Chapter 5

An \((a, c, d)\) Policy State Dependent M/M/1 Single and Batch Service Queue with Multiple Vacations
5.1 Introduction

Server vacation models are useful for the systems in which a server wants to utilize the idle time for different purposes. Application of server vacation models can be found in manufacturing systems, designing of local area networks and data communication systems etc. Introducing vacation in the queueing models make the model more realistic and flexible in studying real world queueing situations. Because of the wide applications of vacation queueing models, considerable attention has paid to analyze the queueing systems with vacation policies. A wide class of policies for governing vacation mechanism have been discussed in the literature survey on queueing systems with server vacations can be found in Doshi (1986,1990), Takagi (1991), Medhi (2003), Tian and Zhang (2006). An \( M/M(a,b)/1 \) queue with multiple vacation and change over time is considered in Reddy and Anitha (1998) and have obtained the steady state distributions of the queue size and waiting time distribution of an arriving customer. N-policy bulk service queue with multiple vacations and setup times was considered in Reddy et al (1998). Madhu Jain and Poonam Singh (2005) have studied the steady state behavior of a state dependent bulk service queue with delayed vacations.

In this investigation, we study a state dependent \( M/M(a,c,d)/1 \) single and batch service queue with multiple vacations. Baburaj and Surendranath (2005) have considered an \( (a,c,d) \) policy bulk service queue where the server begins service only if the number of units in the queue is at least \( c \) and serves a maximum of \( d \) units in a batch. If after a service completion epoch, the queue size is less than \( c \) but not less than a secondary limit \( a \), the server continues the batch service. In this model, if the queue size is less than \( c \) the server serves the unit manually according to FCFS rule and
the service time distribution is assumed to be exponential with parameter $\mu_1$ (Type I service). If $n \geq c$, the server serves a maximum of $d$ units in a batch and the service time distribution is exponential with parameter $\mu_2$ (Type II service). If after a Type II service completion epoch, when the number of units in the queue is less than $c$ but not less than a secondary limit $a$ ($a \leq c \leq d$), the server serves them altogether ia batch and the service time distribution is assumed to be exponential with parameter $\mu_3$ (Type III service). After a batch service completion epoch, when the queue size is less than $a$, the server will go for a vacation which is exponentially distributed with parameter $\beta$. After a vacation, when he returns, if the queue length is less than $c$, he leaves for another vacation and so on until he finds at least $c$ customers waiting in the queue. The arrival process is assumed to be Poisson with parameter $\lambda$. The model is analyzed and obtained the transient and steady state probability distributions and explicit expressions for the performance measures.

### 5.2 Description of the Model

Here we consider a State dependent $M/M/1$ single and batch service queueing system with multiple vacations under the policy $(a, c, d)$.

- The customers arrive according to poisson process with parameter $\lambda$.
- The customers are served either manually or in batches with different service rates depending on the number of units in the queue.
- If the number of units in the queue is less than $c$, the server serves the units manually and if the queue size $n \geq c$, the server serves a maximum of $d$ units
in a batch.

- If after a service completion epoch, the queue size is less than $c$ but not less than a secondary limit $a$, the server serves them altogether in a batch.

- After a service completion epoch, if the number of customers in the queue is less than $a$, the server will go for a vacation, which is assumed to be exponentially distributed with rate $\beta$.

- On returning from a vacation, if the queue length is less than $c$, he leaves for another vacation and so on, until he finds at least $c$ customers waiting in the queue.

5.3 Analysis of the Model

Let $Y(t)$ and $X(t)$ respectively denote the state of the server and the queue size at time $t$. Here $Y(t)$ can assume the values 0, 1, 2, 3 or 4 according as the server is idle, busy with a manual service, busy with a Type II service, busy with a Type III service or in vacation. Then the two-dimensional stochastic process $\{Y(t), X(t) \geq 0\}$ forms a Markov process with state space

$$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$$

where

- $S_1 = \{(0,0)\}$,
- $S_2 = \{(1,n), n = 0, 1, 2, \ldots, c - 1\}$,
- $S_3 = \{(2,n), n = 0, 1, 2, \ldots\}$,
\[ S_4 = \{(3, n), n = 0, 1, 2, \ldots\}, \]

and \[ S_5 = \{(4, n), n = 0, 1, 2, \ldots\} \]

Let \[ P(i, n, t) = P\{Y(t) = i, X(t) = n\} \]

Following are the transitions that can be occurred during \((t, t+h]\):

<table>
<thead>
<tr>
<th>Transitions during ((t, t+h])</th>
<th>Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0) \rightarrow (1, 0))</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1, n) \rightarrow (1, n+1),)</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1, c-1) \rightarrow (2, 0))</td>
<td>(\lambda h + O(h))</td>
</tr>
<tr>
<td>((1, 0) \rightarrow (0, 0),)</td>
<td>(\mu_1 h + O(h))</td>
</tr>
<tr>
<td>((1, n) \rightarrow (1, n-1),)</td>
<td>(\mu_1 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (2, n+1),)</td>
<td>(\lambda h + O(h); n = 0, 1, 2, \ldots)</td>
</tr>
<tr>
<td>((2, 0) \rightarrow (4, 0))</td>
<td>(\mu_2 h + O(h))</td>
</tr>
<tr>
<td>((2, n) \rightarrow (4, n),)</td>
<td>(\mu_2 h + O(h); 1 \leq n \leq a - 1)</td>
</tr>
<tr>
<td>((2, n) \rightarrow (3, 0),)</td>
<td>(\mu_2 h + O(h); a \leq n \leq c - 1)</td>
</tr>
<tr>
<td>((2, n) \rightarrow (2, 0),)</td>
<td>(\mu_2 h + O(h); c \leq n \leq d)</td>
</tr>
<tr>
<td>((2, n) \rightarrow (2, n-d),)</td>
<td>(\mu_2 h + O(h); n \geq d + 1)</td>
</tr>
<tr>
<td>((3, n) \rightarrow (3, n+1),)</td>
<td>(\lambda h + O(h); n = 0, 1, 2, \ldots)</td>
</tr>
<tr>
<td>((3, 0) \rightarrow (4, 0))</td>
<td>(\mu_3 h + O(h))</td>
</tr>
<tr>
<td>((3, n) \rightarrow (4, n),)</td>
<td>(\mu_3 h + O(h); 1 \leq n \leq a - 1)</td>
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<td>((3, n) \rightarrow (3, 0),)</td>
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</tr>
<tr>
<td>((3, n) \rightarrow (2, 0),)</td>
<td>(\mu_3 h + O(h); c \leq n \leq d)</td>
</tr>
<tr>
<td>((3, n) \rightarrow (2, n-d),)</td>
<td>(\mu_3 h + O(h); n \geq d)</td>
</tr>
<tr>
<td>((4, n) \rightarrow (4, n+1),)</td>
<td>(\lambda h + O(h); n \geq 0)</td>
</tr>
<tr>
<td>((4, n) \rightarrow (4, n),)</td>
<td>(\beta h + O(h); 0 \leq n \leq c - 1)</td>
</tr>
<tr>
<td>((4, n) \rightarrow (2, 0),)</td>
<td>(\beta h + O(h); c \leq n \leq d)</td>
</tr>
<tr>
<td>((4, n) \rightarrow (2, n-d),)</td>
<td>(\beta h + O(h); n \geq d + 1)</td>
</tr>
</tbody>
</table>
Hence the difference differential equations governing the transitions are,

\[
P'(0,0,t) = -\lambda P(0,0,t) + \mu_1 P(1,0,t)
\]

(5.3.1)

\[
P'(1,0,t) = -(\lambda + \mu_1) P(1,0,t) + \lambda P(0,0,t) + \mu_1 P(1,1,t)
\]

(5.3.2)

\[
P'(1,n,t) = -(\lambda + \mu_1) P(1,n,t) + \lambda P(1,n-1,t) + \mu_1 P(1,n+1,t),
\]

\[1 \leq n \leq c - 2\]

(5.3.3)

\[
P'(1,c-1,t) = -(\lambda + \mu_1) P(1,c-1,t) + \lambda P(1,c-2,t)
\]

(5.3.4)

\[
P'(2,0,t) = -(\lambda + \mu_2) P(2,0,t) + \lambda P(1,c-1,t) + \mu_2 \sum_{n=c}^{d} P(2,n,t) + \mu_3 \sum_{n=c}^{d} P(3,n,t) + \beta \sum_{n=c}^{d} P(4,n,t)
\]

(5.3.5)

\[
P'(2,n,t) = -(\lambda + \mu_2) P(2,n,t) + \lambda P(2,n-1,t) + \mu_2 P(2,n+d,t) + \mu_3 P(3,n+d,t) + \beta P(4,n+d,t), \; n = 1, 2, 3, \ldots
\]

(5.3.6)

\[
P'(3,0,t) = -(\lambda + \mu_3) P(3,0,t) + \mu_2 \sum_{n=a}^{c-1} P(2,n,t) + \mu_3 \sum_{n=a}^{c-1} P(3,n,t)
\]

(5.3.7)

\[
P'(3,n,t) = -(\lambda + \mu_3) P(3,n,t) + \lambda P(3,n-1,t), \; n = 1, 2, 3, \ldots
\]

(5.3.8)

\[
P'(4,0,t) = -(\lambda + \beta) P(4,0,t) + \mu_2 P(2,0,t) + \mu_3 P(3,0,t) + \beta P(4,0,t)
\]

(5.3.9)

\[
P'(4,n,t) = -(\lambda + \beta) P(4,n,t) + \lambda P(4,n-1,t) + \beta P(4,n,t) + \mu_2 P(2,n,t) + \mu_3 P(3,n,t), \; 1 \leq n \leq a - 1
\]

(5.3.10)

\[
P'(4,n,t) = -(\lambda + \beta) P(4,n,t) + \lambda P(4,n-1,t) + \beta P(4,n,t), \; a \leq n \leq c - 1
\]

(5.3.11)

\[
P'(4,n,t) = -(\lambda + \beta) P(4,n,t) + \lambda P(4,n-1,t), \; n \geq c
\]

