Chapter 7

APPLICATIONS OF FIXED POINT
THEOREMS IN FUZZY DIFFERENTIAL
EQUATIONS

7.1 Introduction

A physical problem, when transformed into a deterministic initial value problem,

\[ x'(t) = f(t, x(t)) \]  \hspace{2cm} (7.1)

\[ x(0) = x_0 \]  \hspace{2cm} (7.2)

usually cannot be sure that the above modelling is perfect. The initial value may not
be known exactly and the function \( f \) may contain unknown parameters. Especially,
if they are known through some measurements they necessarily be subject to errors.
The analysis of the effect of these errors leads to the study of qualitative behaviour
of solution to above equation. If the nature of errors is random, then instead of
an above deterministic problem, a random differential equation with random initial
value and/or random coefficients will be arrived. But if the underlying structure is not
probabilistic, for example, because of subjective choices, then it may be appropriate
to use fuzzy differential equations.

Today there are a lot of works devoted to use these fuzzy differential equations,
important from the theoretical and practical point of view. The theoretical investi-
gations are emerged on min-max (max-min) relational equations and studies on its use in practical problems such as medical diagnosis, diagnostics of technical devices or fuzzy control of industrial processes.

Integrodifferential equations play an important role in characterizing of many social, physical, biological and engineering problems. For example, Volterra [105] was investigating the population growth, focusing his study on the hereditary influences and several authors (see [3], [7] and [41]) discussed the integrodifferential modeled integral equations in the field of heat transfer and diffusion process in general neutron diffusion. For fuzzy concepts recently the authors (see [17] and [101]) established respectively the theory of metric space of fuzzy sets and fuzzy Volterra integral equations. In particular, Kloeden [55] studied the fuzzy dynamic system and Kaleva [47] researched the fuzzy differential equations and Cauchy problem. Seikkala [96] proved the existence and uniqueness of the fuzzy solution for the systems (7.1)-(7.2) in which \( f \) is a continuous mapping from \( \mathbb{R}^+ \times \mathbb{R} \) into \( \mathbb{R} \) and \( x_0 \) is a fuzzy number.

This Chapter is classified into three sections.

In section 7.2, some needed definitions and results are given.

The section 7.3, investigation of the existence and uniqueness of the fuzzy solution for the nonlinear fuzzy integrodifferential equations and Volterra integrodifferential equations are carried out.

In section 7.4, the existence and uniqueness of a fuzzy solution for the nonlinear fuzzy neutral functional differential equation is obtained.

### 7.2 Preliminaries

A fuzzy subset of \( \mathbb{R}^n \) is defined in terms of a membership function which assigns to each point \( x \in \mathbb{R}^n \) a grade of membership in the fuzzy set. Such a membership function is denoted by

\[
u : \mathbb{R}^n \rightarrow [0, 1].\]
Throughout this chapter, it is assumed that $u$ maps $R^n$ onto $[0, 1]$, $[u]^b$ is a bounded subset of $R^n$, $u$ is upper semicontinuous, and $u$ is fuzzy convex.

Denote by $E^u$ the space of all fuzzy subsets $u$ of $R^n$ which are normal, fuzzy convex and upper semicontinuous fuzzy sets with bounded supports. In particular, $E^1$ denotes the space of all fuzzy subsets $u$ of $R$.

A fuzzy number $a$ in the real line $R$ is a fuzzy set characterized by a membership function $\mu_a$ as

$$\mu_a : R \rightarrow [0, 1].$$

A fuzzy number $a$ is expressed as

$$a = \int_{x \in R} \mu_a(x) / x$$

with the understanding that $\mu_a(x) \in [0, 1]$ represents the grade of membership of $x$ in $a$ and $\int$ denotes the union of $\mu_a(x) / x$.

**Definition 7.1 (Klir and Yuan [54])** A fuzzy number $a$ in $R$ is said to be convex if for any real numbers $x, y, z$ in $R$ with $x \leq y \leq z$,

$$\mu_a(y) \geq \min\{\mu_a(x), \mu_a(z)\}.$$  

**Definition 7.2 (Klir and Yuan [54])** The height of a fuzzy set is the largest membership value attained by any point in the set.

**Definition 7.3 (Klir and Yuan [54])** If the height of a fuzzy set equals one then the fuzzy set is called a normal fuzzy set.

