Chapter 1

INTRODUCTION

In real life models, many systems are related to 'uncertainty' and/or 'inexactness'. The problem of 'inexactness' is considered in general as exact science, whereas that of 'uncertainty' is vague or fuzzy and accidental. Fuzzy concept is an innovative technology that enhances conventional systems design with engineering expertise. The application of fuzzy set theory matches day-to-day realities. It is better than the standard set theory because all the phenomena and observations can have more than two definite states.

Fuzzy Mathematics as proposed by Zadeh [106] has proved to be one of the powerful fields with which several useful avenues can be explored for the benefit of mankind. He used it as a tool for dealing with uncertainty arising out of lack of information about certain complex systems. Thus fuzzy set is a collection of objects with grade membership in continuum with each object being assigned a value between 0 and 1. The only membership possibilities for an ordinary or crisp subset are non membership or full membership. Such a set $A$ can thus be identified with the fuzzy set given by the characteristic function $\chi_A : X \to [0, 1]$.

Example 1.1 Let $A$ be a mapping from $R$ to $[0, 1]$ defined by

$$A(x) = \begin{cases} 
0 & \text{if } x \leq 1 \\
\frac{1}{50}(x - 1) & \text{if } 1 < x < 50 \\
1 & \text{if } 50 \leq x.
\end{cases}$$

Then $A$ is a fuzzy set.
Fuzzy sets are taken up with enthusiasm by engineers, computer scientists and operations researchers, particularly in Japan where fuzzy controllers are now an integral part of many manufacturing devices. A notable reason is the relationship that the fuzzy sets have a multi-valued logic, offering decision possibilities such as 'may be true' and 'may be false', suitably quantified in addition to the traditional dichotomy of true or false.

Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. The study of Kramosil and Michalek’s [57] of fuzzy metric space paved the way for very soothing machinery to develop fixed point theorems for contractive type maps. Applications of fuzzy fixed points emerges in the fields approximation theory, min-max problems, mathematical economics, variational inequalities, eigen value problems and boundary value problems.

The objective of the thesis is to study the fixed point theorems in fuzzy metric spaces. To study the fixed point theorems in fuzzy metric spaces, the following types are analyzed:

I) methods based on the contraction mappings.

II) methods based on the nonexpansive mappings.

III) methods based on the pointwise $\psi$-weakly commuting mappings.

IV) methods based on the compatible mappings.

Further, the application of fixed point theorems are applied in various fuzzy differential equations.
1.1 Fixed Point Theorems in Metric Spaces

In this section, following definitions and the results of the fixed point theorems are analyzed.

**Definition 1.1 (Limaye [63])** A metric on a set \( \mathcal{X} \) is a function \( d: \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) such that for \( x, y, z \in \mathcal{X} \),

(i) \( d(x, y) \geq 0 \) and \( d(x, y) = 0 \) i\( f i f \) \( x = y \),

(ii) \( d(x, y) = d(y, x) \), and

(iii) \( d(x, y) \leq d(x, z) + d(z, y) \)

where \( d(x, y) \) denoting the distance between \( x \) and \( y \).

A **metric space** is a set \( \mathcal{X} \) together with a metric \( d \) on it, and it is denoted as \( (\mathcal{X}, d) \).

**Example 1.2** If \( \mathcal{X} \) is any set, let for \( x, y \in \mathcal{X} \)

\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y.
\end{cases}
\]

This metric is called the **discrete metric** on the set \( \mathcal{X} \).

**Definition 1.2 (Limaye [63])** A sequence is said to be a **Cauchy sequence** if for every \( \epsilon > 0 \), there exists \( n_0 \) such that for all \( n, m \geq n_0 \), \( d(x_n, x_m) < \epsilon \).

**Definition 1.3 (Limaye [63])** A metric space \( \mathcal{X} \) is said to be **complete** if every Cauchy sequence in \( \mathcal{X} \) is converges in \( \mathcal{X} \).

**Definition 1.4** Let \( \mathcal{X} \) be a metric space and \( f \) be a self-mapping on \( \mathcal{X} \). A solution of the equation \( f(x) = x \), if it exists, is called **fixed point** of \( f \).
Existence of problems in this type arises frequently in analysis. For example, the problem of solving the equation \( p(x) = 0 \) is equivalent to finding a fixed point of the mapping \( x \mapsto x - p(x) \), provided it is a self-mapping of \( \mathcal{X} \). More generally, if \( f : K \to \mathcal{X} \) is any mapping on a subset \( K \) of \( \mathcal{X} \), to show that the equation \( p(x) = 0 \) has a solution is equivalent to showing that the mapping \( y \mapsto y - p(y) \) has a fixed point in \( K \). Thus conditions on a mapping or on its domain of definition which guarantee the existence of fixed point can be frequently reinterpreted as existence theorem in analysis and therefore have considerable interest.

