p=1 \)

From equation (3.5.1) to (3.5.10), The steady state probabilities are

\[
P(0,0) = P(1,c)T_6 \quad \text{(3.9.1)}
\]

\[
P(0,n) = P(1,c) \left[ T_6 + \theta_3 T_4 + \theta_5 T_4 r^2 \frac{1 - r^{n-1}}{1 - r} \right], 1 \leq n \leq a - 1 \quad \text{(3.9.2)}
\]

\[
P(0,n) = P(1,c) \left[ T_6 + \theta_3 T_4 + \theta_5 T_4 r^2 \frac{1 - r^{a-2}}{1 - r} \right], a \leq n \leq c - 1 \quad \text{(3.9.3)}
\]

\[
P(1,a) = P(1,c) \theta_2 r^a T_4 \quad \text{(3.9.4)}
\]

\[
P(1,n) = P(1,c) \theta_2 r^a T_4 \left[ \frac{\theta_1^{n-a+1} - r^{n-a+1}}{\theta_1 - r} \right], a + 1 \leq n \leq c - 1 \quad \text{(3.9.5)}
\]

\[
P(1,n) = P(1,c) \left[ \theta_1^{n-c} + \theta_2 T_4 r^{c+1} \left( \frac{\theta_1^{n-c} - r^{n-c}}{\theta_1 - r} \right) \right],
\quad c + 1 \leq n \leq b - 1 \quad \text{(3.9.6)}
\]

\[
P(2,0) = T_5 P(1,c) \quad \text{(3.9.7)}
\]

\[
P(2,1) = T_4 P(1,c) \quad \text{(3.9.8)}
\]

\[
P(2,n) = r^n T_4 P(1,c) \quad \text{and} \quad \text{(3.9.9)}
\]

\[
P(1,c) = T^{-1} \quad \text{(3.9.10)}
\]

where

\[
\theta_1 = \frac{\lambda}{\lambda + \mu}, \quad \theta_2 = \frac{\mu}{\lambda + \mu}, \quad \theta_3 = \frac{\mu}{\lambda}
\]

\[
T_1 = \theta_2 \left( \frac{r^b - r^{d+1}}{1 - r} + \theta_1 r^{c+1} \frac{\theta_1^{b-c-1} - r^{b-c-1}}{\theta_1 - r} \right)
\]

\[
T_2 = T_1 \theta_1
\]

\[
T_3 = \theta_1^{b-c+1}
\]
The expected busy period of the server is given by

\[ T_4 = \frac{T_3}{1 - T_2} \]

\[ T_5 = T_1 T_4 + \theta_1 \]

\[ T_6 = T_4 \theta_3 \left[ \frac{\theta_2 r^a + \theta_2 r^a}{\theta_1 - r} \left( \theta_1 \frac{1 - (\theta_1)^{c-a-1}}{1 - \theta_1} - r^2 \frac{1 - r^{c-a-1}}{1 - r} \right) \right. \]

\[ + \frac{\theta_2 r^{c+1}}{\theta_1 - r} \left( \theta_1 \frac{1 - (\theta_1)^{b-c-1}}{1 - \theta_1} - r \frac{1 - r^{b-c-1}}{1 - r} \right) \right] + 1 + \theta_1 \frac{1 - (\theta_1)^{b-c-1}}{1 - \theta_1} + T_5 \]

\[ T = a T_6 + (a-1) \theta_3 T_4 + \frac{\theta_3 T_4 r^2}{1 - r} \left( \frac{a(a - 1)}{2} - \frac{1 - r^a}{1 - r} \right) \]

\[ + (a-1) \left[ T_6 + \frac{\theta_3 T_4 r^2}{1 - r} \left( \theta_1 \frac{1 - (\theta_1)^{c-a-1}}{1 - \theta_1} - r^2 \frac{1 - r^{c-a-1}}{1 - r} \right) \right. \]

\[ + \frac{\theta_2 r^a T_4}{\theta_1 - r} \left( \frac{1 - (\theta_1)^{c-a-1}}{1 - \theta_1} - r^2 \frac{1 - r^{c-a-1}}{1 - r} \right) \]

\[ + \frac{\theta_2 T_4 r^{c+1}}{\theta_1 - r} \left( \frac{1 - (\theta_1)^{b-c-1}}{1 - \theta_1} - r \frac{1 - r^{b-c-1}}{1 - r} \right) \right] + T_5 + T_4 + \frac{T_4 r^2}{1 - r} \]