Thus, a fuzzy number $a$ in $R$ is called normal if the following holds

$$\max_x \mu_a(x) = 1.$$  

**Definition 7.4 (Das and Choudhury [16])** A fuzzy real number $\beta : R \rightarrow [0, 1]$ is upper semicontinuous if for each $\epsilon > 0$, $\beta^{-1}([0, a + \epsilon])$ is open in $R$. 
Result 7.1 Let $E_N$ be the set of all upper semicontinuous convex normal fuzzy numbers with bounded $\alpha$-level intervals (see [70]). This means that if $a \in E_N$, then the $\alpha$-level set

$$[a]^\alpha = \{x \in R : a(x) \geq \alpha, 0 < \alpha \leq 1\}$$

is a closed bounded interval, which is denoted by

$$[a]^\alpha = [a_1^\alpha, a_2^\alpha],$$

and there exists a $t_0 \in R$ such that $a(t_0) = 1$.

Result 7.2 Two fuzzy numbers $a$ and $b$ are called equal $a = b$, if $\mu_a(x) = \mu_b(x)$ for all $x \in R$. It follows that

$$a = b \iff [a]^\alpha = [b]^\alpha \text{ for all } \alpha \in (0, 1).$$

Result 7.3 A fuzzy number $a$ may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha[a]^\alpha,$$

where $\alpha[a]^\alpha$ is the product of a scalar $\alpha$ with the set $[a]^\alpha$ and $\int$ is the union of $[a]^\alpha$ with $\alpha$ ranging from 0 to 1.

Definition 7.5 (Klir and Yuan [54]) The support of a fuzzy set $A$ in the universal set $U$ is a crisp set that contains all the elements of $U$ that have nonzero membership values in $A$, that is,

$$\text{supp}(A) = \{x \in U : \mu_a(x) > 0\}$$

where $\text{supp}(A)$ denotes the support of the fuzzy set $A$. Hence, the support $\Gamma_a$ of a fuzzy number $a$ is defined, as a special case of level set, by the following

$$\Gamma_a = \{x : \mu_a(x) > 0\}.$$

Definition 7.6 (Klir and Yuan [54]) A fuzzy number $a$ in $R$ is said to be positive if $0 < a_1 < a_2$ holds for the support $\Gamma_a = [a_1, a_2]$ of $a$, that is, $\Gamma_a$ is in the positive real line. Similarly, $a$ is called negative if $a_1 < a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$. 
Lemma 7.1 (Seikkala [96]) If \( a, b \in E_N \), then for \( \alpha \in (0, 1) \),

\[
[a + b]^\alpha = [a_q^\alpha + b_q^\alpha, a_r^\alpha + b_r^\alpha].
\]

\[
[a \cdot b]^\alpha = [\min\{a_q^\alpha b_j^\alpha, a_r^\alpha b_j^\alpha\}, \max\{a_q^\alpha b_j^\alpha, a_r^\alpha b_j^\alpha\}] (i, j = q, r),
\]

\[
[a - b]^\alpha = [a_q^\alpha - b_q^\alpha, a_r^\alpha - b_r^\alpha].
\]

Lemma 7.2 (Seikkala [96]) Let \([a_q^\alpha, a_r^\alpha]\), \(0 < \alpha \leq 1\), be a given family of nonempty intervals. If

\[
[a_q^\alpha, a_r^\alpha] \subset [a_q^{\alpha_k}, a_r^{\alpha_k}] \text{ for } 0 < \alpha \leq \beta \tag{7.3}
\]

and \([\lim_{k \to \infty} a_q^{\alpha_k}, \lim_{k \to \infty} a_r^{\alpha_k}] = [a_q^\alpha, a_r^\alpha]\), \(0 < \alpha \leq 1\), are the \(\alpha\) - level sets of a fuzzy number \(a \in E_N\). Conversely, if \([a_q^\alpha, a_r^\alpha]\), \(0 < \alpha \leq 1\), are the \(\alpha\) - level sets of a fuzzy number \(a \in E_N\), then conditions (7.3) and (7.4) are valid.