(5.3.12)
5.4 Method of Solution

Let $P^*(i, n, s)$ denote the Laplace transform of $P(i, n, t)$ and assume that $P(0, 0, 0) = 1$. Then the following are the laplace transform of the transient probabilities corresponding to system of equations (5.3.1) to (5.3.12)

\[(s + \lambda)P^*(0, 0, s) - 1 = \mu_1 P^*(1, 0, s) \quad (5.4.1)\]
\[(s + \lambda + \mu_1)P^*(1, 0, s) = \lambda P^*(0, 0, s) + \mu_1 P^*(1, 1, s) \quad (5.4.2)\]
\[(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n - 1, s) + \mu_1 P^*(1, n + 1, s), \quad 1 \leq n \leq c - 2 \quad (5.4.3)\]
\[(s + \lambda + \mu_1)P^*(1, c - 1, s) = \lambda P^*(1, c - 2, s) \quad (5.4.4)\]
\[(s + \lambda + \mu_2)P^*(2, 0, s) = \lambda P^*(1, c - 1, s) + \mu_2 \sum_{n=c}^{d} P^*(2, n, s) \]
\[+ \mu_3 \sum_{n=c}^{d} P^*(3, n, s) + \beta \sum_{n=c}^{d} P^*(4, n, s) \quad (5.4.5)\]
\[(s + \lambda + \mu_2)P^*(2, n, s) = \lambda P^*(2, n - 1, s) + \mu_2 P^*(2, n + d, s) \]
\[+ \mu_3 P^*(3, n + d, s) + \beta P^*(4, n + d, s), \quad n \geq 1 \quad (5.4.6)\]
\[(s + \lambda + \mu_3)P^*(3, 0, s) = \mu_2 \sum_{n=a}^{c-1} P^*(2, n, s) + \mu_3 \sum_{n=a}^{c-1} P^*(3, n, s) \quad (5.4.7)\]
\[(s + \lambda + \mu_3)P^*(3, n, s) = \lambda P^*(3, n - 1, s), \quad n \geq 1 \quad (5.4.8)\]
\[(s + \lambda)P^*(4, 0, s) = \mu_2 P^*(2, 0, s) + \mu_3 P^*(3, 0, s) \quad (5.4.9)\]
\[(s + \lambda)P^*(4, n, s) = \lambda P^*(4, n - 1, s) + \mu_2 P^*(2, n, s) \]
\[+ \mu_3 P^*(3, n, s), \quad 1 \leq n \leq a - 1 \quad (5.4.10)\]
\[(s + \lambda)P^*(4, n, s) = \lambda P^*(4, n - 1, s), \quad a \leq n \leq c - 1 \quad (5.4.11)\]
\[(s + \lambda + \beta)P^*(4, n, s) = \lambda P^*(4, n - 1, s), \quad n \geq c \quad (5.4.12)\]
From (5.4.1)  \( P^*(0, 0, s) = P^*(1, 0, s)e_2 + \frac{1}{s + \lambda} \)

From (5.4.3)  \( P^*(1, n, s) = P^*(1, 0, s)R_1^n \);  \( 1 \leq n \leq c-2 \)

From (5.4.4)  \( P^*(1, c-1, s) = P^*(1, 0, s)e_5R_1^{c-2} \)

From (5.4.12)  \( P^*(4, n, s) = e_5^{n-c+1}P^*(4, c-1, s) \);  \( n \geq c \)

From (5.4.8)  \( P^*(3, n, s) = e_8^nP^*(3, 0, s) \);  \( n = 1, 2, 3, \cdots \)

From (5.4.6)  \( P^*(2, n, s) = P^*(2, 0, s)R^n - \frac{e_16e_8^{n+d}P^*(3, 0, s)}{K(e_8)} \)

\[ - \frac{e_18e_5^{n+d+1-c}P^*(4, c-1, s)}{K(e_5)} \];  \( n \geq 1 \)

Where  \( K(z) = \mu_2z^{d+1}-(s+\lambda+\mu_2)z+\lambda \)

\( R_1 \) is the solution less than unity of the equation \( \mu_1z^2-(s+\lambda+\mu_1)z+\lambda = 0 \) and  \( R \) is the unique positive real root less than unity of the equation  \( K(z) = 0 \).

From (5.4.9)  \( P^*(4, 0, s) = e_2P^*(2, 0, s)+e_4P^*(3, 0, s) \)

From (5.4.10)  \( P^*(4, n, s) = e_3P^*(2, 0, s) \left[ \frac{e_1^{n+1} - R^{n+1}}{e_1 - R} \right] \)

\[ + P^*(3, 0, s) \left\{ e_4 \left( \frac{e_1^{n+1} - e_8^{n+1}}{e_1 - e_8} \right) - \frac{e_3e_8^{d+1}e_16}{K(e_8)} \left( \frac{e_1^n - e_8^n}{e_1 - e_8} \right) \} \]

\[ - P^*(4, c-1, s) \left( \frac{e_3e_8^{d-c+2}}{K(e_5)} \left( \frac{e_1^n - e_5^n}{e_1 - e_5} \right) \right) \];  \( 1 \leq n \leq a-1 \)

From (5.4.11)  \( P^*(4, n, s) = e_1^{n-a+1} \left\{ L_4P^*(2, 0, s) + L_5P^*(3, 0, s) \right\} - L_6P^*(4, c-1, s) \right\} \);  \( a \leq n \leq c-1 \)

From (5.4.7)

\( P^*(3, 0, s) = L_1 \left[ e_14 \left( \frac{R^a - R^c}{1 - R} \right) P^*(2, 0, s) - \frac{e_14e_8^{d-c+1}}{K(e_5)} \left( \frac{e_5^n - e_8^n}{1 - e_5} \right) P^*(4, c-1, s) \right] \)
From (5.4.5) \( P^*(2,0,s) = P^*(1,0,s) e_7 e_5 R_1^{c-2} L_3 - L_2 L_4 P^*(4,c-1,s) \)

Where \( L_1 = \left[ 1 + \left( \frac{e_8^a - e_8^c}{1 - e_8} \right) \left( \frac{e_{14} e_6^d}{K(e_8)} - e_{17} \right) \right]^{-1} \)

\[
L_2 = \left( \frac{L_1 e_{14} e_8 e_5^{d-c+1} e_{16}}{K(e_5)} \right) \left( \frac{e_5^a - e_5^c}{1 - e_5} \right) \left( \frac{e_8^a - e_8^{d+1}}{1 - e_8} \right) \left( 1 - \frac{e_{13} e_5^d}{K(e_5)} \right)
- e_8 e_5 \left( \frac{1 - e_5^{d-c+1}}{1 - e_5} \right) \left( 1 - \frac{e_{13} e_5^d}{K(e_5)} \right)
\]

\[
L_3 = \left[ 1 - e_{13} \left( \frac{R_c - R^{d+1}}{1 - R} \right) - L_1 e_{14} \left( \frac{R_a - R^d}{1 - R} \right) \right] e_{16} \left( \frac{e_8^a - e_8^{d+1}}{1 - e_8} \right) \left( 1 - \frac{e_{13} e_8^d}{K(e_8)} \right) \]^{-1}
\]

\[
L_4 = e_3 \left( \frac{e_1^a - R^a}{e_1 - R} \right)
\]

\[
L_5 = e_4 \left( \frac{e_1^a - e_8^a}{e_1 - e_8} \right) - e_3 e_8^{d+1} e_{16} \left( \frac{e_1^{a-1} - e_8^{a-1}}{e_1 - e_8} \right)
\]

\[
L_6 = \frac{e_3 e_8 e_5^{d-c+2}}{K(e_5)} \left( \frac{e_1^{a-1} - e_5^{a-1}}{e_1 - e_5} \right)
\]

\[
L_7 = \frac{e_3}{e_1 - R} \left[ \frac{e_1^2 (1 - e_1^{a-1})}{1 - e_1} - \frac{R^2 (1 - R^{a-1})}{1 - R} \right]
\]

\[
L_8 = \frac{1}{e_1 - e_8} \left[ e_4 \left\{ \frac{e_1^2 (1 - e_1^{a-1})}{1 - e_1} - \frac{e_8^2 (1 - e_8^{a-1})}{1 - e_8} \right\}
- \frac{e_3 e_8^{d+1} e_{16}}{K(e_8)(1 - e_8)} \left\{ \frac{e_1 (1 - e_1^{a-1})}{1 - e_1} - \frac{e_8 (1 - e_8^{a-1})}{1 - e_8} \right\} \right]
\]

\[
L_9 = \frac{e_3 e_8 e_5^{d-c+2}}{K(e_5)(e_1 - e_5)} \left\{ \frac{e_1 (1 - e_1^{c-1})}{1 - e_1} - \frac{e_5 (1 - e_5^{c-1})}{1 - e_5} \right\}
\]

\[
L_{10} = e_1^{c-a} \left( L_4 + L_5 e_{14} \left( \frac{R^a - R^c}{1 - R} \right) \right) e_1 e_5 R_1^{c-2}
\]

\[
L_{11} = \left[ 1 + \frac{e_{14} e_8 e_5^{d-c+1}}{K(e_5)} \left( \frac{e_5^a - e_5^c}{1 - e_5} \right) + L_6 - L_2 L_3 \left( L_4 + L_5 e_{14} \left( \frac{R^a - R^c}{1 - R} \right) \right) \right]^{-1}
\]

and

\[
e_1 = \frac{\lambda}{s + \lambda} \quad e_2 = \frac{\mu_1}{s + \lambda} \quad e_3 = \frac{\mu_2}{s + \lambda}
\]
\[ e_4 = \frac{\mu_3}{s + \lambda} \quad e_5 = \frac{\lambda}{s + \lambda + \beta} \quad e_6 = \frac{\lambda}{s + \lambda + \mu_1} \]
\[ e_7 = \frac{\lambda}{s + \lambda + \mu_2} \quad e_8 = \frac{\lambda}{s + \lambda + \mu_3} \quad e_9 = \frac{\mu_1}{s + \lambda + \mu_1} \]
\[ e_{10} = \frac{\mu_1}{s + \lambda + \mu_2} \quad e_{11} = \frac{\mu_1}{s + \lambda + \mu_3} \quad e_{12} = \frac{\mu_2}{s + \lambda + \mu_1} \]
\[ e_{13} = \frac{\mu_2}{s + \lambda + \mu_2} \quad e_{14} = \frac{\mu_2}{s + \lambda + \mu_3} \quad e_{15} = \frac{\mu_3}{s + \lambda + \mu_1} \]
\[ e_{16} = \frac{\mu_3}{s + \lambda + \mu_2} \quad e_{17} = \frac{\mu_3}{s + \lambda + \mu_3} \quad e_{18} = \frac{\beta}{s + \lambda + \mu_2} \]

