Chapter 2

An \((a, c, d)\) policy M/M/1 Bulk Service Queue with Bernoulli Schedule

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2.1 Introduction

Single server bulk service queueing systems are considered by many researchers in the past (see for example: Bailey(1954), Bhat(1964), Chaudhry and Templeton (1972), Neuts (1967), Medhi (1975), etc.). Neuts (1967) considered a model with general bulk service rule, where the server starts service only when a specified number of units are present in the queue and if after a service completion epoch, the queue size is less than that limit, then the server stops service. An $M/M/1$ queue under $(M, N)$ policy is considered in Bhom and Mohanty (1993), where the server begins service if the queue size is at least $N$ and continue to serve even when the queue size is less than $M$, after a service completion epoch.

There are many real life situations where service is rendered with a control limit policy $(a, c, d)$. In these models the server begins service when the queue size accumulated to a specified level $c$ and serves a maximum of $d$ units in a batch. Here the server continue to serve even when the queue size is less than $c$ but not less than a secondary limit $a$, after a service completion epoch. If the queue size is below $a$ after a service completion epoch, then the server becomes idle. Suppose a set up cost is required for initiating the batch service. Once the batch service is started no set up cost is required for further service batches. So the server continue to serve even when the queue size is less than the specified level but not less than a secondary limit $a$.

Baburaj and Surendranath (2005) considered an $M/M/1$ queue under the policy $(a, c, d)$. In this study we consider an $(a, c, d)$ policy bulk service queue under Bernoulli schedule. Here the server begins service only if the number of units in the queue $n$ is at least $c$ and serves a maximum of $d$ units in a batch. If after a service completion epoch the queue size $n$ is less than $c$ but not less than a secondary limit $a$ ($a \leq c \leq d$), then
with probability $p$ the server serves them all together in a batch and with probability $1-p$ the server becomes idle.

After a service completion epoch the server may find the system in any of the following cases (i) $n < a$, (ii) $n \geq a$. In case(i), the server becomes idle and in case(ii), he serves a maximum of $d$ units in a batch. The service time distribution is assumed to be exponential with parameter $\mu$ and the arrival process is assumed to be Poisson with parameter $\lambda$. The transient and steady state behaviour of the model is studied. Explicit expressions for the steady state distribution, expected queue length, expected busy period, expected waiting time in the queue, are obtained. The method of determining the optimal control limits $a$ and $c$ is also discussed and obtained expressions for the optimal control limits $a$ and $c$.

2.2 Analysis of the Model

Let the random variables $Y(t)$ and $X(t)$ respectively denote the state of the server and the queue size at time $t$. Here $Y(t)$ can assume the values 0 or 1 according as the server is idle or busy at time $t$.

The stochastic process $\{Y(t), X(t), t \geq 0\}$ forms a Markov process with state space

$$S = S_1 \cup S_2,$$

where

$$S_1 = \{(0, n), n = 0, 1, ..., c - 1\},$$

$$S_2 = \{(1, n), n = 0, 1, 2, ...\}$$

Let $P(i, n, t) = P\{Y(t) = i, X(t) = n\}$
Following are the transitions that can be occurred during \((t, t+h]\) with the indicated probabilities

<table>
<thead>
<tr>
<th>Transitions during ((t, t+h])</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,n) \rightarrow (0,n+1), \quad 0 \leq n \leq c-2)</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((0,c-1) \rightarrow (1,0))</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (1,n+1), \quad n \geq 0)</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (0,n), \quad 0 \leq n \leq a-1)</td>
<td>(\mu h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (1,0), \quad a \leq n \leq c-1)</td>
<td>(p\mu h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (0,0), \quad a \leq n \leq c-1)</td>
<td>((1-p)\mu h + O(h))</td>
</tr>
<tr>
<td>((1,n) \rightarrow (1,n-d), \quad n &gt; d)</td>
<td>(\mu h + O(h))</td>
</tr>
</tbody>
</table>