Let \(x\) be a point in \(R^n\) and \(A\) be a nonempty subset of \(R^n\). Define the distance \(d(x, A)\) from \(x\) to \(A\) by

\[
d(x, A) = \inf\{\|x - a\| : a \in A\}. \tag{7.5}
\]

Now let \(A\) and \(B\) be nonempty subsets of \(R^n\). Define the Hausdorff separation of \(B\) from \(A\) by

\[
d^*_H(B, A) = \sup\{d(b, A) : b \in B\}. \tag{7.6}
\]

In general,

\[
d^*_H(A, B) \neq d^*_H(B, A).
\]

Define the Hausdorff distance between nonempty subsets of \(A\) and \(B\) of \(R^n\) by

\[
d_H(A, B) = \max\{d^*_H(A, B), d^*_H(B, A)\}. \tag{7.7}
\]

This is symmetric in \(A\) and \(B\). Consequently,
(i) \( d_H(A, B) \geq 0 \) with \( d_H(A, B) = 0 \) if and only if \( A = B \).

(ii) \( d_H(A, B) = d_H(B, A) \),

(iii) \( d_H(A, B) \leq d_H(A, C) + d_H(C, B) \),

for any nonempty subsets of \( A, B \) and \( C \) of \( \mathbb{R}^n \). The Hausdorff distance \((7.7)\) is a metric, called the Hausdorff metric.

The supremum metric \( d_\infty \) on \( E^n \) is defined by

\[
d_\infty(u, v) = \sup\{d_H([u]_\alpha, [v]_\alpha) : \alpha \in (0, 1]\}
\]  \hspace{1cm} (7.8)

for all \( u, v \in E^n \), and it is obviously a metric on \( E^n \). The supremum metric \( H_1 \) on \( C(J, E^n) \) is defined by

\[
H_1(x, y) = \sup\{d_\infty(x(t), y(t)) : t \in J\}
\]  \hspace{1cm} (7.9)

for all \( x, y \in C(J : E^n) \).

Let \( I \) be a real interval. A mapping \( x : I \to E_N \) is called a fuzzy process.

Denote \( [x(t)]_\alpha = [x_q^\alpha(t), x_\alpha^\alpha(t)], t \in I, \ 0 < \alpha \leq 1 \).

The derivative \( x'(t) \in E_N \) of a fuzzy process \( x \), satisfies

\[
[x'(t)]_\alpha = [(x_q^\alpha)'(t), (x_\alpha^\alpha)'(t)], \ 0 < \alpha \leq 1.
\]  \hspace{1cm} (7.10)

The fuzzy integral

\[
\int_a^b x(t)dt, \ a, b \in I,
\]

is defined by

\[
\int_a^b [x(t)dt]_\alpha = \left[ \int_a^b x_q^\alpha(t)dt, \int_a^b x_\alpha^\alpha(t)dt \right]
\]

provided that the Lebesgue integrals on the right exist. The relation between fuzzy derivative and fuzzy integral is obvious; first

\[
\frac{d}{dt} \int_a^t x(s)ds = x(t), \ a.e \ t \in I.
\]

and if the end point functions \( (x_q^\alpha)' \) and \( (x_\alpha^\alpha)' \) in \((7.10)\) are integrable, then

\[
x(t) = x(a) + \int_a^t x'(s)ds, \ t \in I.
\]
Definition 7.7 The fuzzy process \( x : J \to E_N \) is a solution of the equation \( x'(t) = a(t)x(t); \ x(0) = x_0 \in E_N \) if and only if

\[
\begin{align*}
(x_q^\alpha)'(t) & = \min\{a_i^\alpha(t)x_j^\alpha(t), \ i, j = q, r\}, \\
(x_r^\alpha)'(t) & = \max\{a_i^\alpha(t)x_j^\alpha(t), \ i, j = q, r\}, \text{ and} \\
(x_q^\alpha)(0) & = x_{0q}^\alpha, \\
(x_r^\alpha)(0) & = x_{0r}^\alpha.
\end{align*}
\]

7.3 Fuzzy Integrodifferential Equation

The existence and uniqueness of the fuzzy solution for the following nonlinear fuzzy integrodifferential equation is proved in this section.

\[
x'(t) = a(t)x(t) + \int_0^t k(s, t, x(s))ds + f(t, x(t)), \quad t \in J = [0, T], \quad (7.11)
\]

\[
x(0) = x_0 \in E_N, \quad (7.12)
\]

where \( a : J \to E_N \), is fuzzy coefficient, \( E_N \) is the set of all upper semicontinuous convex normal fuzzy numbers with bounded \( \alpha \)-level intervals, \( k : J \times J \times E_N \to E_N \), and \( f : J \times E_N \to E_N \) are nonlinear continuous functions.