Hence the Laplace transform of transient probabilities are

\[ P^*(0, 0, s) = P^*(1, 0, s)e_2 + \frac{1}{s + \lambda} \quad (5.4.13) \]
\[ P^*(1, n, s) = P^*(1, 0, s)R^n_1 ; \quad 1 \leq n \leq c - 2 \quad (5.4.14) \]
\[ P^*(1, c - 1, s) = P^*(1, 0, s)e_5R^{c-2}_1 \quad (5.4.15) \]
\[ P^*(2, 0, s) = P^*(1, 0, s)e_7e_5R^{c-2}_1L_3 - P^*(4, c - 1, s)L_2L_4 \quad (5.4.16) \]
\[ P^*(2, n, s) = P^*(2, 0, s)R^n - P^*(3, 0, s)e_16e_8^{n+d}K(e_8) \]
\[- P^*(4, c - 1, s)\frac{e_18e_5^n}{K(e_5)} ; \quad n \geq 1 \quad (5.4.17) \]
\[ P^*(3, 0, s) = P^*(2, 0, s)L_1 \left[ e_{14} \left( \frac{R^n - R^c}{1 - R} \right) \right. \]
\[- P^*(4, c - 1, s)\frac{e_14e_18e_5^{d+c+1}}{K(e_5)} \left( \frac{e_5 - e_8^c}{1 - e_5} \right) \right] \quad (5.4.18) \]
\[ P^*(3, n, s) = P^*(3, 0, s)e_8^n ; \quad n \geq 1 \quad (5.4.19) \]
\[ P^*(4, 0, s) = P^*(2, 0, s)e_2 + P^*(3, 0, s)e_4 \quad (5.4.20) \]
\[ P^*(4, n, s) = P^*(2, 0, s) e_3 \left[ \frac{e_1^{n+1} - R_{s+1}}{e_1 - R} \right] + P^*(3, 0, s) \]
\[
\begin{align*}
\left\{ e_4 \left( \frac{e_1^{n+1} - e_8^{n+1}}{e_1 - e_8} \right) - \frac{e_3 e_8^{d+1}}{e_8} \left( \frac{e_1^n - e_8^n}{e_1 - e_8} \right) \right\} - P^*(4, c - 1, s) \\
\left( \frac{e_3 e_18 e_5^{d-c+2}}{K(e_5)} \left( \frac{e_1^n - e_5^n}{e_1 - e_5} \right) \right) ; \quad 1 \leq n \leq a - 1
\end{align*}
\]
\[ P^*(4, n, s) = e_1^{n-a+1} \{ P^*(2, 0, sL_4) + P^*(3, 0, s)L_5 - P^*(4, c - 1, s)L_6 \} ; \quad a \leq n \leq c - 1 \] (5.4.21)
\[ P^*(4, n, s) = e_5^{n-c+1} P^*(4, c - 1, s) ; \quad n \geq c \] (5.4.22)

and \( P^*(1, 0, s) \) can be obtained by using the normalizing condition
\[
\sum_i \sum_n P^*(i, n, s) = \frac{1}{s} \quad \text{as}
\]
\[ P^*(1, 0, s) = \left( \frac{1}{s} - \frac{1}{s + \lambda} \right) L_{12} \] (5.4.24)

where
\[
L_{12} = \left\{ 1 + e_2 + \frac{R_1(1 - R_1^{c-2})}{1 - R_1} + e_5 R_1^{c-2} + L_3 e_7 e_5 R_1^{c-2} \left( 1 + e_3 + L_7 + e_1^{c-a} L_4 \right) \right. \\
+ \left[ L_1 e_{14} \left( \frac{R^a - R^c}{1 - R_2} \right) L_3 e_7 e_5 R_1^{c-2} - \frac{L_{10} L_1 L_1 e_{14} e_8 e_5^{d-c+1}}{K(e_5)} \left( \frac{e_5^2 - e_5^c}{1 - e_5} \right) \right. \\
- L_{10} L_{11} L_1 e_{14} \left( \frac{R^a - R^c}{1 - R} \right) L_2 L_3 \left[ 1 + \frac{e_8}{1 - e_8} + e_4 + L_8 + e_1^{c-a} L_5 \right] \\
- L_{10} L_{11} \left[ L_9 + L_6 e_1^{c-a} - \frac{e_5}{1 - e_5} \right] \left\}^{-1}
\]
5.5 Steady State Probabilities

The steady state probabilities of the system states can be obtained by using final value theorem on Laplace transforms as

\[ P(i, n) = \lim_{t \to \infty} P(i, n, t) = \lim_{s \to 0} s P^*(i, n, s) \]

Hence from (3.13) to (3.24), the steady state probabilities can be obtained as

\[
P(0, 0) = P(1, 0) \theta_2 \tag{5.5.1}
\]

\[
P(1, n) = P(1, 0) r_1^n ; \quad 1 \leq n \leq c - 1 \tag{5.5.2}
\]

\[
P(1, c - 1) = P(1, 0) \theta_5 r_1^{c-2} \tag{5.5.3}
\]

\[
P(2, 0) = P(1, 0) \theta_7 \theta_5 r_1^{c-2} T_3 - P(4, c - 1) T_2 T_4 \tag{5.5.4}
\]

\[
P(2, n) = P(2, 0) r^n - P(3, 0) \frac{\theta_1 \theta_8^{n+d}}{K'(\theta_8)} - P(4, c - 1) \frac{\theta_1 \theta_5^{n+d+1-c}}{K'(\theta_5)} ; \quad n \geq 1 \tag{5.5.5}
\]

\[
P(3, 0) = T_1 \left[ P(2, 0) \theta_{14} \left( \frac{r^a - r^c}{1 - r} \right) - P(4, c - 1) \frac{\theta_1 \theta_{18} \theta_5^{d-c+1}}{K'(\theta_5)} \left( \frac{\theta_5^a - \theta_5^c}{1 - \theta_5} \right) \right] \tag{5.5.6}
\]

\[
P(3, n) = P(3, 0) \theta_8^n ; \quad n \geq 1 \tag{5.5.7}
\]

\[
P(4, 0) = P(2, 0) \theta_2 + P(3, 0) \theta_4 \tag{5.5.8}
\]

\[
P(4, n) = P(2, 0) \theta_3 \left( \frac{1 - r^{n+1}}{1 - r} \right) + P(3, 0) \left[ \theta_4 \left( \frac{1 - \theta_8^{n+1}}{1 - \theta_8} \right) - \frac{\theta_3 \theta_5^{d+1} \theta_{16}}{K'(\theta_8)} \left( \frac{1 - \theta_8^n}{1 - \theta_8} \right) \right] - P(4, c - 1) \left( \frac{\theta_3 \theta_{18} \theta_5^{d-c+2}}{K'(\theta_5)} \left( \frac{1 - \theta_5^n}{1 - \theta_5} \right) \right) ; \quad 1 \leq n \leq a - 1 \tag{5.5.9}
\]

\[
P(4, n) = P(2, 0) T_4 + P(3, 0) T_5 - P(4, c - 1) T_6 ; \quad a \leq n \leq c - 1 \tag{5.5.10}
\]

\[
P(4, n) = P(4, c - 1) \theta_5^{n-c+1} ; \quad n \geq c \tag{5.5.11}
\]
\[
P(1, 0) = \left\{ 1 + \theta_2 + \frac{r_1(1 - r_1^{c-2})}{1 - r_1} + \theta_5 r_1^{c-2} + T_3 \theta_7 \theta_5 r_1^{c-2} (1 + \theta_3 + T_7 + \theta_1^{c-a} T_4) + T_1 \theta_4 \left( \frac{r^a - r^c}{1 - r} \right) T_3 \theta_7 \theta_5 r_1^{c-2} - \frac{T_1 \theta_4 T_3 \theta_1 \theta_8 \theta_5^{d-c+1}}{K'(\theta_5)} \left( \frac{\theta_5 - \theta_5^d}{1 - \theta_5} \right) - T_1 \theta_4 T_3 \theta_5 \left( \frac{r^a - r^c}{1 - r} \right) T_3 T_3 \left[ 1 + \frac{\theta_8}{1 - \theta_8} + \theta_4 + T_8 + \theta_1^{c-a} T_5 \right] - T_1 \theta_4 T_3 \left[ T_1 + T_6 \theta_1^{c-a} - \frac{\theta_5}{1 - \theta_5} \right] \right\}^{-1} \quad (5.5.12)
\]

where

\[
\theta_1 = 1 \quad \theta_2 = \frac{\mu_1}{\lambda} \quad \theta_3 = \frac{\mu_2}{\lambda} \quad \theta_4 = \frac{\mu_3}{\lambda}
\]

\[
\theta_5 = \frac{\lambda}{\lambda + \beta} \quad \theta_6 = \frac{\lambda}{\lambda + \mu_1} \quad \theta_7 = \frac{\lambda}{\lambda + \mu_2} \quad \theta_8 = \frac{\lambda}{\lambda + \mu_3}
\]

\[
\theta_9 = \frac{\mu_1}{\lambda + \mu_1} \quad \theta_{10} = \frac{\mu_1}{\lambda + \mu_2} \quad \theta_{11} = \frac{\mu_1}{\lambda + \mu_3} \quad \theta_{12} = \frac{\mu_2}{\lambda + \mu_1}
\]

\[
\theta_{13} = \frac{\mu_2}{\lambda + \mu_2} \quad \theta_{14} = \frac{\mu_2}{\lambda + \mu_3} \quad \theta_{15} = \frac{\mu_3}{\lambda + \mu_1} \quad \theta_{16} = \frac{\mu_3}{\lambda + \mu_2}
\]

\[
\theta_{17} = \frac{\mu_3}{\lambda + \mu_3} \quad \theta_{18} = \frac{\beta}{\lambda + \mu_2}
\]

\[
T_1 = \left[ 1 + \left( \frac{\theta_8^a - \theta_8^c}{1 - \theta_8} \right) \left( \frac{\theta_1 \theta_4 \theta_5 \theta_8^d}{K(\theta_5)} - \theta_1 \right) \right]^{-1}
\]