Hence the forward equations governing the transitions are

\[
P'(0,0,t) = -\lambda P(0,0,t) + \mu P(1,0,t) \quad (2.2.1)
\]

\[
P'(0,n,t) = -\lambda P(0,n,t) + \lambda P(0,n-1,t) + \mu P(1,n,t), \quad 1 \leq n \leq a - 1 \quad (2.2.2)
\]

\[
P'(0,n,t) = -\lambda P(0,n,t) + \lambda P(0,n-1,t) + (1-p)\mu P(1,n,t),
\quad a \leq n \leq c - 1
\]

\[
P'(1,0,t) = -(\lambda + \mu)P(1,0,t) + \lambda P(0,c-1,t) + \mu \sum_{n=a}^{c-1} P(1,n,t)
\quad + \mu \sum_{n=c}^{d} P(1,n,t) \quad (2.2.4)
\]

\[
P'(1,n,t) = -(\lambda + \mu)P(1,n,t) + \lambda P(1,n-1,t) + \mu P(1,n+d,t), n \geq 1 \quad (2.2.5)
\]
2.3 Method of Solution

Let $P^*(i,n,s)$ denote the Laplace transform of $P(i,n,t)$. Here we assume that $P(0,0,0)=1$.

Hence the Laplace transforms for the transient distribution of the system states are

\[(s + \lambda)P^*(0,0,s) - 1 = \mu P^*(1,0,s)\] (2.3.1)

\[(s + \lambda)P^*(0,n,s) = \lambda P^*(0,n-1,s) + \mu P^*(1,n,s), \quad 1 \leq n \leq a - 1\] (2.3.2)

\[(s + \lambda)P^*(0,n,s) = \lambda P^*(0,n-1,s) + (1 - p)\mu P^*(1,n,s), \quad a \leq n \leq c - 1\] (2.3.3)

\[(s + \lambda + \mu)P^*(1,0,s) = \lambda P^*(0,c - 1,s) + p\mu \sum_{n=a}^{c-1} P^*(1,n,s) + \mu \sum_{n=c}^{d} P^*(1,n,s)\] (2.3.4)

\[(s + \lambda + \mu)P^*(1,n,s) = \lambda P^*(1,n - 1,s) + \mu P^*(1,n + d,s), \quad n \geq 1\] (2.3.5)

Invoking Rouche’s theorem and solving (2.3.5) as a difference equation in $P^*(1,n,s)$, we get

\[P^*(1,n,s) = P^*(1,0,s).R^n, \quad n \geq 1,\]

where $R \approx R(s)$ is the unique positive real root less than unity of the equation

\[\mu z^{d+1} - (s + \lambda + \mu)z + \lambda = 0.\]

From (2.3.1) to (2.3.3)

\[P^*(0,0,s) = P^*(1,0,s).e_2 + \frac{1}{s + \lambda}\]

\[P^*(0,n,s) = P^*(1,0,s).e_2 \left\{ \frac{e_1^n - R^n}{e_1 - R} \right\} + \frac{e_1^n}{s + \lambda}, \quad 1 \leq n \leq a - 1\]

where

\[e_1 = \frac{\lambda}{s + \lambda} \quad \text{and} \quad e_2 = \frac{\mu}{s + \lambda}.\]
\[
P^*(0, n, s) = P^*(1, 0, s) e_2 \left\{ e^{a-1 - R^{a-1}/e_1 - R} e^{n+1-a} + (1-p) R^a \left[ e^{n-a - R^{n-a}}/e_1 - R \right] \right\} \\
+ e^{n}/s + \lambda, \quad a \leq n \leq c - 1
\]

Hence the Laplace transform of the transient probabilities are

\[
P^*(0, 0, s) = P^*(1, 0, s) e_2 + \frac{1}{s + \lambda} \tag{2.3.6}
\]

\[
P^*(0, n, s) = P^*(1, 0, s) e_2 \left\{ e^{n+1 - R^{n+1}/e_1 - R} \right\} + e^{n}/s + \lambda, \quad 1 \leq n \leq a - 1 \tag{2.3.7}
\]

\[
P^*(0, n, s) = P^*(1, 0, s) e_2 \left\{ e^{a - R^{a}}/e_1 - R e^{n+1-a} \right\} \\
+ (1-p) R^a \left[ e^{n+1-a - R^{n+1-a}}/e_1 - R \right] \right\} + e^{n}/s + \lambda, \quad a \leq n \leq c - 1 \tag{2.3.8}
\]

\[
P^*(1, n, s) = P^*(1, 0, s) R^n, \quad n \geq 1 \tag{2.3.9}
\]

and \(P^*(1, 0, s)\) can be obtained by using the normalizing condition

\[
\sum_{i} \sum_{n} P^*(i, n, s) = \frac{1}{s}, \quad \text{as}
\]

\[
P^*(1, 0, s) = T_1 T_2 \tag{2.3.10}
\]
where

\[ T_1 = \left\{ \frac{e_2}{e_1 - R} \left[ e_1 - e_1^{c+1} - pR^a e_1 \frac{1 - e_1^{c-a}}{1 - e_1} - \frac{R}{1 - R} (1 - pR^a - R^c (1 - p)) \right] \right\}^{-1} + \frac{1}{1 - R}, \]

\[ T_2 = \frac{\lambda(1 - e_1) - s.e_1 + s.e_1^c}{s(s + \lambda)(1 - e_1)}. \]