Similar to the fuzzy solution discussed by Kwun et al [58] for fuzzy differential equation. We take the fuzzy solution for the above nonlinear fuzzy integrodifferential equations (7.11)-(7.12) as

\[
x(t) = S(t)x_0 + \int_0^t S(t-s)\left(\int_0^s k(s, \tau, x(\tau))d\tau\right)ds + \int_0^t S(t-s)f(s, x(s))ds, \quad t \in J.
\]

(7.13)

To prove the existence and uniqueness of the fuzzy solution, assume the following hypotheses:
(H1) The inhomogeneous terms \( f : J \times E_N \to E_N \) and \( k : J \times J \times E_N \to E_N \) are continuous functions that satisfy global Lipschitz conditions.

\[
d_H ([f(s, \xi_1(s))]^\alpha, [f(s, \xi_2(s))]^\alpha) \leq \eta_1 d_H ([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha),
\]

\[
d_H ([k(t, s, \xi_1(s))]^\alpha, [k(t, s, \xi_2(s))]^\alpha) \leq \eta_1 d_H ([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)
\]

for all \( \xi_1(s), \xi_2(s) \in E_N \) with a finite constant \( \eta_1 > 0 \).

(H2) \( S(t) \) is a fuzzy number such that

\[
[S(t)]^\alpha = [S^q(t), S^r(t)]
\]

\[
= \left[ \exp\left( \int_0^t a^q_\xi(s) ds \right), \exp\left( \int_0^t a^r_\xi(s) ds \right) \right]
\]

where \( S^i(t), i = q, r, \) are continuous. That is, there exists a constants \( \eta_2 > 0 \) such that \( |S^i(t)| \leq \eta_2 \) and for all \( t \in J \).

**Theorem 7.1** Let \( T > 0 \), and assume the hypothesis satisfies (H1)-(H2) hold, then for every \( x_0 \in E_N \), the fuzzy integrodifferential equation has a unique solution \( x \in C(J : E_N) \).

**Proof.** For each \( \xi(t) \in E_N, t \in J \). Define

\[
(\Phi \xi)(t) = S(t)x_0 + \int_0^t S(t-s)(\int_0^s k(s, r, \xi(r))dr)ds + \int_0^t S(t-s)f(s, \xi(s))ds.
\]

Thus, \( (\Phi \xi) : J \to E_N \) is continuous, and \( \Phi : C(J : E_N) \to C(J : E_N) \). For \( \xi_1, \xi_2 \in C(J : E_N) \)

\[
d_H \left( [([\Phi \xi_1](t))]^\alpha, [([\Phi \xi_2](t))]^\alpha \right)
\]

\[
= d_H \left( [S(t)x_0 + \int_0^t S(t-s)(\int_0^s k(s, r, \xi_1(r))dr)ds + \int_0^t S(t-s)f(s, \xi_1(s))ds]^\alpha, 
\]