The expected busy period of the server is given by

\[ E(B) = \bar{B} \]

\[ = \frac{-d}{ds} b^*(s) \big|_{s=0} \]

\[ = \left\{ B \mu(1 - r)(\lambda + \mu)^{b-c+1} - \mu r^a \left[ (\lambda + \mu)^{b-c+1} - (\lambda)^{b-c+1} \right] \right\} \]

\[ \left\{ \mu^2(\lambda + \mu)^{b-c+1} [B(1 - r) - r^a] \right\}^{-1} \]

(3.9.11)

where

\[ B = 1 + \left( \frac{\mu}{\lambda + \mu} \right) \left( \frac{r^b - r^{d+1}}{1 - r} \right) + \frac{\mu}{(\lambda + \mu)^{b-a+1}} r^a \lambda \frac{(\lambda)^{b-a} - ((\lambda + \mu)r)^{b-a}}{(\lambda - r(\lambda + \mu))} \]

\[ + \frac{\mu}{\lambda + \mu} r^a + \frac{\mu}{\lambda + \mu} r^a \left[ \frac{(\lambda)^2}{(\lambda + \mu)^{b-a+2}} \frac{(\lambda + \mu)^{b-a+1} - (\lambda)^{b-a+1}}{\mu} \right] \]

\[ - r^2 \frac{1 - r^{b-a+1}}{1 - r} \right] + \frac{r^2}{1 - r} \]
These agrees with the corresponding results of the $M/M(a, c, d)/1$ model with accessible and non-accessible batches considered by Baburaj and Surendranath (2006).

**Case 2:** When $p=0$ and $b=d$

Then from (3.5.1) to (3.5.10) the steady state probabilities are

\[
P(0, 0) = P(1, c)T_5 \quad \text{(3.9.12)}
\]
\[
P(0, n) = P(1, c) \left[ T_6 + \theta_3 T_4 + \theta_3 T_4 r^2 \frac{1 - r^{n-1}}{1 - r} \right],
\]
\[
1 \leq n \leq a - 1 \quad \text{(3.9.13)}
\]
\[
P(0, n) = P(1, c) \left[ T_6 + \theta_3 T_4 + \theta_3 T_4 r^2 \frac{1 - r^{a-2}}{1 - r} \right],
\]
\[
a \leq n \leq c - 1 \quad \text{(3.9.14)}
\]
\[
P(1, n) = P(1, c) \theta_2 r^n T_4, a \leq n \leq c - 1 \quad \text{(3.9.15)}
\]
\[
P(2, 0) = T_5 P(1, c) \quad \text{(3.9.16)}
\]
\[
P(2, 1) = T_4 P(1, c) \quad \text{(3.9.17)}
\]
\[
P(2, n) = r^n T_4 P(1, c) \quad \text{and} \quad \text{(3.9.18)}
\]
\[
P(1, c) = T^{-1} \quad \text{(3.9.19)}
\]

where

\[
\theta_1 = \frac{\lambda}{\lambda + \mu}, \quad \theta_2 = \frac{\mu}{\lambda + \mu}, \quad \theta_3 = \frac{\mu}{\lambda}
\]
\[
T_1 = \theta_2 r^d
\]
\[
T_2 = T_1 \theta_1 + \theta_2 r^\alpha \theta_1 \left[ 1 - \frac{r^2(1 - r^{c-a-1})}{1 - r} \right]
\]
\[
T_4 = \frac{\theta_1}{1 - T_2}
\]
\[ T_5 = T_1 T_4 + (p \theta_1)^{d-c} \]

\[ T_6 = T_4 \theta_3 \left[ \theta_2 T_{a}^2 + \theta_2 T_{a+1} \left( \frac{1 - r^{c-a-1}}{1 - r} \right) + \theta_2 T_{c+1} \left( \frac{1 - r^{d-c-1}}{1 - r} \right) \right] + 1 + T_5 \]

\[ T = aT_6 + (a-1) \theta_3 T_4 + \frac{\theta_3 T_4 T_2}{1 - r} \left( \frac{a(a - 1)}{2} - \frac{1 - r^a}{1 - r} \right) + (a - 1) \left[ T_6 + \theta_3 T_4 r^2 \frac{1 - r^{a-2}}{1 - r} + r^a \theta_2 T_4 + 1 + \theta_2 T_{a+1} \right] + 1 + T_5 + T_4 + \frac{T_4 T_2}{1 - r} \]