\[
T_2 = \left( \frac{T_1 \theta_4 \theta_5 \theta_8^{d-c+1} \theta_1 \theta_6}{K(\theta_5)} \right) \left( \frac{\theta_8^a - \theta_8^c}{1 - \theta_5} \right) \left( \frac{\theta_8^c - \theta_8^{d+1}}{1 - \theta_8} \right) \left( 1 - \frac{\theta_1 \theta_3 \theta_8^d}{K(\theta_5)} \right) - \theta_1 \theta_5 \left( \frac{1 - \theta_5^{d-c+1}}{1 - \theta_5} \right) \left( 1 - \frac{\theta_1 \theta_3 \theta_5^d}{K(\theta_5)} \right)
\]

\[
T_3 = \left[ 1 - \theta_3 \left( \frac{r^c - r^{d+1}}{1 - r} \right) - T_1 \theta_4 \left( \frac{r^a - r^d}{1 - r} \right) \theta_1 \theta_6 \left( \frac{\theta_8^c - \theta_8^{d+1}}{1 - \theta_8} \right) \left( 1 - \frac{\theta_1 \theta_3 \theta_8^d}{K(\theta_5)} \right) \right]^{-1}
\]

\[
T_4 = \theta_3 \left( \frac{1 - r^a}{1 - r} \right)
\]

\[
T_5 = \theta_4 \left( \frac{1 - \theta_8^a}{1 - \theta_8} - \frac{\theta_3 \theta_8^{d+1} \theta_1 \theta_6}{K(\theta_5)} \left( \frac{1 - \theta_8^{a-1}}{1 - \theta_8} \right) \right)
\]
\[ T_6 = \frac{\theta_3 \theta_5^{d-c+2}}{k(e_5)} \left( \frac{1 - \theta_5^{a-1}}{1 - \theta_5} \right) \]

\[ T_7 = \frac{\theta_3}{1-r} \left[ 1 - \frac{r^2(1-r^{a-1})}{1-r} \right] \]

\[ T_8 = \frac{1}{1-\theta_8} \left\{ \theta_4 \left( \frac{1}{1-\theta_8} \right) - \frac{\theta_3 \theta_5^{d+1} \theta_{16}}{K(\theta_8)(1-\theta_8)} \right\} \]

\[ T_9 = \frac{\theta_3 \theta_5^{d-c+2}}{K(\theta_5)(1-\theta_5)} \left[ 1 - \frac{\theta_5(1-\theta_8^{a-1})}{1-\theta_8} \right] \]

\[ T_{10} = \left( T_4 + T_5 \theta_{14} \left( \frac{r^a - r^c}{1-r} \right) \right) \theta_5 r_1^{c-2} \]

\[ T_{11} = \left[ 1 + \frac{\theta_4 \theta_5^{d-c+1}}{K(\theta_5)} \left( \frac{\theta_5 - \theta_5^a}{1-\theta_5} \right) + T_6 - T_2 T_3 \left( T_4 + T_5 \theta_{14} \left( \frac{r^a - r^c}{1-r} \right) \right) \right]^{-1} \]

where \( r \) is the unique positive real root less than unity of the equation

\[ \mu_2 z^{d+1} - (\lambda + \mu_2) z + \lambda = 0 \]

and \( r_1 = \frac{\lambda}{\mu_1} \) Here for the existence of steady state distribution we assume that

\[ \frac{\lambda}{\mu_1} < 1 \quad \text{and} \quad \frac{\lambda}{d \mu_2} < 1 \]

5.6 Expected Queue Length

The expected queue length is given by

\[ L_q = \sum_{n=1}^{c-1} nP(1,n) + \sum_{n=1}^{\infty} nP(2,n) + \sum_{n=1}^{\infty} nP(3,n) + \sum_{n=1}^{\infty} nP(4,n) \]

\[ = P^*(1,0) \left[ T_{14} + \theta_7 \theta_5 r_1^{c-2} T_3 (1 + T_2 T_4 T_{12})^{-1} T_{15} \right] \quad (5.6.1) \]

where

\[ T_{12} = \left[ T_4 + T_5 \theta_{14} \left( \frac{r^a - r^c}{1-r} \right) \right] \left\{ 1 + \frac{T_5 \theta_4 \theta_5^{d-c+1}}{K(\theta_5)} \left( \frac{\theta_5^a - \theta_5^c}{1-\theta_5} \right) + T_6 \right\}^{-1} \]
\[ T_{13} = T_{12} T_3 \theta_7 \theta_5 r_1^{-2} (1 + T_2 T_4 T_{12})^{-1} \left[ \theta_{14} \left( \frac{r^a - r^c}{1 - r} \right) - \frac{\theta_{14} \theta_{18} \theta_5^{d-c+1}}{K(\theta_5)} \left( \frac{\theta_5^c - \theta_5^d}{1 - \theta_5} \right) T_{12} \right] \]

\[ T_{14} = 1 + (c-1) \theta_5 r_1^{c-2} + T_{12} \frac{\theta_8 (1 - \theta_8^{c-1})}{1 - \theta_8} - T_{12} \]

\[ T_{15} = \frac{r}{(1 - r)^2} + T_4 [2(a + 1) - c] - T_{13} \left\{ \frac{\theta_{16} \theta_8^{a+1}}{K(\theta_8)(1 - \theta_8)} + \frac{\theta_8}{(1 - \theta_8)^2} - T_5 [2(a + 1) - c] + 1 - \theta_5 (1 - \theta_5^{a-1}) + \frac{\theta_{18} \theta_5^{a+2-c}}{K(\theta_5)(1 - \theta_5)^2} + T_6 [2(a + 1) - c] - \frac{e_5}{(1 - e_5)^2} \right\} + \left[ 1 - \frac{r^2 (1 - r^{a-1})}{(1 - r)^2} \right] \frac{\theta_3}{1 - r} \]

### 5.7 Busy Period Distribution in Type I service

In this model, the busy period of the server in Type I service begins with the arrival of a unit in the system and lasts till the queue size is less than \( c \) for the first time. In this case, the server serves the units manually with rate \( \mu_1 \).

The distribution of the busy period of the server \( B_1 \) can be obtained by considering the states \((0,0), (2,n), n = 0, 1, 2, \cdots, (3,n), n = 0, 1, 2, \cdots\) and \((4,n), n = 0, 1, 2, \cdots\) are absorbing. Also assume \( P(1, 0, 0) = 1 \)

Let \( f_{0,0}(t) = P(t \leq B_1 < t + dt, Y(t + dt) = 0, X(t + dt) = 0) \)

\[ f_{i,n}(t) = P(t \leq B_1 < t + dt, Y(t + dt) = i, X(t + dt) = n), \quad n = 0, 1, 2, \cdots, i=2,3,4 \]

Then \( f_{0,0}(t) = \frac{d}{dt} P(0,0,t) \)

\[ f_{i,n}(t) = \frac{d}{dt} P(i,n,t), \quad n = 0, 1, 2, \cdots, i=2,3,4 \]

Let the Laplace transform of \( f_{0,0}(t) \), \( f_{i,n}(t), i=2,3,4 \) are,

\[ f_{0,0}^*(s) = s P^*(0,0,s) \]

\[ f_{i,n}^*(s) = s P^*(i,n,s), \quad n = 0, 1, 2, \cdots, i=2,3,4 \]
Hence the Laplace transform of the busy period distribution in Type I service is

\[ b_1^*(s) = f_{0,0}^*(s) + \sum_{n=0}^{\infty} f_{2,n}^*(s) + \sum_{n=0}^{\infty} f_{3,n}^*(s) + \sum_{n=0}^{\infty} f_{4,n}^*(s) \]

\[ = sf_{0,0}^*(s) + \sum_{n=0}^{\infty} sP^*(2, n, s) + \sum_{n=0}^{\infty} sP^*(3, n, s) + \sum_{n=0}^{\infty} sP^*(4, n, s) \]

The Laplace transform of the transient probabilities of the system are given by

\[ sP^*(0, 0, s) = \mu_1 P^*(1, 0, s) \] (5.7.1)

\[ (s + \lambda + \mu_1)P^*(1, 0, s) - 1 = \mu_1 P^*(1, 1, s) \] (5.7.2)

\[ (s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n - 1, s) + \mu_1 P^*(1, n + 1, s), \quad 1 \leq n \leq c - 2 \] (5.7.3)

\[ (s + \lambda + \mu_1)P^*(1, c - 1, s) = \lambda P^*(1, c - 2, s) \] (5.7.4)

\[ sP^*(2, 0, s)) = \lambda P^*(1, c - 1, s) \] (5.7.5)

Solving (5.7.3) as a difference equation in \( P^*(1, n, s) \), we get

\[ P^*(1, n, s) = P^*(1, 0, s)R_1^n, \quad 1 \leq n \leq c - 2 \]

where \( R_1 \) is the unique positive real root less than unity of the equation

\[ \mu_1 z^2 - (s + \lambda + \mu_1)z + \lambda = 0. \]

From (5.7.1)

\[ P^*(0, 0, s) = P^*(1, 0, s)\frac{\mu_1}{s} \]

From (5.7.4) to (5.7.5)

\[ P^*(1, c - 1, s) = P^*(1, 0, s)e_1 R_1^{c-2} \]

\[ P^*(2, 0, s) = P^*(1, 0, s)\frac{\lambda}{s} e_1 R_1^{c-2} \]
and $P^*(1,0,s)$ can be obtained by using the normalizing condition

$$\sum_i \sum_n P^*(i,n,s) = \frac{1}{s},$$

as

$$P^*(1,0,s) = \left\{ \mu_1 + \frac{sR_1[1 - R_1^{-2}]}{1 - R_1} + se_1R_1^{-2} + \lambda e_1 R_1^{-2} \right\}^{-1}$$

Hence the expected busy period of the server is given by

$$\bar{B}_1 = \frac{-d}{ds}b_1^*(s)\bigg|_{s=0} = \frac{\lambda(1 + r_1) - 2r_1^c}{(1 - r_1)(\lambda + \mu_1 - r_1 c)} \quad (5.7.6)$$

### 5.8 Busy Period Distribution in Batch service

In this model, the busy period of the server in batch service begins when there are at least $c$ units in the queue and lasts till the queue size is less than $a$ for the first time after the service completion epoch of a Type II service.