### 2.4 Steady State Probabilities

The steady state probabilities of the system states can be obtained by using final value theorem on Laplace transforms as

\[ P(i, n) = \lim_{t \to \infty} P(i, n, t) = \lim_{s \to 0} sP^*(i, n, s) \]

Hence from (2.3.6) to (2.3.10), the steady state probabilities can be obtained as

\[ P(0, 0) = P(1, 0) \theta_1 \]  \hspace{1cm} (2.4.1)

\[ P(0, n) = P(1, 0) \theta_1 \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}, \quad 1 \leq n \leq a - 1 \]  \hspace{1cm} (2.4.2)

\[ P(0, n) = P(1, 0) \theta_1 \left\{ \frac{1 - r^a}{1 - r} + (1 - p) \left[ \frac{r^a - r^{n+1}}{1 - r} \right] \right\}, \quad a \leq n \leq c - 1 \]  \hspace{1cm} (2.4.3)

\[ P(1, n) = P(1, 0) r^n, \quad n \geq 1 \]  \hspace{1cm} (2.4.4)

\[ P(1, 0) = \left\{ \frac{\theta_1}{1 - r} \left[ c - pr^a(c - a) - \frac{r}{1 - r} (1 - r^c) + \frac{pr}{1 - r} (r^a - r^c) \right] \right\}^{-1} \]  \hspace{1cm} (2.4.5)

where \( \theta_1 = \frac{\mu}{\lambda} \) and \( r \) is the unique positive real root less than unity of the equation \( \mu z^{d+1} - (\lambda + \mu) z + \lambda = 0 \). Here for the existence of steady state distribution we assume that \( \frac{\lambda}{d\mu} < 1 \) and \( \frac{\lambda}{\mu} = \frac{r - r^{d+1}}{1 - r} \).
2.5 Expected Queue Length

In this model there will be no queue if the system is in states (0,0) or (1,0) and hence the expected queue length is given by

\[
L_q = \sum_{n=1}^{c-1} nP(0,n) + \sum_{n=1}^{\infty} nP(1,n)
\]
\[
= P(1,0) \frac{\theta_1}{1-r} \left\{ \left( \frac{c(c-1)}{2} - \frac{r^2}{(1-r)^2} \right) (1 - pr^a) + apr^a \left( \frac{a-1}{2} + \frac{r}{1-r} \right) \right\} + \frac{(1-p)}{1-r} r^{c+1} \left[ c + \frac{r}{1-r} \right] + P(1,0) r \frac{(1-p) r}{(1-r)^2}
\]

(2.5.1)

2.6 Busy Period Distribution

In this model with probability \( p \) the busy period \( B \) commences with the start of a batch service and lasts till the system size is \( n, n = 0, 1, \cdots, a-1 \) after a service completion epoch and with probability \( 1-p \), the busy period \( B \) commences with the start of a batch service and lasts till the system size is \( n, n = 0, 1, 2, \cdots, c-1 \).

The distribution of the busy period \( B \) can be obtained as follows:

Let \( f_{1,j}(t) = P(t \leq B < t + dt, Y(t + dt) = 0, X(t + dt) = j) \),

\( j = 0, 1, 2, \cdots, a-1 \)

and \( f_{2,j}(t) = P(t \leq B < t + dt, Y(t + dt) = 0, X(t + dt) = j) \),

\( j = 0, 1, 2, \cdots, c-1 \)

Then \( f_{1,j}(t) = \frac{d}{dt} P(0,j,t) \), \( j = 0, 1, 2, \cdots, a-1 \)

and \( f_{2,j}(t) = \frac{d}{dt} P(0,j,t) \), \( j = 0, 1, 2, \cdots, c-1 \)