**Definition 1.5 (Limaye [63])** Let \( (\mathcal{X}, d) \) be a metric space and \( T \) be a self-mapping of \( \mathcal{X} \) into itself. If there exists a non-negative real number \( k \) such that \( k < 1 \) (respectively, \( k = 1 \)) and for any \( x, y \in \mathcal{X} \),

\[
d(Tx, Ty) \leq kd(x, y)
\]

then \( T \) is said to be contraction (respectively, non-expansive) mappings.

Banach contraction mapping principle states that if \( \mathcal{X} \) is complete, then every contraction mapping has a unique fixed point and this point can be obtained by a repeated iteration of the mapping. However non-expansive mapping may exist without fixed points.

The Banach's fixed point theorem [4] is the simplest and one of the most versatile results in the fixed point theory. Based on an iteration process, it can be implemented on a computer to find the fixed point of contraction mapping. It produces approximations of any required accuracy and moreover even the number of iterations needed to get a specified accuracy can be determined.

Banach's fixed point theorem has been generalized by Boyd and Wong [9], Hardy and Rogers [33], Husain and Sehgal [35], Caristi [16], Downing and Kirk [18] in different directions. For a detailed comparison of various definitions and fixed point theorems for contraction and contractive mappings can be referred to the source
article by Rhoades [85]. Kannan [48] obtained existence of common fixed point for
the self-mappings $f$ and $g$ of a complete metric space which satisfy the condition

$$d(f(x), g(y)) \leq \beta[d(x, f(x)) + d(y, g(y))]$$

(1.1)

where $0 < \beta < \frac{1}{2}$. If $f = g$ in (1.1), then $f$ is called a Kannan mapping. This mapping
need not be continuous. Kannan [49] also derived sufficient conditions for existence
of fixed points.

Edelstein [19] has derived a fixed point theorem in the setting of compact metric
spaces for contractive mapping. Sehgal [95] has extended this result to the family of
mappings that satisfy the inequality

$$d(f(x), f(y)) < \max\{d(x, y), d(x, f(x)), d(y, f(y))\}$$

whenever $x \neq y$.

Several authors also have examined many complicated conditions of a contractive type.

**Definition 1.6 (Ciric [14])** A self-map $T$ of a metric space $X$ is a Ciric's contraction if it satisfies the contraction

$$d(Tx, Ty) \leq h \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for some $0 \leq h < 1$ and all $x, y \in X$.

Ciric's theorem states that any Ciric's contraction on a complete metric space
has a fixed point $z$ and, for any $x \in X$, $T^m x \to z$. Fisher [22] generalized the Ciric's contraction to the following and proved the continuous Fisher contraction on a complete metric space has a unique fixed point.

**Definition 1.7 (Fisher [22])** A self-map $T$ of a metric space $X$ is a Fisher contraction, if it satisfies the condition

$$d(T^p x, T^q y) \leq h \max\{d(T^i x, T^j y), d(T^i x, T^j x)\}$$

$$d(T^i y, T^j y) : 0 \leq i, j \leq p \text{ and } 0 \leq i, j' \leq q\}$$
for some fixed \( p, q \in \mathbb{N} \), for some \( 0 \leq b < 1 \) and all \( x, y \in \mathcal{X} \).

Ciric’s [13] analyzed in generalizing contraction mappings, the unique fixed point theorem for the mappings, which are not necessarily continuous and satisfying the following contractive condition:

\[
d(Tx, Ty) \leq a_1d(x, y) + a_2d(x, Tx) + a_3(y, Ty) + a_4[d(x, Ty) + d(y, Tx)]
\]

for all \( x, y \in \mathcal{X} \), where \( a_i = a_i(x, y)(i = 1, 2, 3, 4) \) are non-negative real functions such that

\[
(a_1 + a_2 + a_3 + 2a_4)(x, y) \leq a < 1
\]

where \( a \) is a constant real number. The class of contractions, were further extended and investigated by Pal and Maiti [72], Rhoades [86, 87], Iskii [37].