\[
[S(t)x_0 + \int_0^t S(t-s)(\int_0^s k(s, r, \xi_2(r))dr)ds + \int_0^t S(t-s)f(s, \xi_2(s))ds]^\alpha \right)
\]
\[
\begin{align*}
&= \, d_H\left([S(t)x_0]^\alpha + \int_0^t S(t-s)(\int_0^s k(s,r,\xi_1(r))dr)ds\right]^\alpha + \int_0^t S(t-s)f(s,\xi_1(s))ds]^{\alpha} + \left[\int_0^t S(t-s)f(s,\xi_2(s))ds\right]^{\alpha} \\
&= \, d_H\left([\int_0^t S(t-s)(\int_0^s k(s,r,\xi_1(r))dr)ds]^{\alpha}, \left[\int_0^t S(t-s)(\int_0^s k(s,r,\xi_2(r))dr)ds\right]^{\alpha}\right) \\
&\quad + d_H\left([\int_0^t S(t-s)f(s,\xi_1(s))ds]^{\alpha}, \left[\int_0^t S(t-s)f(s,\xi_2(s))ds\right]^{\alpha}\right) \\
&= \, d_H\left([\int_0^t S(t-s)(\int_0^s k(s,r,\xi_1(r))dr)ds]^{\alpha}, \left[\int_0^t S(t-s)(\int_0^s k(s,r,\xi_2(r))dr)ds\right]^{\alpha}\right) \\
&\quad + d_H\left([\int_0^t S(t-s)f(s,\xi_1(s))ds]^{\alpha}, \left[\int_0^t S(t-s)f(s,\xi_2(s))ds\right]^{\alpha}\right) \\
\leq \, \int_0^t d_H\left([S(t-s)(\int_0^s k(s,r,\xi_1(r))dr)]^{\alpha}, \left[S(t-s)(\int_0^s k(s,r,\xi_2(r))dr)\right]^{\alpha}\right)ds \\
&\quad + \int_0^t d_H\left([S(t-s)f(s,\xi_1(s))]^{\alpha}, [S(t-s)f(s,\xi_2(s))]^{\alpha}\right)ds \\
&= \, \int_0^t d_H\left([S^\alpha(t-s)(\int_0^s k^\alpha(s,r,\xi_1(r))dr)]^{\alpha}, S^\alpha(t-s)(\int_0^s k^\alpha(s,r,\xi_1(r))dr)\right)ds \\
&\quad + \int_0^t d_H\left([S^\alpha(t-s)f^\alpha(s,\xi_1(s))]^{\alpha}, [S^\alpha(t-s)f^\alpha(s,\xi_2(s))]^{\alpha}\right)ds \\
&= \, \int_0^t \max\left\{\left|\int_0^s k^\alpha(s,r,\xi_1(r))dr - S^\alpha(t-s)(\int_0^s k^\alpha(s,r,\xi_1(r))dr)\right|ds, \right. \\
&\quad \left. \int_0^s k^\alpha(s,r,\xi_2(r))dr - S^\alpha(t-s)(\int_0^s k^\alpha(s,r,\xi_1(r))dr)\right|ds \\
&\quad + \int_0^t \max\left\{\left|S^\alpha(t-s)f^\alpha(s,\xi_2(s)) - S^\alpha(t-s)f^\alpha(s,\xi_1(s))\right|ds, \right. \\
&\quad \left. \int_0^s f^\alpha(s,\xi_2(s)) - S^\alpha(t-s)f^\alpha(s,\xi_1(s))\right|ds \\
&\quad = \, \int_0^t \max\left\{\left|S^\alpha(t-s)(\int_0^s k^\alpha(s,r,\xi_2(r))dr) - \left(\int_0^s k^\alpha(s,r,\xi_1(r))dr\right)\right|ds, \right. \\
&\quad \left. \int_0^s k^\alpha(s,r,\xi_2(r))dr - \left(\int_0^s k^\alpha(s,r,\xi_1(r))dr\right)\right|ds \right\}. 
\end{align*}
\]
\[ S_0^\alpha(t - s) \left[ \int_0^t k_\alpha^\alpha(s, r, \xi_2(r)) dr - \int_0^t k_\alpha^\alpha(s, r, \xi_1(r)) dr \right] \right] ds + \int_0^t \max \left( \left| S_0^\alpha(t - s) \left[ f_\alpha^\alpha(s, \xi_2(s)) - f_\alpha^\alpha(s, \xi_1(s)) \right] \right| ds \]

\[ \leq \eta_2 \int_0^t \max \left( \left| \int_0^s k_\alpha^\alpha(s, r, \xi_2(r)) dr - \int_0^s k_\alpha^\alpha(s, r, \xi_1(r)) dr \right| \right) ds + \n_2 \int_0^t \max \left( \left| f_\alpha^\alpha(s, \xi_2(s)) - f_\alpha^\alpha(s, \xi_1(s)) \right| \right) ds \]

\[ = \eta_2 \int_0^t d_H \left( \left| \int_0^t k_\alpha^\alpha(s, r, \xi_1(r)) dr - \int_0^t k_\alpha^\alpha(s, r, \xi_1(r)) dr \right| \right) ds + \n_2 \int_0^t \max \left( \left| f_\alpha^\alpha(s, \xi_2(s)) - f_\alpha^\alpha(s, \xi_1(s)) \right| \right) ds \]