Let \( f_{i,j}^*(s) \) denote the Laplace transform of \( f_{i,j}(t) \). Then \( f_{1,j}^*(s) = sP^*(0,j,s) \), \( j = 0, 1, 2, \cdots, a-1 \) and
\[ f_{2,j}^*(s) = sP^*(0,j,s), \quad j = 0, 1, 2, \cdots, c - 1 \]

Hence the Laplace transform of the busy period distribution is

\[
b^*(s) = p \sum_{j=0}^{a-1} f_{1,j}^*(s) + (1-p) \sum_{j=0}^{c-1} f_{2,j}^*(s)
\]

\[
= \sum_{j=0}^{a-1} sP^*(0,j,s) + (1-p) \sum_{j=a}^{c-1} sP^*(0,j,s) \tag{2.6.1}
\]

The Laplace transform of the transient probabilities of the system are given by

\[
sP^*(0,0,s) = \mu P^*(1,0,s) \tag{2.6.2}
\]

\[
sP^*(0,n,s) = \mu P^*(1,n,s), 1 \leq n \leq a - 1 \tag{2.6.3}
\]

\[
sP^*(0,n,s) = (1-p)\mu P^*(1,n,s), a \leq n \leq c - 1 \tag{2.6.4}
\]

\[
(s + \lambda + \mu)P^*(1,0,s) - 1 = p\mu \sum_{n=a}^{c-1} P^*(1,n,s) + \mu \sum_{n=c}^{d} P^*(1,n,s) \tag{2.6.5}
\]

\[
(s + \lambda + \mu)P^*(1,n,s) = \lambda P^*(1,n-1,s) + \mu P^*(1,n+d,s) \tag{2.6.6}
\]

Invoking Rouche’s theorem and solving (2.6.6) as a difference equation in \( P^*(1,n,s) \), we get

\[
P^*(1,n,s) = P^*(1,0,s).R^n, n \geq 1,
\]

where \( R \cong R(s) \) is the unique positive real root less than unity of the equation

\[
\mu z^{d+1} - (s + \lambda + \mu)z + \lambda = 0.
\]

From (2.6.2) to (2.6.4)

\[
P^*(0,0,s) = P^*(1,0,s)\frac{\mu}{s}
\]

\[
P^*(0,n,s) = P^*(1,0,s)\frac{\mu}{s}.R^n, 1 \leq n \leq a - 1
\]

\[
P^*(0,n,s) = P^*(1,0,s)(1-p)\frac{\mu}{s}.R^n, a \leq n \leq c - 1
\]
and \( P^*(1, 0, s) \) can be obtained by using the normalizing condition

\[
\sum_i \sum_n P^*(i, n, s) = \frac{1}{s}, \quad \text{as}
\]

\[
P^*(1, 0, s) = \left\{ \mu \cdot \left( \frac{1 - R^c - pr^a - R^c}{1 - R} \right) + \frac{s}{1 - R} \right\}^{-1}
\]

where \( e_3 = \frac{\mu}{s + \lambda + \mu} \) and \( e_4 = \frac{1}{s + \lambda + \mu} \).

In this model for \( p = 1 \), the terms \( P^*(0, j, s), j = a, a + 1, ..., c - 1 \) vanish and these values exist only when \( p = 0 \).

Hence the Laplace transform of the busy period distribution is given by

\[
b^*(s) = \sum_{j=0}^{a-1} f_j^*(s) + (1 - p) \sum_{j=a}^{c-1} f_j^*(s)
\]

\[
= P^*(1, 0, s) \cdot \mu \left\{ \frac{1 - R^a}{1 - R} + (1 - p)^2 \cdot \frac{R^a - R^c}{1 - R} \right\}
\]

(2.6.7)

The expected busy period of the server \( E(B) \) is given by

\[
B = \frac{-d}{ds} b^*(s) \bigg|_{s=0} = \frac{1 - r^a + (1 - p)^2(r^a - r^c)}{\mu [1 - r^c - pr^a - r^c]^2}
\]

(2.6.8)
2.7 Waiting Time Distribution

Let the random variable \( W \) denote the waiting time of an arriving unit in the queue.

An arriving unit may find the system in any of the following cases

i) \((0, n), 0 \leq n \leq c - 2\)

ii) \((0, c - 1)\)

iii) \((1, n), n = kd + m; 0 \leq m \leq a - 2, k = 0, 1, 2, \cdots\)

iv) \((1, n), n = kd + m; a - 1 \leq m \leq c - 2, k = 0, 1, 2, \cdots\)

v) \((1, n), n = kd + m; c - 1 \leq m \leq d - 1, k = 0, 1, 2, \cdots\)

In case (ii), the arriving unit does not have to wait. In all other cases the arriving unit has to wait.