In many fixed point theorems the contractive definition is strong enough to guarantee the existence and uniqueness of a fixed point, and for which the fixed point can be obtained by function iteration. If the contractive definition is weak enough, however, then additional restrictions either on the function or on the space, or both, are necessary in order to obtain a fixed point. One of the main additional assumption is an involution (i.e) \( T^2 = I \), \( I \) identity map on the space. Gornickii and Rhoades [29] proved the same.

Chen and Shih [11] obtained a fixed point theorem for a self-mapping \( T \) of a compact metric space \( (\mathcal{X}, d) \) satisfying the following contractive definition:

\[
d(Tx, Ty) \leq \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}
\]

for all \( x, y \in \mathcal{X} \) with \( x \neq y \).

The above result has been further improved by Kasahara and Rhoades [51].

In 1969, Meir and Keeler [66] established a fixed point theorem for self-maps of a metric space \( (\mathcal{X}, d) \) which satisfy the following condition.

Given an \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \epsilon \leq d(x, y) < \epsilon + \delta \) implies \( d(f(x), f(y)) < \epsilon \).
In 1978, Maiti and Pal [64] established a fixed point theorem for maps satisfying the following generalization of [66].

Given an $\epsilon > 0$, there exists a $\delta > 0$ such that

$$
eq \leq \max\{d(x, y), d(x, y) + d(y, y), \frac{1}{2}[d(x, y) + d(y, y)]\} \leq \epsilon + \delta \implies d(fx, fy) < \epsilon.
$$

Park and Rhoades [80] obtained fixed point theorems for pairs of mappings $f, g$ satisfying contractive definitions which reduce to the following generalization of [66] when $f = g$.

Given an $\epsilon > 0$, there exists $\delta > 0$ such that

$$
\epsilon \leq \max\{d(x, y), d(x, y) + d(y, y), \frac{1}{2}[d(x, y) + d(y, y)]\} < \epsilon + \delta \implies d(fx, fy) < \epsilon.
$$

**Fixed points for compatible and noncompatible mappings**

In 1976, Jungck [43] initially proved a common fixed point theorem for commuting mappings which generalizes the well known Banach's fixed point theorem. Sessa [97] introduced a generalization of commutativity, which is called weak commutativity, and proved some common fixed point theorems for weakly commuting which generalize the results of Das and Naik [15].

Jungck [44] introduced the concept of more generalized commutativity so called compatibility, which is more general than that of weak commutativity. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of Park and Bac [81]. Since then many interesting fixed point theorems for compatible mappings satisfying contractive type conditions have been obtained by various authors (see [73, 76, 89]).

**Definition 1.8 (Jungck [44])** Two self-maps $f$ and $g$ of a metric space $(\mathcal{X}, d)$ are called compatible if $\lim_{n \to \infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that

$$
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = x \text{ for some } x \in \mathcal{X}.
$$
These common fixed point theorems invariably require a commutativity condition besides a contractive condition and a majority of the successive generalizations are aimed at weakening either or both of these conditions. Pant [77] obtained common fixed point theorems for a family of maps under minimal type commutativity and contractive conditions. Pant [73] introduced the following definitions:

**Definition 1.9** Two self-maps $A$ and $S$ of a metric space $(X,d)$ are called $R$-weakly commuting at a point $x$ in $X$ if $d(ASx,SAx) \leq R d(Ax,Sx)$ for some $R > 0$.

**Definition 1.10** The maps $A$ and $S$ are called pointwise $R$-weakly commuting on $X$ if given $x \in X$ there exists $R > 0$ such that $d(ASx,SAx) \leq R d(Ax,Sx)$.

It is obvious that $A$ and $S$ can fail to be pointwise $R$-weakly commuting only if there is some $x$ in $X$ such that $Ax = Sx$ but $ASx \neq SAx$, that is, only if they possess a coincidence point at which they do not commute. This means that

- a contractive type mappings pair cannot possess a common fixed point without being pointwise $R$-weakly commuting since a common fixed point is also a coincidence point at which the mappings commute and since contractive conditions exclude the possibility of coincidence points, and

- compatible maps are necessarily pointwise $R$-weakly commuting since compatible maps commute at coincidence points.

Pant [79] established a common fixed point theorem by using new continuity conditions which generalized a multitude of common fixed point theorems and is, perhaps, the first theorem of [79] guaranteeing the existence of a common fixed point even when all the mappings may be discontinuous and some of the mappings may not satisfy the compatibility condition.

