CHAPTER III

$(J, p_n)$ SUMMABILITY OF FOURIER SERIES.

$(33 - 40)$
3.1. Let \( p_n > 0 \) be such that \( \sum_{n=0}^{\infty} p_n \) diverges, and the radius of convergence of the power series,

\[
(3.1.1) \quad p(x) = \sum_{n=0}^{\infty} p_n x^n
\]

be unity. Given any infinite series \( \sum a_n \) with the sequence of partial sums \( \{S_n\} \), we shall use the notations

\[
(3.1.2) \quad \frac{p_s(x)}{p(x)} = \sum_{n=0}^{\infty} p_n S_n x^n
\]

and

\[
(3.1.3) \quad J_s(x) = \frac{p_s(x)}{p(x)}
\]

If the series on the right of (3.1.2) is convergent in the right open interval \([0,1)\), and if

\[
\lim_{x \to 1^-} J_s(x) = S,
\]

we say that the series \( \sum a_n \) or the sequence \( \{S_n\} \) is summable \((J,p_n)\) to \(S\), where \(S\) is finite. 1

**Particular cases of \((J,p_n)\) method of summability:**

(i) \((L)\) method of summability: when \(p_n = \frac{1}{n+1}\), \((J,p_n)\) method reduces to \((L)\) method of summability which was first introduced for the \(\alpha\) time by Borwein. 2,3

(ii) The abel method: when \(p_n = 1\) for all \(n = 1,2,\ldots\)

1. Hardy, G.H. (H).
3.2 Let \( f(t) \) be Lebesgue integrable in \((-\alpha, \alpha)\) and be periodic with period \( 2\alpha \); let the Fourier series of \( f(t) \) at \( t=y \) be

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\gamma + b_n \sin n\gamma \right)
\]

Let \( S_n(y) \) be the partial sum of the series (3.2.1)

We shall use following notations:

\[
(3.2.2) \quad \phi(t) = \frac{1}{2} \left\{ f(y+t) + f(y-t) - 2S \right\}
\]

\[
(3.2.3) \quad \phi_0(t) = \phi(t)
\]

and

\[
(3.2.4) \quad \phi_\alpha(t) = \frac{\alpha}{t^\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u) \, du; \quad \alpha > 0
\]

\[
(3.2.5) \quad H_\alpha(t) = \int_0^t \frac{\phi_\alpha(u)}{u} \, du
\]

and

\[
(3.2.6) \quad \gamma = 1 - \chi
\]

3.3 Generalizing the results of Nanda\(^1\), and Nanda and Das\(^2\)

Nanda and Das\(^3\) have proved the following theorems

**Theorem ND1** Let for \( \alpha > 0 \)

\[
(3.3.1) \quad H_\alpha(t) = o(\log^{1/\chi} t) \quad (t \to \infty),
\]

then the Fourier series (3.2.1) is summable \( (L) \) to the sum \( S \) at \( t=y \).

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1. Nanda, M.M. (N)
THEOREM ND2  Let $\beta > \gamma > 0$; then

\[(3.3.2)\quad H_\alpha(t) = o(\log^{1\over \beta} t) \quad (t \to +0)\]
implies.

\[(3.3.3)\quad H_\beta(t) = o(\log^{1\over \beta} t) \quad (t \to +0)\]

In theorem ND1; when $\alpha = 0$ we get the result of Nanda\(^1\). When $\alpha = 1$ we get the result of Nanda and Das\(^2\). For the first time Khan\(^3\) has applied $(J, p_n)$-method of summability to Fourier series and proved the following theorem.

THEOREM K  Let the sequence $\{p_n\}$ be positive and decreasing steadily to zero such that $\{np_n\}$ is founded.

If

\[(3.3.4)\quad \int_0^t \left| \phi(u) \right| \, du = o(t \, p(1-t)), \quad (t \to +0)\]
and

\[(3.3.5)\quad \int_0^t \frac{\left| \phi(u) \right|}{u} \, du = o(p(1-t)),\]
as $t \to +0$, for any arbitrary $\delta$, $0 < \delta \leq \pi$, then the Fourier series (3.2.1) is summable $(J, p_n)$ to $S$ at $t = y$.

The object of the present chapter is two fold.

Firstly to obtain a theorem which is $(J, p_n)$ analogue of Theorem ND2 and Secondly to generalize Theorem ND1 and to improve Theorem K.