Hence the probability of a no delay is

\[
P(W = 0) = P(0, c - 1) = P(1, 0) \theta_1 \left\{ \frac{1 - r^a}{1 - r} + (1 - p) \left[ \frac{r^a - r^c}{1 - r} \right] \right\}^{(2.7.1)}
\]

and \( P(\text{no delay}) = 1 - P(0, c - 1) \)

In case (i) the arriving unit has to wait for the arrival of \((c - n - 1)\) more units. This duration has a gamma distribution with parameters \( \lambda \) and \( c - n - 1 \).

In case (iii) the arriving unit has to wait till the service of \((k + 1)\) batches are over and \((a - m - 1)\) units arrive with probability \( p \) or \((c - m - 1)\) units arrive with probability \((1-p)\), whichever occurs later. Let the random variable \( Z \) denote this waiting time. Then \( Z \) is the maximum of two gamma variables.
\[ Z = \max \{ \text{Gamma variate with parameter } \mu, k+1; \] 
\[ \text{Gamma variate with parameter } \lambda, (a - m - 1) \text{with probability } p \text{ or } (c - m - 1) \text{with probability } (1 - p) \} \]

and

\[
G_z(t) = P(z \leq t) = \Gamma_t(\mu, k + 1) \{ p \Gamma_t(\lambda, a - m - 1) + (1 - p) \Gamma_t(\lambda, c - m - 1) \},
\]

where

\[
f(\lambda, k; t) = \frac{\lambda^k}{\Gamma(k)} e^{-\lambda t} t^{k-1}, t > 0, k = 1, 2, \ldots
\]

and

\[
\Gamma_t(\lambda, k) = \int_0^t f(\lambda, k; t) dt,
\]

is the incomplete gamma function. Hence

\[
g_z(t) = \frac{d}{dt} G_z(t) = \Gamma_t(\mu, k + 1) \{ p f(\lambda, a - m - 1; t) + (1 - p) f(\lambda, c - m - 1; t) \} \\
+ \{ p \Gamma_t(\lambda, a - m - 1) + (1 - p) \Gamma_t(\lambda, c - m - 1) \} \cdot f(\mu, k + 1; t) \quad (2.7.2)
\]

In case (iv), with probability \( p \) the arriving unit has to wait till the service of \( k + 1 \) batches are over and with probability \( (1 - p) \) the arriving unit has to wait till the service of \( k \) batches are over and \( (c - m - 1) \) units arrive whichever occurs later.

Let the random variable \( Z_1 \) denote this waiting time. Then

\[
h_{z_1}(t) = p f(\mu, k + 1; t) + \Gamma_t(\mu, k). (1 - p) f(\lambda, c - m - 1; t) \\
+ (1 - p) \Gamma_t(\lambda, c - m - 1) f(\mu, k; t) \quad (2.7.3)
\]

In case (v), the arriving unit has to wait for the completion of services of \( k + 1 \) batches and this duration has a gamma distribution with parameter \( \mu \) and \( k + 1 \).
Hence the p.d.f of the r.v W can be obtained as

\[ V(t) = \sum_{n=0}^{a-1} f(\lambda, c - n - 1; t)P(0, n) + \sum_{n=a}^{c-2} f(\lambda, c - n - 1; t)P(0, n) \]

\[ + \sum_{k=0}^{\infty} \sum_{m=0}^{a-2} P(1, kd + m)g_z(t) \]

\[ + \sum_{k=0}^{\infty} \sum_{m=a-1}^{c-2} P(1, kd + m)h_z(t) \]

\[ + \sum_{k=0}^{\infty} \sum_{m=c-1}^{d-1} P(1, kd + m)f(\lambda, k + 1; t) \]  \hspace{1cm} (2.7.4)

