Chapter 6

Common Fixed Points of Compatible Maps in Generalized Fuzzy Metric Space

6.1 Introduction

In this chapter, the concept of compatible mappings is introduced in generalized fuzzy metric space. The common fixed point theorems for compatible mapping is obtained in the setting of generalized fuzzy metric space. Further, relations between compatible mappings and compatible maps of type ($\beta$) in generalized fuzzy metric spaces are discussed.

Definition 6.1 (Bijendra Singh and Chauhan [5]) The 3-tuple $(X, S, \ast)$ is said to be a $S$-fuzzy metric space if $X$ is an arbitrary set, $\ast$ is continuous $t$-norm and $S$ is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions:

\begin{align*}
\tag{2.1} S(x, y, z, t) &> 0 \quad \text{(non-negativity)} \\
\tag{2.2} S(x, y, z, t) &= 1 \quad \Leftrightarrow x = y = z \quad \text{(coincidence)} \\
\tag{2.3} S(x, y, z, t) &= S(y, z, x, t) = S(z, y, x, t) \quad \text{(symmetry)} \\
\tag{2.4} S(x, y, z, r + s + t) &\geq S(x, y, r) \ast S(y, z, s) \ast S(w, y, z, t) \quad \text{(tetrahedral inequality)}
\end{align*}
(2.5) $S(x, y, z, t) : (0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y, z, w \in X$ and $r, s, t > 0$.

Definition 6.2 (Bijendra Singh and Chauhan [5]) A sequence $\{x_n\}$ in a $S$-fuzzy metric space $(X, S, *)$ is Cauchy sequence if and only if for each $\epsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $S(x_n, x_m, x_p, t) > 1 - \epsilon$ for all $n, m, p \geq n_0$.

Definition 6.3 (Bijendra Singh and Chauhan [5]) A $S$-fuzzy metric space in which every Cauchy sequence is convergent is called a complete $S$-fuzzy metric space.

Definition 6.4 (Bijendra Singh and Chauhan [5]) Let $(X, S, *)$ be a $S$-fuzzy metric space. Then we define an open ball with centre $x_0 \in X$ and radius $r$, $0 < r < 1$, $t > 0$ as

$$B(x_0, r, t) = \{y \in X : S(x_0, y, y, t) > 1 - r\}$$

Example 6.1 Let $S$ be fuzzy set $X^2 \times (0, \infty)$ defined by

$$S(x, y, z, t) = \min\{M(x, y, t), M(y, z, t), M(x, z, t)\}$$

for all $x, y, z \in X$, $t > 0$. $M(x, y, t)$ is fuzzy set in the sense of Kramosil and Michalek [57] and defined as $M(x, y, t) = \frac{1}{t}(t + |x - y|)$. Then $(X, S, *)$ is $S$-fuzzy metric space (see [5]).

Definition 6.5 (Bijendra Singh and Chauhan [5]) A mapping $A$ from a $S$-fuzzy metric space $(X, S, *)$ into itself is said to be sequentially continuous at $x$ in $X$ if for every sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} S(x_n, x, z, t) = 1$ for all $z \in X$ and $t > 0$, $\lim_{n \to \infty} S(Ax_n, Ax, z, t) = 1$.

Definition 6.6 Let $F$ and $G$ be a $S$-fuzzy metric space $(X, S, *)$ into itself. $F$ and $G$ are said to be compatible if $\lim_{n \to \infty} S(FGx_n, GFx_n, z, t) \to 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $Fx_n \to y$, $Gx_n \to y$ for some $y \in X$.