The expected busy period of the server is,

\[ E(T) = \frac{1}{\mu(1 - r^a)} \quad (3.9.20) \]

These agrees with the corresponding results of the \( M/M(a, c, d)/1 \) model considered by Baburaj and Surendranath (2005) .
3.10 Numerical Illustration

The table giving the values of Steady state probabilities for the values of $\lambda=3.5$, $\mu=1.9$, $a=8$, $c=26$, $b=30$ and $d=40$.

Table 3.10.1

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<th>p(0,n)</th>
<th>p(1,n)</th>
<th>p(2,n)</th>
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</tr>
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</tr>
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When $p=0.3$

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<th>p(2,n)</th>
</tr>
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<td>0</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

When $p=0.9$
The expected queue length plotted for $\lambda=3.5$, $\mu=1.9$, $a=5$, $b=30$, $d=40$ and for different values of $p$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.10.1.png}
\caption{Figure 3.10.1}
\end{figure}

**Remark**: From Figure 3.10.1 we can see that as the value of $p$ increases, the expected queue length decreases.
The expected queue length plotted for $\lambda=3.5$, $\mu=1.9$, $c=25$, $b=30$, $d=40$ and for different values of $p$

![Figure 3.10.2](image)

**Remark**: From Figure 3.10.2 we can see that as the value of $p$ increases, the expected queue length decreases.
The expected busy period plotted for $\lambda=3.5$, $\mu=1.9$, $a=5$, $b=30$, $d=40$ and for different values of $p$.

Figure 3.10.3

Remark: From Figure 3.10.3 we can see that as the value of $p$ increases, the expected busy period increases.
The expected busy period plotted for $\lambda=3.5$, $\mu=1.9$, $c=25$, $b=30$, $d=40$ and for different values of $p$

**Figure 3.10.4**

**Remark**: From Figure 3.10.4 we can see that as the value of $p$ increases, the expected busy period increases.
The table giving the values of the expected cost function for the values of $\lambda=3.5,$ $\mu=1.9$, $p=0.1$, $b=30$, $d=40$, $C_h=50$, $C_0=400$, $C_s=50$, $C_a=30$.

Table 3.10.2

<table>
<thead>
<tr>
<th>a\c</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
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<td>1.5372</td>
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<td>1.5367</td>
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<td>1.4705</td>
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<td>1.4668</td>
<td>1.4677</td>
<td>1.4714</td>
<td>1.4813</td>
</tr>
<tr>
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<td>1.4490</td>
<td>1.4464</td>
<td>1.4455</td>
<td>1.4465</td>
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<td>1.4582</td>
</tr>
<tr>
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<td>1.4426</td>
<td>1.4399</td>
<td>1.4390</td>
<td>1.4399</td>
<td>1.4430</td>
<td>1.4502</td>
</tr>
<tr>
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<td>1.4462</td>
<td>1.4415</td>
<td>1.4388</td>
<td><strong>1.4378</strong></td>
<td>1.4386</td>
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<tr>
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<td>1.4422</td>
<td>1.4394</td>
<td>1.4384</td>
<td>1.4392</td>
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<td>1.4476</td>
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<tr>
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<td>1.4435</td>
<td>1.4406</td>
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</tr>
<tr>
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<td>1.4422</td>
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</tbody>
</table>

**Remark:** Here $TC_1(a, c)$ is minimum for $a = 5$ and $c = 12$. 
The table giving the values of the expected cost function for the values of \( \lambda = 20 \), \( \mu = 2 \), \( p = 0.8 \), \( b = 30 \), \( d = 40 \), \( C_h = 50 \), \( C_0 = 400 \), \( C_s = 50 \), \( C_a = 30 \).

Table 3.10.3

<table>
<thead>
<tr>
<th>a ( \text{c} )</th>
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<th>12</th>
<th>13</th>
<th>14</th>
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<tbody>
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</tr>
<tr>
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<td>1.3141</td>
<td>1.3100</td>
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<td>1.3151</td>
<td>1.3312</td>
</tr>
</tbody>
</table>

**Remark:** Here \( TC_1(a, c) \) is minimum for \( a = 6 \) and \( c = 12 \).