\[ = \lambda P(1, 0) \left\{ \frac{\theta_1}{1 - r} \left[ \left( E(c - 1; \lambda t) - r^{c-1}e^{-\lambda t}e(c - 1; \frac{\lambda t}{r}) \right) \\
-p \left( E(c - a - 1; \lambda t)r^a + r^{c-1}e^{-\lambda t}e(c - a - 1; \frac{\lambda t}{r}) \right) \right] \\
+ \frac{e^{-\lambda t}}{1 - r^d} \left[ \left( 1 - e^{-\mu t(1-r^d)} \right) \left( \frac{pe(a - m - 1, \frac{\lambda t}{r})}{r^{2-a}} + \frac{(1 - p)e(c - 1, \frac{\lambda t}{r})}{r^{2-c}} \right) \\
- \frac{r^d e^{-\mu t(1-r^d)}}{r^{2-c}} \left( e(c - a, \frac{\lambda t}{r}) \right) (1 - p) + \frac{e^{-\lambda t(1-r^d)}}{1 - r} \cdot (r^{c-1} - r^d) \right] \\
+ \frac{\mu P(1, 0)e^{-\mu t(1-r^d)}}{1 - r} \left\{ \left[ e^{-\lambda t} \left( pr^{a-1}e(a - 1, \frac{\lambda t}{r}) + r^{c-1}(1 - p)e(c - 1, \frac{\lambda t}{r}) \right) \right] \\
+ (1 - r^{c-1}) + (1 - p)(1 - r^d) \left[ r^{a-1}E(c - a - 1, \lambda t) - e^{-\lambda t}r^{c-1}e(c - a + 1, \lambda t) \right] \\
- [pE(a - 1, \lambda t) + (1 - p)E(c - 1, \lambda t)] \right\} \right\} \]  \hspace{1cm} (2.7.5)

where

\[ e(k; x) = \sum_{j=0}^{k-1} \frac{x^j}{j!} \] and \[ E(k; x) = e^{-x}e(k; x). \]

The conditional waiting time distribution of a customer, who has to wait is

\[ dV_1(t) = \frac{V(t)}{1 - P(0, c - 1)} \]  \hspace{1cm} (2.7.6)
The expected waiting time in the queue is given by

\[ E(W) = \int_0^\infty tdV(t) \]

\[ = \frac{P(1,0)}{\lambda(1-r)(1-r^d)} \left\{ \frac{c(c-1)(1-r)}{2r} - \frac{pr^{a-1}(c-a)(c-a-1)(1-r)}{2} \right. \]
\[ -pc(1+r^c) - r^c \left( \frac{1}{1-r} - c \right) - p(a-m)r^m(1-r^2) + \frac{pr^{a-3}(1+r^2)}{1-r} \]
\[ +par^2 - (r^{c-1} - r^d) \left( \frac{1}{1-r} \right) + r^a(c-r^d) - (1-p) \left[ (c-a+1)r^{d+a-1} \right. \]
\[ + (c-a)r^{c-1}(1-r^d) + r^{d+c-1}(1+r) - \frac{r^a(1-r^d) - r^c}{1-r} + \frac{r^c(1+r + r^{d+1})}{1-r} \]
\[ \left. - r^a \left( r^d(a-c-1) + 2 - a \right) \right\} \]

\[ (2.7.7) \]

In this model the server is idle when he is in the state \((0, n); n = 0, 1, 2, \ldots\).

The expected length of an idle period \(E(I)\) is given by

\[ \bar{I} = \sum_{n=1}^{c-1} \sum_{n=1}^{c-1} \frac{(c-n)P(0, n)}{\lambda \sum_{n=1}^{c-1} P(0, n)} \]

\[ = \frac{c}{\lambda} - \frac{1}{\lambda(c-1-(c-a-(\frac{r}{1-r}))))pr^a - (\frac{r^a}{1-r}) + (\frac{c-1}{1-r})}(1-p) \]
\[ \left\{ \frac{r^2}{(1-r)^2} + \frac{r^a}{2}(c+a(a-1)) + \frac{c(c-1)}{2} + ar^a(c-a)(1-p) + \frac{apr^{(a+1)}}{1-r} \right. \]
\[ + (c-a)(1-p)r^a(c+a-1) + \frac{(1-p)}{(1-r)^2}(r^{c+2} - r^{a+2}) \]
\[ + \frac{(cr^{c+1})}{(1-r)}(1-p) \} \]

\[ (2.7.8) \]

Hence the expected length of a Busy cycle \(E(c)\) is given by

\[ \bar{c} = B + I \]
2.8 Optimal Policy

The design of an optimal policy for a queueing system has received a lot of attention, as shown by the survey conducted by Tadj and Choudhury (2005). This is known in queueing theory as the optimal control of the system. The aim is to find the best values that the decision maker would implement in order to minimize the total expected cost per unit of time.