Throughout this chapter $X$ denoted by $S$-fuzzy metric space.
6.2 Fixed Point Theorem

Theorem 6.1 Let $A, B, F, G$ be self-maps of a complete $S$-fuzzy metric space $X$ with continuous $t$-norm $\ast$ defined by $a \ast b = \min \{a, b\}$, $a, b \in [0, 1]$ satisfying the following conditions:

(c1) $A(X) \subseteq G(X)$, $B(X) \subseteq F(X)$

(c2) One of $A, B, F, G$ are continuous

(c3) $[A, F], [B, G]$ are compatible pairs of maps

(c4) for all $x, y, z \in X$, $k \in (0, 1)$, $t > 0$

$$S(Ax, By, z, kt) \geq \min\{S(Fx, Gy, z, t), S(Ax, Fx, z, t),$$

$$S(By, Gy, z, t), S(By, Fx, z, 2t), S(Ax, Gy, z, t)\}$$

(c5) For all $x, y, z$ in $X$, $\lim_{n \to \infty} S(x, y, z, t) \to 1$ as $n \to \infty$.

Then $A, B, F$ and $G$ have a unique common fixed point in $X$.

Proof. Let $x_0$ be an arbitrary point in $X$. Construct a sequence $\{y_n\}$ in $X$ such that

$y_{2n-1} = Gx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Fx_{2n} = Bx_{2n-1}$; $n = 1, 2, ...$

From (c4) we have,

$$S(y_{2n+1}, y_{2n+2}, z, kt) = S(Ax_{2n+1}, Bx_{2n+1}, z, kt)$$

$$\geq \min \{S(Fx_{2n}, Gx_{2n+1}, z, t), S(Ax_{2n}, Fx_{2n}, z, t),$$

$$S(Bx_{2n+1}, Gx_{2n+1}, z, t), S(Bx_{2n+1}, Fx_{2n}, z, 2t), S(Ax_{2n}, Gx_{2n+1}, z, t)\}$$

$$\geq \min \{S(y_{2n}, y_{2n+1}, z, t), S(y_{2n+1}, y_{2n}, z, t),$$

$$S(y_{2n+2}, y_{2n+1}, z, t), S(y_{2n+2}, y_{2n}, z, 2t), S(y_{2n+1}, y_{2n+1}, z, t)\}$$

$$\geq \min \{S(y_{2n}, y_{2n+1}, z, t), S(y_{2n+1}, y_{2n+2}, z, t), 1\}$$
which implies
\[ S(y_{2n+1}, y_{2n+2}, z, kt) \geq S(y_{2n}, y_{2n+1}, z, t). \]

In general
\[ S(y_{n}, y_{n+1}, z, kt) \geq S(y_{n-1}, y_{n}, z, t). \]  \hspace{1cm} (6.1)

To prove that \( \{y_n\} \) is a Cauchy sequence, we prove (6.2) is true for all \( n \geq n_0 \) and for every \( m \in N \).

\[ S(y_{n}, y_{n+m}, z, t) > 1 - \lambda. \]  \hspace{1cm} (6.2)

Here we use induction method. From (6.1), we have

\[ S(y_{n}, y_{n+1}, z, t) \geq S(y_{n-1}, y_{n}, z, \frac{t}{k}) \]
\[ \geq S(y_{n-2}, y_{n-1}, z, \frac{t}{k^2}) \]
\[ \geq \ldots \geq S(y_{n-m}, y_{n-m+1}, z, \frac{t}{k^n}) \to 1 \quad \text{as} \quad n \to \infty. \]

That is, for \( t > 0, \lambda \in (0,1) \), we can choose \( n_0 \in N \) such that

\[ S(y_{n}, y_{n+1}, z, t) > 1 - \lambda. \]  \hspace{1cm} (6.3)

Thus (6.2) is true for \( m = 1 \), suppose (6.2) is true for \( m \) then we shall show that it is also true for \( m + 1 \). Using the definition of \( S \)-fuzzy metric space (6.1) and (6.2), we have

\[ S(y_{n}, y_{n+m+1}, z, t) \geq \min \{ S(y_{n}, y_{n+m}, z, \frac{t}{2}), S(y_{n-m}, y_{n-m+1}, z, \frac{t}{2}) \} \]
\[ > 1 - \lambda. \]