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In fact we shall prove following theorems:

**THEOREM 1**  
Let the sequence \( \{ p_n \} \) be positive and decreasing steadily to zero such that \( \{ np_n \} \) is bounded \( \beta \geq \alpha > 0 \); then

\[(3.3.6) \quad H_\alpha(t) = o(p(1-t)) \quad (t \to +0)\]

implies

\[(3.3.7) \quad H_\beta(t) = o(p(1-t)) \quad (t \to +0)\]

**THEOREM 2**  
Let for \( \alpha > 0 \)

\[(3.3.8) \quad H_\alpha(t) = o(p(1-t)) \quad (t \to +0), \]

then the Fourier series (3.2.1) is summable \( (j, p_n) \) to \( S \) at \( t = y \), provided the sequence \( \{ p_n \} \) be same defined in theorem 1.

3.4 Proof of Theorem 1

Write

\[ K = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1) \Gamma(\beta-\alpha)} \]

Then,

\[ \phi(\alpha) = \frac{\beta}{u^\beta} \int_0^u (u-v)^{\beta-1} \phi(v) dv \]

\[ = Ku^{-\beta} \int_0^u (u-v)^{\beta-\alpha-1} v^\alpha \phi(v) dv \]

and so

\[ H_\beta(t) = K \int_t^\infty u^{-\beta-1} du \int_0^t (u-v)^{\beta-\alpha-1} v^\alpha \phi(v) dv \]

\[ = K \int_0^\infty v^\alpha \phi(v) dv \int_t^\infty u^{-\beta-1} (u-v)^{\beta-\alpha-1} du \]

\[ + K \int_t^\infty v^\alpha \phi(v) dv \int_0^t u^{-\beta} (u-v)^{\beta-\alpha-1} du \]

\[(3.4.1) \quad \ldots \quad = I_1 + I_2, \text{ say} \]

Now write
\[
\mathcal{K}(\nu) = \nu^{\alpha+1} \int_{t}^{\bar{\alpha}} \omega^{-\beta-1}(\omega-1)^{\beta-\alpha-1} d\omega
\]

(3.4.2)

It is obvious that \(\mathcal{K}(\nu)\) is positive decreasing and bounded for \(\nu > 0\). Hence, by the second mean value theorem,

\[
|I_2| = K \frac{\mathcal{K}(t)}{\mathcal{K}(\nu)} \int_{t}^{\bar{\alpha}} \frac{\phi(\nu)}{\nu} d\nu \text{ for some } \xi \text{ between } t \text{ and } \lambda
\]

(3.4.3)

\(= o(p(1-t))\) by (3.3.6)

(3.4.4)

\[
|I_1| = K \int_{t}^{\bar{\alpha}} \nu^{\alpha+1} dH_{\alpha}(\nu) \int_{t}^{\bar{\alpha}} \omega^{-\beta-1}(\omega-1)^{\beta-\alpha-1} d\omega
\]

It is evident from the line above that the inner integral in (3.4.4) for fixed \(t\), is \(O(\nu^{\alpha+1})\) as \(\nu \to 0\). So on integration by parts:

\[
|I_1| < K \int_{t}^{\bar{\alpha}} \mathcal{K}(t) H_{\alpha}(t) d\nu
\]

\[
+ \bar{\alpha}^{-\beta} \int_{t}^{\bar{\alpha}} \nu^{\alpha} (\bar{\alpha}-\nu)^{\beta-\alpha-1} d\nu
\]

\[
+ t^{-\beta} \int_{t}^{\bar{\alpha}} \nu^{\alpha} (t-\nu)^{\beta-\alpha-1} H_{\alpha}(\nu) d\nu
\]

Now use the boundedness property of \(\mathcal{K}(t)\). Then it is obvious that the first two terms on the right hand side of \(I_1\) are \(o(p(1-t))\) and the third term is

\[
o(t^{-\alpha-1} \int_{t}^{\bar{\alpha}} \nu^{\alpha} \{ p(1-\nu) \} d\nu
\]

\[
+ o(t^{\alpha-\beta} \{ p(1-t) \} \int_{t}^{\bar{\alpha}} (t-\nu)^{\beta-\alpha-1} d\nu)
\]
(3.4.5) \[ = \mathcal{o}(p(1-t)) \]

Collection of (3.4.1), (3.4.3) and (3.4.5) completes the proof of Theorem 1.

3.5

We shall require the following lemmas for the proof of Theorem 2.