Let $C_h$: The holding cost per unit time in the system.

$C_a$: Start up cost per unit time for the preparatory work of the server before starting the service.

$C_s$: Set up cost per busy cycle.

$C_0$: Cost per unit time for keeping the server on and in operation. In our model, the start up cost $C_a$ is charged only for starting the service. There is no start up cost for the subsequent service batches in a busy period.

The linear cost function constructed for determining the optimal control limits $c$ and $a$ is

$$TC1(a, c) = C_h L_q + C_0 \frac{\bar{B}}{C} + C_s \frac{1}{C} + C_a \frac{\bar{I}}{C}$$
2.9 Particular Cases

Case 1: When $p=1$

From equation (2.4.1) to (2.4.5), The steady state probabilities are

\[
P(0, 0) = P(1, 0)\theta_1 \tag{2.9.1}
\]

\[
P(0, n) = P(1, 0)\theta_1 \left\{\frac{1 - r^{n+1}}{1 - r}\right\}; 1 \leq n \leq a - 1 \tag{2.9.2}
\]

\[
P(0, n) = P(1, 0)\theta_1 \left\{\frac{1 - r^a}{1 - r}\right\}; a \leq n \leq c - 1 \tag{2.9.3}
\]

\[
P(1, n) = P(1, 0)r^n; n \geq 1 \tag{2.9.4}
\]

\[
P(1, 0) = \left\{\frac{\theta_1}{1 - r}\left[c - r^a(c - a) - \frac{r}{1 - r}(1 - r^a)\right] + \frac{1}{1 - r}\right\}^{-1} \tag{2.9.5}
\]

From (2.5.1), Expected Queue length

\[
L_q = \frac{P(1, 0)\theta_1}{1 - r} \left\{(1 - r^a)\left[\frac{c(c - 1)}{2} - \frac{r^2}{(1 - r)^2}\right] + ar^a\left(\frac{a - 1}{2} + \frac{r}{1 - r}\right)\right\}
\]

\[
+ \frac{P(1, 0)r}{(1 - r)^2} \tag{2.9.6}
\]

and from equation (2.6.8) The expected busy period of the server is,

\[
E(T) = \frac{1}{\mu(1 - r^a)} \tag{2.9.7}
\]

These agrees with the corresponding results of the $M/M(a, c, d)/1$ model considered by Baburaj and Surendranath (2005).
Case 2: When \( p=0, \ c=a \) and \( d=b \)

Then from (2.4.1) to (2.4.5) the steady state probabilities are

\[
P(0, 0) = P(1, 0)\theta_1 \tag{2.9.8}
\]

\[
P(0, n) = P(1, 0)\theta_1 \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}; 1 \leq n \leq a - 1. \tag{2.9.9}
\]

\[
P(1, n) = P(1, 0)r^n; n \geq 1 \tag{2.9.10}
\]

\[
P(1, 0) = \left\{ \frac{\theta_1}{1 - r} \left[ a - \frac{r}{1 - r} (1 - r^a) \right] + \frac{1}{1 - r} \right\}^{-1} \tag{2.9.11}
\]

From equation (2.5.1) The expected Queue length is

\[
L_q = P(1, 0)\frac{\theta_1}{1 - r} \left\{ \frac{a(a - 1)}{2} - \frac{r^2}{(1 - r)^2} + \frac{r^{a+1}}{1 - r} \left[ a + \frac{r}{1 - r} \right] \right\} + \frac{P(1, 0)r}{(1 - r)^2} \tag{2.9.12}
\]

The expected busy period of the server is,

\[
E(T) = \frac{1}{\mu(1 - r^a)} \tag{2.9.13}
\]

These agrees with the corresponding results of the standard \( M/M(a, b)/1 \) model.