Hence (6.2) is true for \( m + 1 \). Thus \( \{y_n\} \) is a Cauchy sequence. By completeness of \( X \), \( \{y_n\} \) converges to some point \( v \) in \( X \). Thus \( \{Ax_{2n}\}, \{F_{2n}\}, \{B_{2n-1}\} \) and \( \{G_{2n-1}\} \) also converge to \( v \). Now \( Ax_{2n} \to v \) and \( S \) is continuous hence \( SAx_{2n} \to Sv \). Thus for \( t > 0, \lambda \in (0,1) \), there exists an \( n_0 \in N \) such that

\[ S(FAx_{2n}, Fv, z, \frac{t}{2}) > 1 - \lambda, \quad \text{for all} \quad n \geq n_0. \]
Using (6.3), we have

\[ S(AF_{x_{2n}}, FA_{x_{2n}}, z, \frac{t}{2}) \to 1. \]

\[ S(AF_{x_{2n}}, Fv, z, t) \geq \min \{ S(AF_{x_{2n}}, FA_{x_{2n}}, z, \frac{t}{2}), S(FA_{x_{2n}}, Fv, z, \frac{t}{2}) \} \]

\[ > 1 - \lambda \text{ for all } n \geq n_0. \]

Hence \( AF_{x_{2n}} \to Fv. \) \hfill (6.4)

Similarly,

\[ BG_{x_{2n-1}} \to Gv. \] \hfill (6.5)

Using (c4), we have

\[ S(AF_{x_{2n}}, BG_{x_{2n-1}}, z, kt) \geq \min \{ S(F^2_{x_{2n}}, G^2_{x_{2n-1}}, z, t), S(AF_{x_{2n}}, F^2_{x_{2n}}, z, t), S(BG_{x_{2n-1}}, G^2_{x_{2n-1}}, z, t), S(AF_{x_{2n}}, G^2_{x_{2n-1}}, z, 2t), S(AF_{x_{2n}}, F^2_{x_{2n-1}}, z, t) \}. \]

Taking limit as \( n \to \infty, \) and using (6.4) and (6.5), we get

\[ S(Fv, Gv, z, kt) \geq S(Fv, Gv, z, t) \]

which implies

\[ Fv = Gv. \] \hfill (6.6)

Now

\[ S(Ay, BG_{x_{2n-1}}, z, kt) \geq \min \{ S(Fy, G^2_{x_{2n-1}}, z, t), S(Ay, Fy, z, t), S(BG_{x_{2n-1}}, G^2_{x_{2n-1}}, z, t), S(BG_{x_{2n-1}}, Fy, z, 2t), S(Ay, G^2_{x_{2n-1}}, z, t) \}. \]

Taking the limit as \( n \to \infty, \) and using (6.4)-(6.6), we get

\[ Ay = Gv. \] \hfill (6.7)
Now using (6.6) and (6.7),
\[
S(Av, Bv, z, kt) \geq \min \left\{ S(Fv, Gv, z, t), S(Av, Fv, z, t),
S(Bv, Gv, z, t), S(Bv, Fv, z, 2t), S(Av, Gv, z, t) \right\}
\]
\[
= \min \left\{ S(Gv, Gv, z, t), S(Av, Av, z, t), S(Av, Bv, z, t),
S(Av, Bv, z, 2t), S(Av, Av, z, t) \right\}
\]
\[
\geq \min \left\{ S(Av, Bv, z, t) \right\}
\]
which implies
\[
Av = Bv. \tag{6.8}
\]
Using (6.6)–(6.8), we get
\[
Av = Bv = Fv = Gv. \tag{6.9}
\]
Now
\[
S(Av_{2n}, Bv, z, kt) \geq \min \left\{ S(Fv_{2n}, Gv_{2n}, z, t), S(Av_{2n}, Fv_{2n}, z, t),
S(Bv, Gv, z, t), S(Bv, Fv_{2n}, z, 2t), S(Av_{2n}, Gv_{2n}, z, t) \right\}
\]
Taking limit and using (6.9) we get \( v = Bv \). Thus \( v \) is a common fixed point of \( A, B, F \) and \( G \).