\[ \leq \eta_2 \int_0^t d_H \left( \left| \int_0^t k(s, r, \xi_1(r)) dr \right| \right) ds + \n_2 \int_0^t \max \left( \left| f(s, \xi_2(s)) - f(s, \xi_1(s)) \right| \right) ds \]

\[ \leq \eta_2 \int_0^t d_H \left( \left| \int_0^t k(s, r, \xi_1(r)) dr \right| \right) ds + \n_2 \int_0^t \max \left( \left| f(s, \xi_2(s)) - f(s, \xi_1(s)) \right| \right) ds \]

\[ \leq \eta_2 \int_0^t \left( \left| \xi_1(s) \right|^{\alpha} + \left| \xi_2(s) \right|^{\alpha} \right) ds + \eta_2 \int_0^t \left( \left| \xi_1(s) \right|^{\alpha} + \left| \xi_2(s) \right|^{\alpha} \right) ds \]
\[ \leq \eta_1 \eta_2 (T + 1) \int_0^t d_H \left( \left[ \xi_1(s) \right]^\alpha, \left[ \xi_2(s) \right]^\alpha \right) ds \]

\[ \leq k \eta_2 \int_0^t d_H \left( \left[ \xi_1(s) \right]^\alpha, \left[ \xi_2(s) \right]^\alpha \right) ds, \quad \text{where } k = \eta_1 (T + 1). \]

Therefore

\[ d_\infty \left( \Phi \xi_1(t), (\Phi \xi_2)(t) \right) = \sup_{t \in [0,1]} d_H \left( \left[ (\Phi \xi_1)(t) \right]^\alpha, \left[ (\Phi \xi_2)(t) \right]^\alpha \right), \]

\[ \leq (k \eta_2) \int_0^t \sup_{\alpha \in [0,1]} d_H \left( \left[ \xi_1(s) \right]^\alpha, \left[ \xi_2(s) \right]^\alpha \right) ds, \]

\[ = (k \eta_2) \int_0^t d_\infty (\xi_1(s), \xi_2(s)) ds. \]

Hence

\[ H_1 (\Phi \xi_1, \Phi \xi_2) = \sup_{t \in J} d_\infty \left( \Phi \xi_1(t), (\Phi \xi_2)(t) \right), \]

\[ \leq (k \eta_2) \sup_{t \in J} \int_0^t d_\infty (\xi_1(s), \xi_2(s)) ds, \]

\[ \leq (k \eta_2) T H_1 (\xi_1, \xi_2). \]

We take sufficiently small \( T; (k \eta_2 T) < 1 \). Hence \( \Phi \) is a contraction mapping. By the Banach fixed point theorem, fuzzy integrodifferential equation has a unique fixed point \( x \in C(J : E_N) \).

**Example 7.1** Consider the fuzzy solution of the nonlinear fuzzy integrodifferential equation

\[ x' = 2x + 2tx(t)^2 + 2tx^2, \quad t \in J, \]

\[ x(0) = 2 \in E_N. \]

**Solution.** The \( \alpha \)-level set of fuzzy number 2 is \( [2]^\alpha = [\alpha + 1, 3 - \alpha] \) for all \( \alpha \in [0,1] \).

Let \( \int_0^t k(t, s, x(s)) ds = 2tx(t)^2 \),
\[ f(t, x(t)) = 2tx(t)^2 \]

and \( \eta_2 = 3T[|x_\alpha^\alpha(t) + y_\alpha^\alpha(t)|] > 0. \)

Then the \( \alpha \)-level set of \( \int_0^t k(t, s, x(s))ds \) is

\[
\left[ \int_0^t k(t, s, x(s))ds \right]^\alpha = \left[ 2tx(t)^2 \right]^\alpha \\
= t[2]^\alpha \left[ x(t)^2 \right]^\alpha \\
= t[\alpha + 1, 3 - \alpha] \left[ (x_\alpha^\alpha(t))^2, (x_\pi^\alpha(t))^2 \right] \\
= t[\alpha + 1](x_\alpha^\alpha(t))^2, (3 - \alpha)(x_\pi^\alpha(t))^2
\]

where \( [x(t)]^\alpha = [x_\alpha^\alpha(t), x_\pi^\alpha(t)] \) and \( [2]^\alpha = [\alpha + 1, 3 - \alpha] \) for all \( \alpha \in [0, 1] \) and the \( \alpha \)-level set of \( f(t, x(t)) \) is