The distribution of the busy period of the server $B_2$ can be obtained by considering the states $(0,0),(1,n), \ n = 0, 1, 2, \cdots, a-1. (4,n), \ n = 0, 1, 2, \cdots$ Also assume $P(2,0,0)=1$

Let $f_{0,0}(t) = P(t \leq B_2 < t + dt, Y(t + dt) = 0, \ X(t + dt) = 0)$

$f_{1,n}(t) = P(t \leq B_2 < t + dt, \ Y(t + dt) = 1, \ X(t + dt) = n, \ n = 0, 1, 2...a - 1)$

$f_{4,n}(t) = P(t \leq B_2 < t + dt, \ Y(t + dt) = 4, \ X(t + dt) = n, \ n = 0, 1, 2...)$

Then $f_{0,0}(t) = \frac{d}{dt}P(0,0,t)$

$f_{1,n}(t) = \frac{d}{dt}P(1,n,t), \ n = 0, 1, 2...a - 1$

$f_{4,n}(t) = \frac{d}{dt}P(4,n,t), \ n = 0, 1, 2...$

Let the Laplace transform of $f_{0,0}(t), f_{1,n}(t)$ and $f_{4,n}(t)$ are,

$f_{0,0}^*(s) = sP^*(0,0,s)$
\( f_{1,n}(s) = sP^*(1, n, s), n = 0, 1, 2 \ldots a - 1 \)

\( f_{4,n}(s) = sP^*(4, n, s), n = 0, 1, 2 \ldots \)

Hence the Laplace transform of the busy period distribution in batch service is

\[
\begin{align*}
    b_2^*(s) & = f_{0,0}^*(s) + \sum_{n=0}^{a-1} f_{1,n}^*(s) + \sum_{n=0}^{\infty} f_{4,n}^*(s) \\
    & = sf_{0,0}^*(s) + \sum_{n=0}^{a-1} sP^*(1, n, s) + \sum_{n=0}^{\infty} sP^*(4, n, s)
\end{align*}
\]

The Laplace transform of the transient probabilities of the system are given by

\[
\begin{align*}
    (s + \lambda + \mu_2)P^*(2, 0, s) - 1 & = \mu_2 \sum_{n=c}^{d} P^*(2, n, s) + \mu_3 \sum_{n=c}^{d} P^*(3, n, s) \quad (5.8.1) \\
    (s + \lambda + \mu_2)P^*(2, n, s) & = \lambda P^*(2, n - 1, s) + \mu_2 P^*(2, n + d, s) + \mu_3 P^*(3, n + d, s), \quad n \geq 1 \quad (5.8.2) \\
    (s + \lambda + \mu_3)P^*(3, 0, s) & = \mu_2 \sum_{n=a}^{c-1} P^*(2, n, s) + \mu_3 \sum_{n=a}^{c-1} P^*(3, n, s) \quad (5.8.3) \\
    (s + \lambda + \mu_3)P^*(3, n, s) & = \lambda P^*(3, n - 1, s), \quad n \geq 1 \quad (5.8.4) \\
    sP^*(4, 0, s) & = \mu_2 P^*(2, 0, s) + \mu_3 P^*(3, 0, s) \quad (5.8.5) \\
    sP^*(4, n, s) & = \mu_2 P^*(2, n, s) + \mu_3 P^*(3, n, s), \quad 1 \leq n \leq a - 1 \quad (5.8.6)
\end{align*}
\]

From (5.8.4)

\[
P^*(3, n, s) = P^*(3, 0, s)e_8^n, \quad n \geq 1
\]

Invoking Rouche’s theorem and solving (5.8.2) as a difference equation in \( P^*(2, n, s) \), we get

\[
P^*(2, n, s) = P^*(2, 0, s)R^n - \frac{e_{16}e_8^{n+d}P^*(3, 0, s)}{K(e_8)}, n \geq 1
\]

where

\[
K(z) = \mu_2 z^{d+1} - (s + \lambda + \mu_2)z + \lambda
\]
and $R$ is the unique positive real root less than unity of the equation $K(z) = 0$

From (5.8.3)

$$P^*(3, 0, s) = P^*(2, 0, s)A_1$$

From (5.8.5 and (5.8.6)

$$P^*(4, 0, s) = P^*(2, 0, s)\left\{ \frac{\mu_2}{s} + \frac{\mu_3 A_1}{s} \right\}$$

$$P^*(4, n, s) = P^*(2, 0, s)\left\{ \frac{\mu_2}{s} \left\{ R^n - \frac{e_16e_8^{n+d} A_1}{K(e_8)} \right\} + \frac{\mu_3 e_8 A_1}{s} \right\}, \quad 1 \leq n \leq a - 1$$

and $P^*(2, 0, s)$ can be obtained by using the normalizing condition

$$\sum_i \sum_n P^*(i, n, s) = \frac{1}{s}$$

as

$$P^*(2, 0, s) = \left\{ \mu_2 + \mu_3 A_1 + sA_1 + \frac{\mu_2 R(1 - R^{a-1})}{1 - R} - \frac{\mu_2 e_16e_8^{d+1} A_1(1 - e_8^{a-1})}{K(e_8)(1 - e_8)} \right\}$$

$$+ \frac{\mu_3 A_1(1 - e_8^{a-1})}{1 - e_8} + \frac{s}{1 - R} - \frac{se_16e_8^{d+1} A_1}{K(e_8)(1 - e_8)} + \frac{se_8 A_1}{1 - e_8}\right\}^{-1}$$

Hence the expected busy period of the server is given by

$$\bar{B}_2 = -\frac{d}{ds} b_2^*(s)\bigg|_{s=0}$$

$$= \left( (1 - \theta_8) K(\theta_8) + D_1(1 - r)[k(\theta_8) - \theta_10\theta_8^{d+1}] \right)$$

$$\times \left( \mu_3 D_1 K(\theta_8)(1 - r)(1 - \theta_8^{a+1}) + \mu_2(1 - r^{a-1})(1 - \theta_8)K(\theta_8) \right.$$

$$- \mu_2\theta_10\theta_8^{d+1} D_1(1 - \theta_8^{a-1})(1 - r))^{-1}$$

(5.8.7)

where

$$D_1 = \theta_{14} \frac{(r^a - r^c)}{1 - r} \left[ 1 + \left( \frac{\theta_{14}\theta_8^d}{K(\theta_8)} - \theta_{17} \right) \left( \frac{\theta_8^a - \theta_8^c}{1 - \theta_8} \right) \right]^{-1}$$
5.9 Busy Period Distribution in Type II service

In this model, the busy period of the server in Type II service begins if there are at least $c$ units in the queue and lasts till the queue size is less than $c$ for the first time. The server renders Type II service with rate $\mu_2$.

The distribution of the busy period of the server $B_3$ can be obtained by considering the states $(0,0),(1,n)$, $n=0,1,2,...a-1$ $(3,n)$, $n=0,1,2,...$ and $(4,n)$, $n=0,1,2,...$ are absorbing. Also assume $P(2,0,0)=1$

Let $f_{0,0}(t) = P(t \leq B_3 < t + dt, Y(t + dt) = 0, X(t + dt) = 0)$

$f_{1,n}(t) = P(t \leq B_3 < t + dt, Y(t + dt) = 1, X(t + dt) = n, \ n = 0, 1, 2...a - 1)$

and

$f_{i,n}(t) = P(t \leq B_3 < t + dt, Y(t + dt) = i, X(t + dt) = n, \ n = 0, 1, 2...), \ i=3, 4$

Then $f_{0,0}(t) = \frac{d}{dt}P(0,0,t)$

$f_{1,n}(t) = \frac{d}{dt}P(1,n,t), \ n = 0, 1, 2...a - 1$

and $f_{i,n}(t) = \frac{d}{dt}P(i,n,t), \ n = 0, 1, 2..., \ i=3, 4$

Let the Laplace transform of $f_{0,0}(t), f_{1,n}(t)$ and $f_{i,n}(t), \ i=3, 4$ are,

$f_{0,0}^{*}(s) = sP^{*}(0,0,s)$

$f_{1,n}^{*}(s) = sP^{*}(1,n,s), \ n = 0, 1, 2...a - 1$

$f_{i,n}^{*}(s) = sP^{*}(i,n,s), \ n = 0, 1, 2..., \ i=3, 4$

Hence the Laplace transform of the busy period distribution in
Type II service is

\[ b_1^*(s) = f_{0,0}^*(s) + \sum_{n=0}^{a-1} f_{1,n}^*(s) + \sum_{n=0}^{\infty} f_{3,n}^*(s) + \sum_{n=0}^{\infty} f_{4,n}^*(s) \]

\[ = s f_{0,0}^*(s) + \sum_{n=0}^{a-1} s P^*(1, n, s) + \sum_{n=0}^{\infty} s P^*(3, n, s) + \sum_{n=0}^{\infty} s P^*(4, n, s) \]

The Laplace transform of the transient probabilities of the system are given by

\[ (s + \lambda + \mu_2) P^*(2, 0, s) - 1 = \mu_2 \sum_{n=c}^{d} P^*(2, n, s) \]

(5.9.1)

\[ (s + \lambda + \mu_2) P^*(2, n, s) = \lambda P^*(2, n - 1, s) + \mu_2 P^*(2, n + d, s), \quad n \geq 1 \]  

(5.9.2)

\[ s P^*(3, 0, s) = \mu_2 \sum_{n=a}^{c-1} P^*(2, n, s) \]  

(5.9.3)

\[ s P^*(4, 0, s) = \mu_2 P^*(2, 0, s) \]  

(5.9.4)

\[ s P^*(4, n, s) = \mu_2 P^*(2, n, s), \quad 1 \leq n \leq a - 1 \]  

(5.9.5)

Invoking Rouche’s theorem and solving (5.9.2) as a difference equation in \( P^*(2, n, s) \) we get

\[ P^*(2, n, s) = P^*(2, 0, s) R^n, \quad n \geq 1 \]

where \( R \) is the unique positive real root less than unity of the equation

\[ \mu_2 z^{d+1} - (s + \lambda + \mu_2) z + \lambda = 0 \]