For uniqueness, let \( w \) be another common fixed point of said maps. Then we have
\[
S(Av, Bw, z, kt) \geq \min \left\{ S(Fv, Gw, z, t), S(Av, Fv, z, t), S(Bw, Gw, z, t),
S(Bw, Fw, z, 2t), S(Av, Gw, z, t) \right\}
\]
That is
\[
S(v, w, z, kt) \geq S(v, w, z, t).
\]
Hence \( v = w \). This completes the proof.
6.3 Compatible of type ($\beta$)

**Definition 6.7** Let $A$ and $B$ be maps from $S$-fuzzy metric space $X$ into itself. $A$ and $B$ are said to be compatible of type ($\alpha$) if, \( \lim_{n \to \infty} S(ABx_n, Bx_n, z, t) \to 1 \) and \( \lim_{n \to \infty} S(BAx_n, AAx_n, z, t) \to 1 \) for all \( t > 0 \) whenever \( \{x_n\} \) is a sequence in $X$ such that $Ax_n \to y$, $Bx_n \to y$ for some $y$ in $X$.

**Definition 6.8** Let $A$ and $B$ be maps from $S$-fuzzy metric space $X$ into itself. $A$ and $B$ are said to be compatible of type ($\beta$) if, \( \lim_{n \to \infty} S(AAx_n, Bx_n, z, t) \to 1 \) for all \( t > 0 \) whenever \( \{x_n\} \) is a sequence in $X$ such that $Ax_n \to y$, $Bx_n \to y$ for some $y$ in $X$.

**Proposition 6.2** Let $(X, S, \ast)$ be an $S$-fuzzy metric space with $t \ast t \geq t$ for all $t \in [0, 1]$ and let $A$ and $B$ be continuous maps from $X$ into itself. Then $A$ and $B$ are compatible if and only if they are compatible of type($\beta$).

**Proof.** Suppose that $A$ and $B$ are compatible and let \( \{x_n\} \) be a sequence in $X$ such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y \) for some $y$ in $X$. Since $A$ and $B$ are continuous, we have

\[
\lim_{n \to \infty} AAx_n = \lim_{n \to \infty} ABx_n = Ay,
\]

\[
\lim_{n \to \infty} BAx_n = \lim_{n \to \infty} Bx_n = By.
\]

Further, since $A$ and $B$ are compatible,

\[
\lim_{n \to \infty} S(ABx_n, BAx_n, z, t) = 1
\]

for all \( t > 0 \). Now, since we have

\[
S(AAx_n, Bx_n, z, t) \geq S(AAx_n, ABx_n, z, \frac{t}{2}) \ast S(ABx_n, Bx_n, z, \frac{t}{2})
\]

\[
\geq S(AAx_n, ABx_n, z, \frac{t}{2}) \ast S(ABx_n, BAx_n, z, \frac{t}{4}) \ast S(BAx_n, Bx_n, z, \frac{t}{4})
\]

for all \( t > 0 \).
for all $t > 0$, it follows that
\[ \lim_{n \to \infty} S(AAx_n, BBAx_n, z, t) \geq 1 \ast 1 \ast 1 \geq 1 \]
which implies that
\[ \lim_{n \to \infty} S(AAx_n, BBAx_n, z, t) \to 1. \]
Therefore, $A$ and $B$ are compatible of type($\beta$).