\[
\left[ f(t, x(t)) \right]^\alpha = \left[ 2tx(t)^2 \right]^\alpha \\
= t[2]^\alpha \left[ x(t)^2 \right]^\alpha \\
= t[\alpha + 1, 3 - \alpha] \left[ (x_\alpha^\alpha(t))^2, (x_\pi^\alpha(t))^2 \right] \\
= t[\alpha + 1](x_\alpha^\alpha(t))^2, (3 - \alpha)(x_\pi^\alpha(t))^2
\]

where \( [x(t)]^\alpha = [x_\alpha^\alpha(t), x_\pi^\alpha(t)] \) and \( [2]^\alpha = [\alpha + 1, 3 - \alpha] \) for all \( \alpha \in [0, 1] \). Then,

\[
\left[ \int_0^t k(t, s, x(s))ds \right]^\alpha + \left[ f(t, x(t)) \right]^\alpha \\
= t[(\alpha + 1)(x_\alpha^\alpha(t))^2, (3 - \alpha)(x_\pi^\alpha(t))^2] + t[(\alpha + 1)(x_\alpha^\alpha(t))^2, (3 - \alpha)(x_\pi^\alpha(t))^2]
\]

\[
= 2t[(\alpha + 1)(x_\alpha^\alpha(t))^2, (3 - \alpha)(x_\pi^\alpha(t))^2]
\]

Therefore,

\[
d_H\left( \left[ \int_0^t k(t, s, x(s))ds \right]^\alpha + \left[ f(t, x(t)) \right]^\alpha, \left[ \int_0^t k(t, s, y(s))ds \right]^\alpha + \left[ f(t, y(t)) \right]^\alpha \right)
\]
\begin{align*}
  & = \mathcal{d}_{H}(2t[(\alpha + 1)(x^a_q(t))^2, (3 - \alpha)(x^a_q(t))^2], 2t[(\alpha + 1)(y^a_q(t))^2, (3 - \alpha)(y^a_q(t))^2]) \\
  & = 2t \max \left\{ (\alpha + 1)|(x^a_q(t))^2 - (y^a_q(t))^2|, (3 - \alpha)|(x^a_q(t))^2 - (y^a_q(t))^2| \right\} \\
  & \leq T(3 - \alpha) \max \left\{ x^a_q(t) - y^a_q(t), x^a_q(t) + y^a_q(t), x^a_q(t) - y^a_q(t) \right\} \\
  & \leq 3T|\max \left\{ |x^a_q(t) - y^a_q(t)|, |x^a_q(t) + y^a_q(t)| \right\} \\
  & = \eta_2 \mathcal{d}_{H}([x(t)]^a, [y(t)]^a)
\end{align*}

Since \( f \) and \( k \) satisfies a global Lipschitz condition, from Theorem 7.1 fuzzy integrodifferential equation has a unique fuzzy solution.

**Fuzzy Volterra Integrodifferential Equations**

The existence and uniqueness of the fuzzy solution for the following nonlinear fuzzy Volterra integrodifferential equation is proved in this section.

\begin{align*}
  x'(t) &= a(t)x(t) + \int_0^t k(t,s)f(s,x(s))ds, \quad t \in J = [0,T] \tag{7.14} \\
  x(0) &= x_0 \tag{7.15}
\end{align*}

where \( a : J \to E_N \) is fuzzy coefficient, \( E_N \) is the set of all upper semicontinuous convex normal fuzzy numbers with bounded \( \alpha \)-level intervals, \( k : J \times J \to E_N \) is a continuous function and \( f : J \times E_N \to E_N \) is a nonlinear continuous function.

The fuzzy solution for the above nonlinear fuzzy Volterra integrodifferential equations (7.14)-(7.15) is taken as

\[ x(t) = S(t)x_0 + \int_0^t S(t-s)\left(\int_0^s k(t, r)f(r, x(r))dr\right)ds, \quad t \in J. \]

To prove the existence and uniqueness of the fuzzy solution, assume the following hypotheses:
(H1) The inhomogeneous terms \( f : J \times E_N \to E_N \) is continuous functions that satisfy global Lipschitz conditions,

\[
d_H([f(s, \xi_1(s))]^a, [f(s, \xi_2(s))]^a) \leq \eta_1 d_H([\xi_1(s)]^a, [\xi_2(s)]^a),
\]

for all \( \xi_1(s), \xi_2(s) \in E_N \) with a finite constant \( \eta_1 > 0 \).