From (5.9.3)

\[ P^*(3, 0, s) = P^*(2, 0, s) \mu_2 \frac{(R^a - R^c)}{1 - R} \]

From (5.9.4) and (5.9.5)

\[ P^*(4, 0, s) = P^*(2, 0, s) \frac{\mu_2}{s} \]

\[ P^*(4, n, s) = P^*(2, 0, s) R^n \frac{\mu_2}{s}, \quad 1 \leq n \leq a - 1 \]
and \( P^*(2,0,s) \) can be obtained by using the normalizing condition
\[
\sum_i \sum_n P^*(i,n,s) = \frac{1}{s}
\]
as
\[
P^*(2,0,s) = \left\{ \frac{\mu_2}{1-R} + \frac{\mu_2(R^a-R^c)}{1-R} + \frac{s}{1-R} \right\}^{-1}
\]
Hence the expected busy period of the server is given by
\[
\bar{B}_3 = \left. -\frac{db_3^*(s)}{ds} \right|_{s=0}
\]
\[
= \left\{ \mu_2[1+r^a-r^c] \right\}^{-1}
\]
(5.9.6)

5.10 Busy Period Distribution in Type III service

In this model, the server begins Type III service with rate \( \mu_2 \) only if queue size becomes less than \( c \) and not less than a secondary limit \( a \), after a Type II service completion epoch. Hence the busy period of the server in Type III service begins when the number of units in the queue is less than \( c \) but not less than the limit \( a \) (i.e., \( a \leq n \leq c - 1 \)), after a Type II service completion epoch and lasts till the queue size becomes less than \( a \) or above the level \( c \) after a Type II service completion epoch. In this case the server serves the units altogether in a batch with rate \( \mu_3 \).

The distribution of the busy period of the server \( B_4 \) can be obtained by considering the states \((0,0),(1,n), n=0,1,2,...a-1 \) \((2,n), n=0,1,2,... \) and \((4,n), n=0,1,2,... \) are absorbing. Also assume \( P(3,0,0)=1 \)

Let \( f_{0,0}(t) = P(t \leq B_4 < t + dt, Y(t+dt) = 0, X(t+dt) = 0) \)

\( f_{1,n}(t) = P(t \leq B_4 < t + dt, Y(t+dt) = 1, X(t+dt) = n, n = 0,1,2...a-1) \)
and
\[ f_{i,n}(t) = P(t \leq B_4 < t + dt, Y(t + dt) = i, X(t + dt) = n, \quad n = 0, 1, 2..., \quad i=2, 4 ] \]
Then \[ f_{0,0}(t) = \frac{d}{dt} P(0, 0, t) \]
\[ f_{1,n}(t) = \frac{d}{dt} P(1, n, t), \quad n = 0, 1, 2...a - 1 \]
and \[ f_{i,n}(t) = \frac{d}{dt} P(i, n, t), \quad n = 0, 1, 2..., \quad i=2, 4 \]
Let the Laplace transform of \( f_{0,0}(t), f_{1,n}(t) \) and \( f_{i,n}(t), i=2, 4 \) are,
\[ f_{0,0}^*(s) = sP^*(0, 0, s) \]
\[ f_{1,n}^*(s) = sP^*(1, n, s), n = 0, 1, 2...a - 1 \]
\[ f_{i,n}^*(s) = sP^*(i, n, s), n = 0, 1, 2..., \quad i=2, 4 \]

Hence the Laplace transform of the busy period distribution in Type III service is
\[ b_4^*(s) = f_{0,0}^*(s) + \sum_{n=0}^{a-1} f_{1,n}^*(s) + \sum_{n=0}^{\infty} f_{2,n}^*(s) + \sum_{n=0}^{\infty} f_{4,n}^*(s) \]
\[ = sf_{0,0}^*(s) + \sum_{n=0}^{a-1} sP^*(1, n, s) + \sum_{n=0}^{\infty} sP^*(2, n, s) + \sum_{n=0}^{\infty} sP^*(4, n, s) \]

The Laplace transform of the transient probabilities of the system are given by
\[ sP^*(2, 0, s) = \mu_3 \sum_{n=0}^{d} P^*(3, n, s) \] (5.10.1)
\[ sP^*(2, n, s) = \mu_3 P^*(3, n + d, s), \quad n \geq 1 \] (5.10.2)
\[ (s + \lambda + \mu_3)P^*(3, 0, s) - 1 = \mu_3 \sum_{n=0}^{c-1} P^*(3, n, s) \] (5.10.3)
\[ (s + \lambda + \mu_3)P^*(3, n, s) = \lambda P^*(3, n - 1, s), \quad n \geq 1 \] (5.10.4)
\[ sP^*(4, 0, s) = \mu_3 P^*(3, 0, s) \] (5.10.5)
\[ sP^*(4, n, s) = \mu_3 P^*(3, n, s), \quad 1 \leq n \leq a - 1 \] (5.10.6)
From (5.10.4) \( P^*(3, n, s) = P^*(3, 0, s)e_8^n, \quad n \geq 1 \)
From (5.10.2) \( P^*(2, n, s) = P^*(3, 0, s)\frac{\mu_3}{s}e_{8}^{n+d}, \quad n \geq 1 \)
From (5.10.1) \[ P^*(2, 0, s) = P^*(3, 0, s) \frac{\mu_3}{s} \left[ \frac{e_8^c - e_8^{d+1}}{1 - e_8} \right] \]

From (5.10.5) and (5.10.6)

\[ P^*(4, 0, s) = P^*(3, 0, s) \frac{\mu_3}{s} \]
\[ P^*(4, n, s) = P^*(3, 0, s) \frac{\mu_3}{s} e_8^n, \quad 1 \leq n \leq a - 1 \]

and \( P^*(3, 0, s) \) can be obtained by using the normalizing condition

\[
\sum_i \sum_n P^*(i, n, s) = \frac{1}{s}
\]

as

\[
P^*(3, 0, s) = \left\{ \mu_3 + s + \frac{\mu_3 (e_8^c - e_8^{d+1})}{1 - e_8} + \frac{\mu_3 e_8 (1 - e_8^{a-1})}{1 - e_8} \right. \\
\left. + \frac{\mu_3 e_8^{d+1}}{1 - e_8} + \frac{s e_8}{1 - e_8} \right\}^{-1}
\]

Hence the expected busy period of the server is given by

\[
\bar{B}_4 = -\frac{d}{ds} b_4^* (s) \bigg|_{s=0} = \mu_3 \left\{ 1 + \theta_8^c - \theta_8^{a-1} \right\}^{-1} \quad (5.10.7)
\]

In this model the server is idle when he is in the state (0,0) The expected length of an idle period \( E(I) \) is given by

\[
\bar{I} = \frac{1}{\lambda}
\]

Hence the expected length of a Busy cycle \( E(c) \) is given by

\[
\bar{c} = \bar{B} + \bar{I}
\]

where

\[
B = B_1 + B_3 + B_4
\]
The distribution of the busy period of the server in vacation, $B_5$ can be obtained by considering the states $(0,0)$, $(1,n)$; $n=0,1,2,...a-1$, $(2,n)$; $n=0,1,2,...$ and $(3,n)$; $n=0,1,2...$ as absorbing.

The Laplace transform of the transient probabilities of the system are given by

$$sP^*(2,0,s) = \beta \sum_{n=c}^{d} P^*(4,n,s)$$  \hspace{1cm} (5.10.8)

$$sP^*(2,n,s) = \beta P^*(4,n+d,s), \quad n \geq 1$$  \hspace{1cm} (5.10.9)

$$P^*(4,0,s) = 1$$  \hspace{1cm} (5.10.10)

$$(s+\lambda)P^*(4,n,s) = \lambda P^*(4,n-1,s), \quad 1 \leq n \leq c-1$$  \hspace{1cm} (5.10.11)

$$(s+\lambda+\beta)P^*(4,n,s) = \lambda P^*(4,n-1,s), \quad n \geq c$$  \hspace{1cm} (5.10.12)

From (5.10.11) $P^*(4,n,s) = e_1^n$, $1 \leq n \leq c-1$

From (5.10.12) $P^*(4,n,s) = e_5^{n-c+1}e_1^{c-1}$, $n \geq c$

From (5.10.8) $P^*(2,0,s) = \frac{\beta}{s} e_5 e_1^{c-1} \left[ \frac{1 - e_5^{d-c+1}}{1 - e_5} \right]$

From (5.10.9) $P^*(2,n,s) = \frac{\beta}{s} e_1^{c-1} e_5^{n-1} e_1^{d-c+1}$, $n \geq 1$

Hence the Laplace transform of the busy period distribution is given by

$$b_5^*(s) = f_{0,0}^*(s) + \sum_{n=0}^{a-1} f_{1,n}^*(s) + \sum_{n=0}^{\infty} f_{2,n}^*(s) + \sum_{n=0}^{\infty} f_{3,n}^*(s)$$

$$= sf_{0,0}^*(s) + \sum_{n=0}^{a-1} sP^*(1,n,s) + \sum_{n=0}^{\infty} sP^*(2,n,s) + \sum_{n=0}^{\infty} sP^*(3,n,s)$$

and the expected busy period of the server in vacation is given by

$$\bar{B}_5 = \left. \frac{-d}{ds} b_5^*(s) \right|_{s=0} = \frac{\beta [c\lambda (1 - \theta_5) + \beta (c - 1)(1 - \theta_5) + \lambda \theta_5]}{(\lambda + \beta)^2(1 - \theta_5)^2}$$  \hspace{1cm} (5.10.13)
5.11 Optimal Policy

The design of an optimal policy for a queueing system has received a lot of attention, as shown by the survey conducted by Tadj and Choudhury (2005). This is known in queueing theory as the optimal control of the system. The aim is to find the best values that the decision maker would implement in order to minimize the total expected cost per unit of time.

Let $C_h$: The holding cost per unit time in the system.

$C_a$: Start up cost per unit time for the preparatory work of the server before starting the service.

$C_0$: Set up cost per busy cycle.

$C_{s1}$: Cost per unit time for keeping the server in Type I service.

$C_{s3}$: Cost per unit time for keeping the server in Type II service.

$C_{s4}$: Cost per unit time for keeping the server in Type III service.

$C_v$: Cost per unit time for keeping the server in vacation.

In our model, the start up cost $C_a$ is charged only for starting the service. There is no start up cost for the subsequent service batches in a busy period.