Conversely, suppose that $A$ and $B$ are compatible type($\beta$) and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y$ for some $y$ in $X$. Since $A$ and $B$ are continuous, we have
\[ \lim_{n \to \infty} AAx_n = \lim_{n \to \infty} ABBx_n = Ay. \]
\[ \lim_{n \to \infty} BAx_n = \lim_{n \to \infty} BBAx_n = By. \]
Further, since $A$ and $B$ are compatible of type($\beta$), we have for all $t > 0$
\[ \lim_{n \to \infty} S(AAx_n, BBAx_n, z, t) = 1. \]
Thus, from the inequality
\[ S(AAx_n, BBAx_n, z, t) \geq S(AAx_n, AAx_n, z, \frac{t}{2}) \ast S(ABAx_n, BBAx_n, z, \frac{t}{4}) \]
\[ \geq S(AAx_n, AAx_n, z, \frac{t}{2}) \ast S(ABAx_n, BBx_n, z, \frac{t}{4}) \ast S(BBX_n, BBAx_n, z, \frac{t}{4}) \]
it follows that
\[ \lim_{n \to \infty} S(AAx_n, BBAx_n, z, t) \geq 1 \ast 1 \ast 1 \geq 1 \]
for all $t > 0$, which implies that $\lim_{n \to \infty} S(AAx_n, BBAx_n, z, t) = 1$. Therefore, $A$ and $B$ are compatible. This completes the proof.

**Proposition 6.3** Let $X$ be an $S$-fuzzy metric space with $t \ast t \geq$ for all $t \in [0, 1]$ and let $A$ and $B$ be compatible maps of type($\alpha$). If one of $A$ and $B$ is continuous, then $A$ and $B$ are compatible of type($\beta$).
Proof. Suppose that $A$ and $B$ are compatible of type($\alpha$) and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y$ for some $y$ in $X$. Assume, without loss of generality, that $A$ is continuous. Then we have

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} ABx_n = Ay.$$ 

Further, since $A$ and $B$ are compatible of type($\alpha$), we have also, for all $t > 0$,

$$\lim_{n \to \infty} S(ABx_n, BBx_n, z, t) = 1, \quad \lim_{n \to \infty} S(BAx_n, AAx_n, z, t) = 1.$$ 

Thus, from the inequality,

$$S(AAx_n, BBx_n, z, t) \geq S(AAx_n, ABx_n, z, \frac{t}{2}) + S(ABx_n, BBx_n, z, \frac{t}{2})$$

it follows that

$$\lim_{n \to \infty} S(AAx_n, BBx_n, z, t) \geq 1 * 1 \geq 1$$

which implies that

$$\lim_{n \to \infty} S(AAx_n, BBx_n, z, t) \geq 1.$$ 

Therefore, $A$ and $B$ are compatible of type($\beta$). This completes the proof.

**Proposition 6.4** Let $X$ be an $S$-fuzzy metric space with $t * t \geq 1$ for all $t \in [0, 1]$ and let $A$ and $B$ be continuous maps from $X$ into itself. If $A$ and $B$ are compatible of type($\beta$), then they are compatible of type($\alpha$).

Proof. Suppose that $A$ and $B$ are compatible of type($\beta$) and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y$ for some $y$ in $X$. Since $A$ and $B$ are continuous,

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} ABx_n = Ay,$$

$$\lim_{n \to \infty} BBx_n = \lim_{n \to \infty} BAx_n = By.$$ 

Further, since $A$ and $B$ are compatible of type($\beta$), we have, for all $t > 0$

$$\lim_{n \to \infty} S(AAx_n, BBx_n, z, t) = 1.$$
Thus, from the inequality

\[ S(ABx_n, BBx_n, z, t) \geq S(ABx_n, AAx_n, z, \frac{t}{2}) \ast S(AAx_n, BBx_n, z, \frac{t}{2}) \]

it follows that

\[ \lim_{n \to \infty} S(ABx_n, BBx_n, z, t) \geq 1 \ast 1 \geq 1 \]

which implies that, for all \( t > 0 \),

\[ \lim_{n \to \infty} S(ABx_n, BBx_n, z, t) = 1. \]

Similarly, we have also, for all \( t > 0 \)

\[ \lim_{n \to \infty} S(BAx_n, AAX_n, z, t) = 1. \]

Therefore, \( A \) and \( B \) are compatible of type(\( \alpha \)). This completes the proof.