Case 3: When \( p=1, \ c=a \) and \( d=b \)

Then from (2.4.1) to (2.4.5) the steady state probabilities are

\[
P(0, 0) = P(1, 0)\theta_1 \tag{2.9.14}
\]

\[
P(0, n) = P(1, 0)\theta_1 \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}; 1 \leq n \leq a - 1. \tag{2.9.15}
\]

\[
P(1, n) = P(1, 0)r^n; n \geq 1 \tag{2.9.16}
\]

\[
P(1, 0) = \left\{ \frac{\theta_1}{1 - r} \left[ a - \frac{r}{1 - r} (1 - r^a) \right] + \frac{1}{1 - r} \right\}^{-1} \tag{2.9.17}
\]
From equation (2.5.1) the expected Queue length is

\[ L_q = P(1, 0) \cdot \frac{\theta_1}{1 - r} \left\{ \frac{a(a - 1)}{2} - \frac{r^2}{(1 - r)^2} + \frac{r^{a+1}}{1 - r} \left[ a + \frac{r}{1 - r} \right] \right\} \]

\[ + \frac{P(1, 0) r}{(1 - r)^2} \]  \hspace{1cm} (2.9.18)

The expected busy period of the server is,

\[ E(T) = \frac{1}{\mu(1 - r^a)} \]  \hspace{1cm} (2.9.19)

These agree with the corresponding results of the standard \( M/M/(a, b)/1 \) model.

**Case 4:** When \( p=1, a=c \) and \( d=1 \). From (2.4.1) to (2.4.5) we get the steady state probabilities are

\[ P(1, n) = P(1, 0) \left( \frac{\lambda}{\mu} \right)^n \]  \hspace{1cm} (2.9.20)

\[ P(1, 0) = 1 - \frac{\lambda}{\mu} \]  \hspace{1cm} (2.9.21)

\[ \text{and} \quad E(T) = \frac{1}{\mu - \lambda} \]  \hspace{1cm} (2.9.22)

These agree with the results of the standard \( M/M/1 \) model.

**Case 5** When \( p=0, a=c \) and \( d=1 \). From (2.4.1) to (2.4.5) we get the steady state probabilities are

\[ P(1, n) = P(1, 0) \left( \frac{\lambda}{\mu} \right)^n \]  \hspace{1cm} (2.9.23)

\[ P(1, 0) = 1 - \frac{\lambda}{\mu} \]  \hspace{1cm} (2.9.24)

\[ \text{and} \quad E(T) = \frac{1}{\mu - \lambda} \]  \hspace{1cm} (2.9.25)

These agree with the results of the standard \( M/M/1 \) model.
## 2.10 Numerical Illustration

The table giving the values of Steady state probabilities for the values of $\lambda=20$, $\mu=2$, $a=5$, $c=10$ and $d=20$.

### Table 2.10.1

<table>
<thead>
<tr>
<th>n</th>
<th>$p(0,n)$</th>
<th>$p(1,n)$</th>
<th>When $p=0.4$</th>
<th>n</th>
<th>$p(0,n)$</th>
<th>$p(1,n)$</th>
<th>When $p=0.8$</th>
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</table>
The expected queue length plotted for $\lambda=20$, $\mu=2$, $a=5$ and $d = 20$ and for different values of $p$

![Graph showing expected queue length for different values of p](image)

**Figure 2.10.1**

**Remark**: From Figure 2.10.1 we can see that as the value of $p$ increases, the expected queue length decreases.
The expected queue length plotted for $\lambda=20$, $\mu=2$, $c = 25$ $d = 20$ and for different values of $p$

**Remark**: From Figure 2.10.2 we can see that as the value of $p$ increases, the expected queue length decreases.
The expected busy period plotted for $\lambda=20$, $\mu=2$, $a=5$ and $d = 20$ and for different values of $p$.

Remark: From Figure 2.10.3 we can see that as the value of $p$ increases, the expected busy period increases.
The expected busy period plotted for $\lambda=20$, $\mu=2$, $c = 25$ $d = 20$ and for different values of $p$.

**Figure 2.10.4**

**Remark**: From Figure 2.10.4 we can see that as the value of $p$ increases, the expected busy period increases.
The table giving the values of the expected cost function for the values of \( \lambda = 20, \mu = 2, p = 0.1, d = 20, C_h = 50, C_0 = 400, C_s = 50, C_a = 30. \)

### Table 2.10.2

<table>
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<th>a \ c</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
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**Remark:** Here \( TC1(a, c) \) is minimum for \( a = 5 \) and \( c = 12. \)
The table giving the values of the expected cost function for the values of $\lambda=20$, $\mu=2$, $p=0.8$, $d=20$, $C_h = 50$, $C_0 = 400$, $C_s = 50$, $C_a = 30$.

Table 2.10.3

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</table>

**Remark:** Here $TC1(a, c)$ is minimum for $a = 6$ and $c = 12$. 