Khan et al [53] established a new technique to obtain fixed altering distances between the points with use of a certain continuous control functions. Sastry and
Babu [91] worked in this directions. In fact Sastry and Babu [91] discussed and established the existence of fixed points for the orbits of single self-maps and pairs of self-maps by using control functions, with several examples. The purpose of using control functions is that it unifies and generalizes many fixed point theorems.

Further Sastry et al [92, 93] obtained fixed points for self-maps on a complete metric space under more general contraction type condition by using certain control function.

In [74, 77], Pant [78] initiated the study of common fixed points of noncompatible mappings satisfying contractive type conditions. The study of fixed points of noncompatible mappings extends to the class of Lipschitz type mapping pairs even without assuming continuity and completeness. Further, Pant [75], using the notion of pointwise $R$-weak commutativity proved some common fixed point theorem for a pair of noncompatible mappings without assuming completeness of the space or continuity of the mappings involved.

Menger [67] introduced the concept of probabilistic metric spaces, which is a generalization of metric spaces. Further, Schweizer and Sklar [94] extended in the same direction.

The famous Ekeland's [20] variational principle and Caristi's fixed point theorem are forceful tools in nonlinear analysis, control theory, economic theory, and global analysis. Fang [21] generalized the Ekeland's variational principle and Caristi's fixed point theorem. The usual metric spaces, Hausdorff topological vector spaces and Menger probabilistic metric spaces are all special cases of $F$-type topological spaces.

Rhoades [88] developed the concept of 2-metric space of the following:

**Definition 1.11** The 2-metric is a set $X$ with a real valued function $d$ on $X \times X \times X$ satisfying the following conditions:

1. For two distinct points $x, y \in X$,
   there exists a point $z \in X$ such that $d(x, y, z) \neq 0$
(2) \( d(x, y, z) = d(x, z, y) = d(y, z, x) \)

(3) \( d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z) \).

The function \( d \) is called 2-metric for the space \( X \) and \((X, d)\) denotes a 2-metric space.

Naidu and Prasad [71] introduced the concept of weakly commuting pairs of self-mappings on a 2-metric space and the notion of weak continuity of a 2-metric, respectively and they have proved several common fixed point theorems by using the concepts of the weakly commuting pairs of self-mappings in a 2-metric space and the weak continuity of a 2-metric space.

Jeong and Rhoades [40] analyzed the fixed point theorems for more than two maps. Further, it was extended by Pant [76] and Kang and Rhoades [50]. Imdad [36] proved fixed point theorems involving five mappings which in turn generalizes earlier results due to Fisher [22]; Jungck [45] and others.

1.2 Fuzzy Metric Space

Though mathematicians have involved in the development of fuzzy sets from the very beginning, very recently fuzzy sets have received serious consideration from the wider mathematical community. Many mathematical problems are coming to the fore and mathematical foundations of the subject are now becoming firmly established. Topological aspects of fuzzy sets have been investigated intensively, but so far no direct applications have been found on account of its generality. It contrast metric spaces of fuzzy sets are providing a convenient mathematical framework for diverse applications of fuzzy sets. Klir and Yuan [54] is a very good source book for fuzzy sets and its applications. Diamond and Kloeden [17] have given more details on fuzzy metric spaces.

The following concepts and results in the sequel are analyzed.
Definition 1.12 (Menger [67]) A statistical metric space is a nonempty set together with an associated family of distribution functions $F_{pq}$ satisfying the following conditions:

(i) $F_{pq}(0) = 0$

(ii) If $p = q$, then $F_{pq}(x) = 1$ for all $x > 0$.

(iii) If $p \neq q$, then $F_{pq}(x) < 1$ for some $x > 0$.

(iv) $F_{pq} = F_{qp}$

(v) $F_{pr}(x + y) \geq T[F_{pq}(x), F_{qr}(y)]$ for all $p, q, r \in X$ and all real numbers $x$ and $y$.

Here $T$ is a function from the closed unit square $[0, 1] \times [0, 1]$ to closed unit interval $[0, 1]$ satisfying the following conditions:

(a) $T(a, b) = T(b, a)$

(b) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$

(c) $T(a, 1) > 0$ whenever $a > 0$ and $T(1, 1) = 1$.

A distribution function $F$ is a non-decreasing, left continuous mapping from a set of real numbers $R$ to $[0, 1]$ so that $\inf_{x \in R} F(x) = 0$ and $\sup_{x \in R} F(x) = 1$. This statistical metric $F_{pq}$ can be interpreted as the probability that the distance between $p$ and $q$ is less than $x$.