The linear cost function constructed for determining the optimal control limits $c$ and $a$ is

$$TC1(a, c) = C_h L_q + C_{s1} \frac{B_1}{C} + C_{s3} \frac{B_3}{C} + C_{s4} \frac{B_4}{C} + C_0 \frac{1}{C} + C_a \frac{I}{C} + C_v \frac{B_5}{C}$$
5.12 Particular Cases

Case 1: When $\beta = 0$

From equation (5.5.1) to (5.5.13), the steady state probabilities are obtained as follows

\[
P(0, 0) = B_4P(2, 0) \quad (5.12.1)
\]
\[
P(1, 0) = B_2P(2, 0) \quad (5.12.2)
\]
\[
P(1, n) = \left[ B_2r_1^n - \frac{\theta_{12}r_1^{n+1}}{K_1'(r)} - \frac{\theta_{15}\theta_8^{n+1}B_1}{K_1'(\theta_8)} \right] P(2, 0); \quad 1 \leq n \leq a - 1 \quad (5.12.3)
\]
\[
P(1, n) = B_3r_1^n P(2, 0); \quad a \leq n \leq c - 2 \quad (5.12.4)
\]
\[
P(1, c - 1) = \theta_6B_3r_1^{c-2}P(2, 0) \quad (5.12.5)
\]
\[
P(2, 0) = B_5 \quad (5.12.6)
\]
\[
P(2, n) = r^n P(2, 0) - \frac{\theta_{16}\theta_8^{n+d}B_1}{K'(\theta_8)} P(2, 0); \quad n \geq 1 \quad (5.12.7)
\]
\[
P(3, 0) = B_4P(2, 0) \quad (5.12.8)
\]
\[
P(3, n) = \theta_8^n B_4P(2, 0); \quad n \geq 1 \quad (5.12.9)
\]

\[B_1 = \theta_{14} \left( \frac{r^a - r^c}{1 - r} \right) \left[ 1 + \left( \frac{\theta_{14}\theta_{16}\theta_6^d}{K'(\theta_8)} - \theta_{17} \right) \left( \frac{\theta_8^{a} - \theta_8^{c}}{1 - \theta_8} \right) \right]^{-1}
\]
\[B_2 = \frac{\theta_6\theta_8^2B_1 - \theta_9\theta_{12}r_1^2}{K_1'(r)} - \frac{\theta_6\theta_{15}\theta_8^2B_1}{K_1'(\theta_8)} + \theta_{12}r - \frac{\theta_{12}\theta_{16}\theta_8^{d+1}B_1}{K'(\theta_8)} + \theta_{13}\theta_8B_1}{1 - \theta_6\theta_2 - \theta_9r_1}
\]
\[B_3 = B_2r_1^{a-1} - \frac{\theta_{12}r^a}{K_1'(r)} - \frac{\theta_8^2B_1}{K_1'(\theta_8)}
\]
\[B_4 = \theta_2B_2 + \theta_3 + \theta_4B_1
\]
\[ B_5 = \left[ B_4 + B_2 + B_2 r_1 \frac{(1 - r_1^{a-1})}{1 - r_1} - \frac{\theta_{12} r^2 (1 - r^{a-1})}{K'_1(r)(1 - r)} - \frac{\theta_{15} \theta_8^2 B_1 (1 - \theta_8^{a-1})}{K'_1(\theta_8)(1 - \theta_8)} \right]^{-1} \]

\[ + B_3 \left( \frac{r^c - r_1^{c-1}}{1 - r_1} \right) + \theta_6 B_3 r_1^{c-2} + 1 + \frac{r}{1 - r} - \frac{\theta_{16} \theta_8^{d+1} B_1}{K'(\theta_8)(1 - \theta_8)} + B_1 + \frac{B_1 \theta_8}{1 - \theta_8} \]

where \( r \) is the unique positive real root less than unity of the equation

\[ \mu_2 z^{d+1} - (\lambda + \mu_2) z + \lambda = 0 \] and \( r_1 = \frac{\lambda}{\mu_1} \). Here for the existence of steady state distribution we assume that \( \frac{\lambda}{\mu_1} < 1 \) and \( \frac{\lambda}{d \mu_2} < 1 \).

From equation (5.7.6), the expected busy period of the server in Type I service is,

\[ \bar{B}_1 = \frac{\lambda(1 + r_1) - 2 r_1^c}{(1 - r_1)(\lambda + \mu_1 - r_1^c)} \] (5.12.10)

From (5.8.7), the expected busy period of the server in batch service is,

\[ \bar{B}_2 = \frac{-d}{ds} b_2^*(s) \bigg|_{s=0} = \left( (1 - \theta_8) K(\theta_8) + D_1 (1 - r) [k(\theta_8) - \theta_{16} \theta_8^{d+1}] \right) \times \left\{ \mu_3 D_1 K(\theta_8) (1 - r) (1 - \theta_8^{a+1}) + \mu_2 (1 - r^{a+1})(1 - \theta_8) K(\theta_8) \right. \\
\left. - \mu_2 \theta_{16} \theta_8^{d+1} D_1 (1 - \theta_8^a) (1 - r) \right\}^{-1} \] (5.12.11)

where

\[ D_1 = \theta_{14} \frac{(r^a - r_1^c)}{1 - r} \left[ 1 + \left( \frac{\theta_{14} \theta_1 \theta_8^d}{K(\theta_8)} - \theta_{17} \right) \left( \frac{\theta_8^a - \theta_8^c}{1 - \theta_8} \right) \right]^{-1} \]

From (5.9.6), the expected busy period of the server in Type II service is,

\[ \bar{B}_3 = \left\{ \mu_2 [(1 - r^c) + r^a (1 - r)] \right\}^{-1} \] (5.12.12)

From (5.10.7), the expected busy period of the server in Type III service is,

\[ \bar{B}_4 = \mu_3 \left\{ 1 + \theta_8^a - \theta_8^{a+1} \right\}^{-1} \] (5.12.13)
which agrees with the results of 'A state dependent M/M/1 single and batch service queue under the policy \((a, c, d)\).

**Case 2:** When \(\mu_3 = \mu_2\)

From equation (5.5.1) to (5.5.13), the steady state probabilities are obtained as follows

Here \(\theta_3 = \theta_4, \ theta_7 = \theta_8, \ theta_{10} = \theta_{11}, \ theta_{12} = \theta_{15} \ and \theta_{13} = \theta_{14} = \theta_{16} = \theta_{17}\)

\[
P(0, 0) = B_4 P(2, 0) \quad (5.12.14)
\]

\[
P(1, 0) = B_2 P(2, 0) \quad (5.12.15)
\]

\[
P(1, n) = \left[ B_2 r_1^n - \frac{\theta_7 r^{n+1}}{K'_1(r)} - \frac{\theta_{12} \theta_7^{n+1} B_1}{K_1'(\theta_7)} \right] P(2, 0); \quad 1 \leq n \leq a - 1 \quad (5.12.16)
\]

\[
P(1, n) = B_3 r_1^n P(2, 0) \ ; \quad a \leq n \leq c - 2 \quad (5.12.17)
\]

\[
P(1, c - 1) = \theta_6 B_3 r_1^{c-2} P(2, 0) \quad (5.12.18)
\]

\[
P(2, 0) = B_5 \quad (5.12.19)
\]

\[
P(2, n) = r^n P(2, 0) - \frac{\theta_{13} \theta_5^{n+d} B_1}{K'(\theta_7)} P(2, 0); \quad n \geq 1 \quad (5.12.20)
\]

\[
P(3, 0) = B_1 P(2, 0) \quad (5.12.21)
\]

\[
P(3, n) = \theta_7^2 B_1 P(2, 0); \quad n \geq 1 \quad (5.12.22)
\]

\[
B_1 = \theta_{13} \left( \frac{r^a - r^c}{1 - r} \right) \left[ 1 + \left( \frac{\theta_2^2 \theta_7^d}{K'(\theta_7)} - \theta_1 \right) \left( \frac{\theta_7^2 - \theta_7^c}{1 - \theta_7} \right) \right]^{-1}
\]

\[
B_2 = \frac{\theta_9 \theta_8 r^2 - \theta_9 \theta_{12} \theta_7^2 B_1 - \theta_{12} \theta_7^{d+1} B_1 + \theta_{13} \theta_7 B_1}{1 - \theta_6 \theta_2 - \theta_9 r_1}
\]

\[
B_3 = B_2 r_1^{a-1} - \frac{\theta_7 \theta_{12} r_1^a}{K'_1(r)} - \frac{\theta_7 \theta_{12}^2 B_1}{K_1'(\theta_7)}
\]

\[
B_4 = \theta_2 B_2 + \theta_3 + \theta_3 B_1
\]
\[ B_5 = \left[ B_4 + B_2 + B_2 r_1 \frac{(1 - r_1^{-1})}{1 - r_1} - \frac{\theta_1 r^2 (1 - r^a)}{K_1'(r)(1 - r)} - \frac{\theta_1 \theta_2 B_1 (1 - r_1^{-1})}{K_1'(r_1)(1 - r_1)} \right. \\
+ B_3 \left( \frac{r_1^{-a} - r_1^{-c}}{1 - r_1} \right) + \theta_0 B_3 r_1^{-2} + 1 + \frac{r}{1 - r} - \frac{\theta_1 \theta_2^d + B_1}{K'(r)(1 - r_7)} + B_1 + \frac{B_1 \theta_7}{1 - r_7} \right]^{-1} \]

where \( r \) is the unique positive real root less than unity of the equation \( \mu_2 z^{d+1} - (\lambda + \mu_2) z + \lambda = 0 \) and \( r_1 = \frac{\lambda}{\mu_1} \). Here for the existence of steady state distribution we assume that \( \frac{\lambda}{\mu_1} < 1 \) and \( \frac{\lambda}{d \mu_2} < 1 \).