**Example 6.2** Let \( X \) be a fuzzy metric space with \( X = [0, 1] \), and \( S \) be a fuzzy set on \( X^3 \times (0, \infty) \) for all \( x, y, z \in X \) and \( t > 0 \). Define \( Ax = x/81, \) \( Bx = (x/27) \), \( Fx = (x/3) \) and \( Gx = (x/9) \). For let \( k = [(1/9), 1] \).

Define \( S(Ax, Bx, z, t) = \min \{ M(Ax, Bx, t), M(Bx, z, t), M(z, Ax, t) \} \). Then

\[ M(Ax, Bx, z, t) = \frac{kt}{kt + d(Ax, Bx)} \]

\[ = \frac{kt}{kt + \left| \frac{x}{81} - \frac{z}{9} \right|} \]

\[ \geq \frac{t}{t + \left| \frac{1}{9} - \frac{1}{3} \right|} \]

\[ = M(Fx, Gx, t). \]

Similarly, we can prove the others. Then \( A, B, F \) and \( G \) satisfy all the condition of the Theorem 6.1, and zero is the unique fixed point of \( A, B, F \) and \( G \).
6.4 Application

By applying Theorem 6.1, we can show the existence of solutions for an equation of the form $Ax = Px$, where $A$ and $P$ are sequentially continuous mapping from a complete $S$-fuzzy metric space $X$ into itself.

**Theorem 6.5** Let $A$ and $P$ be sequentially continuous mappings from a complete $S$-fuzzy metric space $X$ into itself satisfying the following conditions: there exist $0 < \beta \leq \alpha$, $\mu \in \mathbb{N}$ and $0 < h < 1$ such that

(d1) $S(Ax, Ay, z, ht) \leq \alpha S(x, y, z, t)$

(d2) $P^n(X) \subset AP^{n-1}(X)$

(d3) $S(P^m(x), P^n(y), z, ht) \geq \beta \min \left\{ S(P^{m-1}(x), P^{n-1}(y), z, t), S(A^{-1}P^m(x), P^{m-1}(x), z, t), S(A^{-1}P^n(y), P^{n-1}(y), z, t), S(A^{-1}P^m(x), P^{m-1}(x), z, t), S(A^{-1}P^n(y), P^{n-1}(y), z, t) \right\}$

for all $x, y$ and $z$ in $X$, $t > 0$. Suppose that

(d4) $A$ is Surjective

(d5) the pair $A^{-1}P^m, P^{m-1}$ is compatible.

Then the equation $Ax = Px$ has at least one solution in $X$.

**Proof.** We note that if $Ax = Ay$, then $x = y$ so that $A$ is bijective and hence $A^{-1}$ exists.

From (d1) and (d3), we deduce

$S(A^{-1}P^m x, A^{-1}P^m y, z, ht) \geq \frac{1}{\alpha} S(P^m x, P^m y, z, t)$
\[
\geq \frac{\beta}{\alpha} \min \{S(P^{m-1}x, P^{m-1}y, z, t), S(A^{-1}P^m x, P^{m-1}x, z, t), \\
S(A^{-1}P^m y, P^{m-1}y, z, t), S(A^{-1}P^m x, P^{m-1}y, z, t), \\
S(A^{-1}P^m y, P^{m-1}x, z, t)\} \}
\]

for all \(z \in X\). Now, we see that all the hypotheses of Theorem 6.1 for \(A^{-1}P^m\) and \(P^{m-1}\) are satisfied. Therefore, there exists a unique point \(x_0\) in \(X\) such that

\[
A^{-1}P^m x_0 = P^{m-1}x_0 = x_0.
\]

and so we can deduce \(A^{-1}Px_0 = x_0\), that is \(Ax_0 = Px_0\). This completes the proof.