There are different ways of fuzzifying metric spaces. Kramosil and Michalek [57] introduced the concept of fuzzy metric spaces in terms of $t$-norm.

Definition 1.13 (Schweizer and Sklar [94]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous $t$-norm if $([0, 1], *)$ is a topological monoid with unit $1$ such that $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Example 1.3 The following are standard examples of $t$-norm.
(i) $a * b = ab$

(ii) $a * b = \min (a, b)$

(iii) $a * b = \max \{0, a + b - 1\}$.

Using this concept of \(t\)-norm, Schweizer and Sklar [94] defined probabilistic metric space as follows.

**Definition 1.14** A **probabilistic metric space** is a pair \((X, \mathcal{F})\), where \(X\) is a nonempty set and \(\mathcal{F}\) is a mapping from \(X \times X\) to the set of all distributive mappings which satisfy the following conditions:

(i) \(\mathcal{F}_{xy}(t) = 0\) for all \(t > 0\) iff \(x = y\)

(ii) \(\mathcal{F}_{xy}(0) = 0\)

(iii) \(\mathcal{F}_{xy} = \mathcal{F}_{yx}\)

(iv) If \(\mathcal{F}_{xy}(t) = 1\) and \(\mathcal{F}_{yz}(s) = 1\), then \(\mathcal{F}_{xz}(t + s) = 1\).

Thus the Menger space \((X, \mathcal{F})\) together with a \(t\)-norm \(*\) is a probabilistic metric space if \(\mathcal{F}_{xz}(t + s) \geq \mathcal{F}_{xy}(t) * \mathcal{F}_{yx}(s)\).

Kramosil and Michalek [57] defined fuzzy metric space by generalizing the concept of probabilistic metric spaces to the fuzzy situation which is given below.

**Definition 1.15** (Kramosil and Michalek [57]) The 3-tuple \((X, M, \ast)\) is said to be a fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X^2 \times [0, \infty)\) satisfying the following conditions:

1. \(M(x, y, 0) = 0\)

2. \(M(x, y, t) = 1\) for all \(t > 0\) if and only if \(x = y\)

3. \(M(x, y, t) = M(y, x, t)\)
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

5. $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous

6. $\lim_{h \to \infty} M(x, y, t) = 1$

for all $x, y, z \in X$ and all $t, s > 0$.

Example 1.4 Let $X = \mathbb{R}$ with usual metric defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $*$ be the usual multiplication.

Define

$$M(x, y, t) = \begin{cases} \frac{t}{|x+y-t|} & : \text{if } x \neq y \text{ and } t \geq 0 \\ 0 & : \text{if } x = y \text{ and } t = 0. \end{cases}$$

Then $(\mathbb{R}, M, *)$ is a fuzzy metric space.

In the present thesis the fuzzy metric space $(X, M, *)$ is denoted by $X$.

Kaleva and Seikkala [46] introduced the concept of fuzzy metric spaces where the distance between two points is non-negative, upper semicontinuous, normalized and convex fuzzy number. George and Veeramani [24] modified the definition of fuzzy metric space given by Kramosil and Michalek [57] as follows.

Definition 1.16 (George and Veeramani [24]) The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

1. $M(x, y, t) > 0$

2. $M(x, y, t) = 1$ if and only if $x = y$

3. $M(x, y, t) = M(y, x, t)$

4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

5. $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous $x, y, z \in X$ and $t, s > 0$. 

Definition 1.17 (George and Veeramani [24]) A sequence \( \{x_n\} \) in a fuzzy metric space \((X, M, *)\) is called Cauchy if \( \lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \) for every \( t > 0 \) and \( p > 0 \). \((X, M, *)\) is complete if every Cauchy sequence in \( X \) converges in \( X \). A sequence \( \{x_n\} \) in \( X \) is convergent to \( x \in X \) if \( \lim_{n \to \infty} M(x_n, x, t) = 1 \) for each \( t > 0 \).

George and Veeramani [24] proved fuzzy analogue of some important results in metric space theory. George and Veeramani [25, 26] have also introduced the notions of \( F \)-Cauchy sequence, \( F \)-bounded set, \( F \)-uniform convergence, \( F \)-uniform continuity and \( F \)-equicontinuity in the modified fuzzy metric spaces. Based on these definitions George and Veeramani [25] proved Cantor's intersection theorem, Baire's category theorem, Uniform limit theorem, Metrization theorem and Ascoli-Arzela's theorem.