From equation (5.7.6), the expected busy period of the server in Type I service is,

\[ \bar{B}_1 = \frac{\lambda(1 + r_1) - 2 r_1^c}{(1 - r_1)(\lambda + \mu_1 - r_1^c)} \] (5.12.23)

From (5.8.7), the expected busy period of the server in batch service is,

\[ \bar{B}_2 = \left( (1 - \theta_7) K(\theta_7) + D_1(1 - r)[k(\theta_1) - \theta_1 \theta_7^{d+1}] \right) \times \left\{ \mu_3 D_1 K(\theta_7)(1 - r)(1 - \theta_7^{a+1}) + \mu_2 (1 - r_1^{a+1})(1 - \theta_7) K(\theta_7) \right. \]
\[ - \mu_2 \theta_1 \theta_7^{d+1} D_1(1 - \theta_7^a)(1 - r) \right\}^{-1} \] (5.12.24)

where

\[ D_1 = \frac{\theta_1}{1 - r} \left[ \left( \frac{\theta_1^{d+2}}{K(\theta_1)} - \theta_1 \right) \left( \frac{\theta_1^2 - \theta_7^2}{1 - \theta_7} \right) \right]^{-1} \]

From (5.9.6), the expected busy period of the server in Type II service is,

\[ \bar{B}_3 = \left\{ \mu_2[(1 - r_1^c) + r^{a_1}(1 - r)] \right\}^{-1} \] (5.12.25)

From (5.10.7), the expected busy period of the server in Type III service is,

\[ \bar{B}_4 = \left\{ \mu_3 \left[ 1 + \theta_7^c - \theta_7^{a+1} \right] \right\}^{-1} \] (5.12.26)

which agrees with the results of 'An \( (a, c, d) \) policy M/M/1 queue with single and batch service' considered by C.Baburaj and P.P.Jayakumar (2005).
Case 3: When $\mu_3 = \mu_2$, c=a

Then from (5.4.1) to (5.4.9) the steady state probabilities are obtained as follows

Here $\theta_3 = \theta_4$, $\theta_7 = \theta_8$, $\theta_{10} = \theta_{11}$, $\theta_{12} = \theta_{15}$ and $\theta_{13} = \theta_{14} = \theta_{16} = \theta_{17}$

$$P(0,0) = B_4P(2,0) \quad (5.12.27)$$

$$P(1,0) = B_2P(2,0) \quad (5.12.28)$$

$$P(1,n) = B_3r_1^nP(2,0) ; \quad 1 \leq n \leq c-2 \quad (5.12.29)$$

$$P(1,c-1) = \theta_4B_3r_1^{c-2}P(2,0) \quad (5.12.30)$$

$$P(2,0) = B_5 \quad (5.12.31)$$

$$P(2,n) = r^n P(2,0); \quad n \geq 1 \quad (5.12.32)$$

where

$$B_2 = \frac{\theta_9 \theta_{12} r^2}{K_1'(r)}$$

$$B_3 = B_2 r_1^{a-1} - \frac{\theta_{12} r^a}{K_1'(r)}$$

$$B_4 = \theta_2B_2 + \theta_3$$

$$B_5 = \left[ B_4 + B_2 + B_3 \frac{(1 - r_1^{c-2})}{1 - r_1} + \theta_6 B_3 r_1^{c-2} + 1 + \frac{r}{1-r} \right]^{-1}$$

where $r$ is the unique positive real root less than unity of the equation $\mu_2 z^{d+1} - (\lambda + \mu_2)z + \lambda = 0$ and $r_1 = \frac{\lambda}{\mu_1}$. Here for the existence of steady state distribution we assume that $\frac{\lambda}{\mu_1} < 1$ and $\frac{\lambda}{d\mu_2} < 1$

From equation (5.7.6), the expected busy period of the server in Type I service is

$$\bar{B}_1 = \frac{\lambda(1 + r_1) - 2r_1^a}{(1 - r_1)(\lambda + \mu_1 - r_1^a)} \quad (5.12.33)$$
From (5.8.7), the expected busy period of the server in batch service is,

\[ \bar{B}_2 = \left\{ \mu_2(1 - r^{a+1}) \right\}^{-1} \]

From (5.9.6), the expected busy period of the server in Type II service is,

\[ \bar{B}_3 = \{\mu_2(1 - r)\}^{-1} \quad (5.12.34) \]

From (5.10.7), the expected busy period of the server in Type III service is,

\[ \bar{B}_4 = \mu_3 \left\{ 1 + \theta^e - \theta^{e+1} \right\}^{-1} \quad (5.12.35) \]

which agrees with the results of ’A single and batch service queue with single a control on batch size’ considered by M.Manoharan and C.Baburaj (1999).

**Case 4:** When \( \mu_1 = 0, \mu_2 = \mu_3 = \mu \)

From (5.5.1) to (5.5.13), the steady state probabilities are obtained as follows

\[ P(0,0) = P(2,0).\frac{\mu}{\lambda} \quad (5.12.36) \]

\[ P(0,n) = P(2,0).\frac{\mu}{\lambda} \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}, \quad 1 \leq n \leq a - 1 \quad (5.12.37) \]

\[ P(0,n) = P(2,0).\frac{\mu}{\lambda} \cdot \frac{1 - r^a}{1 - r}, \quad a \leq n \leq c - 1 \quad (5.12.38) \]

\[ P(2,n) = P(2,0).r^n, \quad n \geq 1 \quad (5.12.39) \]

and

\[ P(2,0) = \left\{ \frac{\mu}{\lambda(1 - r)} \left[ a - \frac{1 - r^a}{1 - r} + (1 - r^a)(c - a) \right] + \frac{1}{1 - r} \right\}^{-1} \quad (5.12.40) \]

where \( r \) is the unique positive real root less than unity of the equation

\[ \mu z^{d+1} - (\lambda + \mu) z + \lambda = 0. \]
The expected busy period of the server is,

\[ \bar{B} = \{\mu(1 - r^a)}\}^{-1} \]  

(5.12.41)

These agrees with the corresponding results of the standard \( M/M(a, c, d)/1 \) model considered by Baburaj and Surendranath (2005).

**Case 5:** When \( \mu_1 = 0, \mu_2 = \mu_3 = \mu, a=c \)

From (5.5.1) to (5.5.13), the steady state probabilities are obtained as follows

\[ P(0, 0) = P(2, 0) \frac{\mu}{\lambda} \]  

(5.12.42)

\[ P(0, n) = P(2, 0) \frac{\mu}{\lambda} \left\{ \frac{1 - r^{n+1}}{1 - r} \right\}, \quad 1 \leq n \leq c - 1 \]  

(5.12.43)

\[ P(2, n) = P(2, 0) r^n, \quad n \geq 1 \]  

(5.12.44)

and

\[ P(2, 0) = \left\{ \frac{\mu}{\lambda(1 - r)} \left[ a - \frac{1 - r^a}{1 - r} \right] + \frac{1}{1 - r} \right\}^{-1}, \]  

(5.12.45)

where \( r \) is the unique positive real root less than unity of the equation

\[ \mu z^{d+1} - (\lambda + \mu) z + \lambda = 0. \]

The expected busy period of the server is,

\[ \bar{B} = \{\mu(1 - r^a)}\}^{-1} \]  

(5.12.46)

These agrees with the corresponding results of the standard \( M/M(a, b)/1 \) model.
5.13 Numerical Illustration

The table giving the values of steady state probabilities for the values of $\lambda=1.6$, $\mu_1=3.5$, $\mu_2=2.5$, $\mu_3=1.7$, $\beta=2$, $a=3$, $c=7$ and $d=30$ are $P(0,0) = 0.53976$ and

<table>
<thead>
<tr>
<th>n</th>
<th>P(1,n)</th>
<th>P(2,n)</th>
<th>P(3,n)</th>
<th>P(4,n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.46748E-01</td>
<td>8.57803E-04</td>
<td>6.93351E-05</td>
<td>1.95011E-03</td>
</tr>
<tr>
<td>1</td>
<td>1.12799E-01</td>
<td>3.34752E-04</td>
<td>3.36170E-05</td>
<td>1.97275E-03</td>
</tr>
<tr>
<td>2</td>
<td>5.15654E-02</td>
<td>1.30635E-04</td>
<td>1.62992E-05</td>
<td>2.19419E-03</td>
</tr>
<tr>
<td>3</td>
<td>2.35727E-02</td>
<td>5.09795E-05</td>
<td>7.90262E-06</td>
<td>2.19419E-03</td>
</tr>
<tr>
<td>4</td>
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<td>1.98944E-05</td>
<td>3.83157E-06</td>
<td>2.19419E-03</td>
</tr>
<tr>
<td>5</td>
<td>4.92622E-03</td>
<td>7.76367E-06</td>
<td>1.85773E-06</td>
<td>2.19419E-03</td>
</tr>
<tr>
<td>6</td>
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<td>3.02973E-06</td>
<td>9.00719E-07</td>
<td>1.11329E-09</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1.18233E-06</td>
<td>4.36712E-07</td>
<td>4.94800E-10</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>4.61398E-07</td>
<td>2.11739E-07</td>
<td>2.19910E-10</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>1.80058E-07</td>
<td>1.02661E-07</td>
<td>9.77400E-11</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>7.02663E-08</td>
<td>4.97752E-08</td>
<td>4.34400E-11</td>
</tr>
</tbody>
</table>
The graph of the expected queue length plotted for $\lambda = 1.6$, $\mu_1 = 3.5$, $\mu_2 = 2.5$, $\mu_3 = 1.7$, $\beta = 2$, $d = 30$ and different values $a$

![Graph showing expected queue length](Image)

**Figure 5.13.1**

**Remarks**: The expected queue length decreases for $c = 7$ onwards.
The table giving the values of the expected cost function for the values of \( \lambda=0.9, \)
\( \mu_1=1.0, \mu_2=2.5, \mu_3=5.7, \beta = 3.9, d = 15, C_h = 50, C_0 = 1000, C_{s1} = 250, C_{s3} = 200, C_{s4} = 750, C_a = 660, C_v = -5. \)

**Table 5.13.2**

<table>
<thead>
<tr>
<th>c\a</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
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<tr>
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<td>16949</td>
<td>16949</td>
</tr>
<tr>
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<td>16932</td>
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<td>16932</td>
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<td>16937</td>
<td>16936</td>
<td>16936</td>
<td>16936</td>
</tr>
<tr>
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<td>16945</td>
<td>16943</td>
<td>16943</td>
<td>16943</td>
<td>16943</td>
</tr>
</tbody>
</table>

**Remarks** : The \( TC1(a, c) \) is minimum for \( a = 5 \) and \( c = 6. \)