(H2) For the continuous function \( k : J \times J \to E_N \), let \( \eta_2 > 0 \) be such that

\[
[k(t, r)]^a \leq \eta_2.
\]

(H3) \( S(t) \) is a fuzzy number such that

\[
[S(t)]^a = [S_q^a(t), S_r^a(t)] = [\exp\{\int_0^t e_q^a(s)ds\}, \exp\{\int_0^t e_r^a(s)ds\}]
\]

where \( S_i^a(t), i = q, r \), are continuous. That is, there exists a constants \( \eta_3 > 0 \) such that \( |S_i^a(t)| \leq \eta_3 \) and for all \( t \in J \).

**Theorem 7.2** Let \( T > 0 \), and assume the hypothesis satisfies (H1)-(H3) hold, then for every \( x_0 \in E_N \) the fuzzy Volterra integrodifferential equation has a unique solution \( x \in C(J; E_N) \).

Proof is similar to Theorems 7.1 in Section 7.3.

### 7.4 Fuzzy Neutral Functional Differential Equation

There exists an extensive theory for neutral functional differential equations which includes qualitative behavior of classes of such equations and applications to biological and engineering process, for details (see [32], [56], [84]). However, the concrete example is the radiocardiogram, where the two compartments correspond to
the left and right ventricles of the pulmonary and systematic circulation. Pipes coming out from and returning into the same compartment may represent shunts, and the equation representing this model is a nonlinear neutral Volterra integrodifferential equation in [31]. These class of equations also arise, in the study of problems such as heat conduction in materials with memory or population dynamics for spatially distributed populations see [28].

The existence and uniqueness of fuzzy solution for the following nonlinear fuzzy neutral functional differential equation is proved in this section.

\[
\frac{d}{dt}[x(t) - f(t, x_t)] = Ax(t) + g(t, x_t), \quad t \in J = (0, T]
\]

(7.16)

\[
x(t) = \psi(t), \quad t \in (-\infty, 0],
\]

(7.17)

where \( A : J \to E_N \) is a fuzzy coefficient, \( f, g : J \times E_N \to E_N \) are nonlinear continuous functions and satisfies a global Lipschitz condition.

The fuzzy solution for the above nonlinear fuzzy neutral functional differential equations (7.16)-(7.17) is

\[
x(t) = S(t)[\psi(0) - f(0, \psi)] + f(t, x_t) + \int_0^t AS(t - s)f(s, x_s)ds + \int_0^t S(t - s)g(s, x_s)ds.
\]

For the existence and uniqueness of the fuzzy solution, assume the following hypotheses:

(H1) The inhomogeneous terms \( f, g : J \times E_N \to E_N \) is continuous functions that satisfy global Lipschitz conditions,

\[
d_H([g(s, \xi_s)]^a, [g(s, \zeta_s)]^a) \leq c_1 d_H([\xi_s]^a, [\zeta_s]^a),
\]

\[
d_H([f(s, \xi_s)]^a, [f(s, \zeta_s)]^a) \leq c_1 d_H([\xi_s]^a, [\zeta_s]^a)
\]

for all \( \xi_s, \zeta_s \in E_N \) with finite constant \( c_1 > 0 \).
\( (H2) \) \( S(t) \) is a fuzzy number such that

\[
[S(t)]^a = [S_q^a(t), S_r^a(t)]
\]

\[
= [\exp\{\int_0^t a_q^a(s)ds\}, \exp\{\int_0^t a_r^a(s)ds\}]
\]

where \( S_i^a(t), \ i = q, r, \) are continuous. That is, there exists a constant \( \eta_2 > 0 \) such that \( |S_i^a(t)| \leq \eta_2 \) and for all \( t \in J \).

**Theorem 7.3** Letting \( T > 0 \), and assume the hypothesis satisfies \((H1)-(H2)\) hold, for every \( x_0 \in E_N \), then the fuzzy neutral functional differential equation has a unique solution \( x \in C(J : E_N) \).

Proof is similar to the Theorems 7.1 in section 7.3.