Fixed point theorems in fuzzy metric space

When messages are transmitted over a noisy channel, the received messages will be erroneous. These are fuzzy and therefore it is natural to consider a mapping whose image is fuzzy. Such a mapping is called fuzzy mapping. Thus the application of fixed points to fuzzy mappings has received considerable attention. Heilpern [34] evolved a fixed point theorem for contractive mapping which is a fuzzy analogue of the fixed point theorem for multi-valued mappings. Bose and Sahani[8] established fixed point theorems for contractive type fuzzy mappings and proved fixed point theorem for nonexpansive fuzzy mapping. Som and Mukherjee [99] and Lee et al [60, 62] proved fixed point theorems in this direction. Gregori and Romaguera [30] evolved common fixed point theorems for contractive type mappings.

Forte et al [23] have found that the study of contractive mappings having interesting applications in fractals and animation of images where the fixed points set of fuzzy contractions could be thought of as attractors representing images which are deformed in an apparently continuous manner in time.

Zadeh's [106] investigation of the concept of fuzzy sets has led to a rich growth
of fuzzy mathematics (see for instance [46, 55, 57, 65, 98]). Now it is well recognized
that this system embraces upon complexities and uncertainties in various physical
schemes. Abstract coincidence and fixed point equation are of immense importance
in interpreting various physical formulations. A recent attempt of Grabiec [27] to
formulate the well known Banach contraction principle in fuzzy metric spaces in the
sense of Kramosil and Michalek [57] appears to provide a very soothing platform for
developing Banach type fixed point theory. Fang [21] and Mishra et al [68], following
Grabiec's approach, have obtained Banach type fixed point theorems in fuzzy metric
spaces. Following Grabiec's approach to Banach fixed point theorems in fuzzy metric
spaces Mishra et al [69] obtained some new coincidence theorems for a family of
mappings on an arbitrary set with values in a fuzzy metric space and derives a few
general fixed point theorems for a family of mappings on a fuzzy metric space. These
fixed point theorems are applied to obtain common solutions of fixed point type
equations on product spaces.

In 1994, Lee and Clio [59] established a fixed point theorem for a contractive
type fuzzy map. Further, Arora and sliarma [2] extended the same. Also Lee et al.
[61] generalized the result of [8, 59]. Cho et al [12] obtained some common fixed point
theorems for compatible maps of type(/3) on fuzzy metric space.

Yasiiki [102] proved a common fixed point theorem for a sequence of mappings
in a fuzzy metric space. This results offers a generalization of Grabiec's [27] theorem.
Further, Vasuki [103] introduced /?-weakly commuting maps in fuzzy metric: space
and proved common fixed point theorems. Vijayaraju and Marudai [104], proved the
common fixed point theorems for a pair of self-maps in a fuzzy metric space. This
result offers a generalization of Grabiec [27] findings.

The (ixed point theory for fuzzy mappings in complete metric spaces has peri-
odically received certain attention (see [8, 34, 42, 74, 82, 83]). On the other hand, the
theory of quasi-uniform, quasi-metric spaces has been considered and applied in the
last years, by many authors, studied of hyperspaces, function spaces, fuzzy topology,
fixed point theory, theoretical computer science etc. (see [38, 39]).

Bijendra Singh and Chauhan [6] introduced the concept of compatibility in fuzzy metric space and use it to prove common fixed point theorems for four compatible mappings which generalized the results of [44, 45].

Applications of fixed point theorems in fuzzy differential equations have been discussed in the last chapter 7.

In the light of the above, some significant results have been obtained on the following topics:

* Variational principle of fixed point theorems in certain fuzzy topological spaces.

* Fixed points for compatible mappings in fuzzy metric space.

* Common fixed points for noncompatible mappings in fuzzy metric space.

* Fixed point theorems for a sequence of mappings in fuzzy metric space.

* Common fixed points of compatible maps in generalized fuzzy metric space.

Applications of fixed point theorems are

* Existence and uniqueness of the fuzzy solution for a nonlinear fuzzy integro-differential equation.

* Existence and uniqueness of the fuzzy solution for the nonlinear fuzzy Volterra integro-differential equations.

* Existence and uniqueness of the fuzzy solution for a nonlinear fuzzy neutral functional differential equation.