**Lemma 1**

Let \( \sum_{n=1}^{\infty} p_n x^n \sin nt \)

\((p_n \text{ being same as in the theorem})\)

and

(3.5.2) \[ N_k(t) = t^{k+1}D_t^k \left( \frac{M(t)}{t} \right), \quad D_t^k = \frac{d^k}{dt^k} \]

then for \( k = 1, 2, \ldots \)

(3.5.3) \[ N_k'(t) = \begin{cases} O(t^{k+1}/n^2) & \text{for } t/n < 1-x \\ O(t^{k+1}/n) & \text{for } t/n \end{cases} \]

**Proof:** Using Leibnitz theorem for successive differentiation, we have

(3.5.4) \[ D_t^{k+1} \left( \frac{M(t)}{t} \right) = t D_t^{k+1} \left( \frac{M(t)}{t} \right) + (k+1)D_t^k \left( \frac{M(t)}{t} \right) \]

From (3.5.2) we have

(3.5.5) \[ N_k'(t) = t^k \left\{ D_t^{k+1} \left( \frac{M(t)}{t} \right) \right\} \]

Now \( M(t) = \sum_{n=1}^{\infty} p_n x^n \sin nt \)

\[ = \sum_{n=1}^{\infty} p_n \frac{x^n \sin nt}{n} \]

\[ = \frac{\ln (x e^{it} n)}{n} \]

\[ = 0 \left( \tan^{-1} \frac{x \sin t}{1-x} \right) \]
and Nanda \(^{1}\) gives

\[
D_{t}^{k+1} (M(t)) = \begin{cases} 
0(1/n^{k+1}) & \text{for } t < n \\
0(n/t^{k+2}) & \text{for } t > n \end{cases}
\]

So (3.5.5) and (3.5.6) yields (3.5.3)

**Lemma 2** According to Nanda and Das \(^{2}\)

\[
D_{t}^{k} \left( \frac{M(t)}{t} \right) = O(1) \quad (t \to +0)
\]

for \(k = 0, 1, 2, \ldots \) and fixed \(n\)

**Lemma 3** Nanda and Das \(^{3}\) have given

for fixed \(t \in (0 < t < 2\pi)\) and \(k\)

\[
D_{t}^{k} \left( \frac{M(t)}{t} \right) = O(1) (x \to 1-)
\]

In fact as \(x \to 1\)

\[
D_{t}^{k} \frac{M(t)}{t} \to \frac{(-1)^{k} \pi}{2n^{k+1}}
\]

3.6 **Proof of Theorem 2**

It is well known that

\[
S_{n}(y) - S = \frac{2}{\pi} \int_{0}^{\pi} \frac{\phi(t) \sin t}{t} \, dt + o(1)
\]

So

\[
\sum_{n=1}^{\infty} p_{n} x^{n}(S_{n}(y) - S) = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_{0}^{\pi} \frac{\phi(t)}{t} \, p_{n} x^{n} \sin t \, dt + o(p(x))
\]

\[
= \frac{2}{\pi} \int_{0}^{\pi} \frac{\phi(t)}{t} M(t) \, dt + o(p(x))
\]

\[
= \frac{2}{\pi} I + o(p(x))
\]

Integrating \( \frac{\partial}{\partial \xi} \) by parts \( \xi \) times we have
\[
I = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^F}{r^2} \phi^F(t) \frac{t^{r-1} M(t)}{t} \int_0^\infty \frac{t^{-k}}{r^2} \phi^F(t) \frac{t^{r-1} M(t)}{t} \, dt
\]
\[
= \frac{(-1)^{\xi}}{t^F} \int_0^\infty \frac{t^{\xi} \phi^F(t) \frac{t^{r-1} M(t)}{t}}{t} \, dt
\]
(3.6.2)

\( I_1 = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^F}{r^2} \phi^F(t) \frac{t^{r-1} M(t)}{t} \) at \( t = \bar{\eta} \)

\[ \Rightarrow \sum_{k=1}^{\infty} (-1)^{k-1} \frac{t^F}{r^2} \phi^F(\bar{\eta}) \left[ (-1)^{k-1} \frac{r-1}{2^{\bar{\eta}}} (x \to 1^-) \right] \]

by (3.5.9)

\[
= \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{\phi^F(\bar{\eta})}{r}
\]
(3.6.3)

\( = o(p(x)) \)

Now on integration by parts,
\[
I_2 = \int_0^\infty N_{\alpha}(t) \, dH_{\alpha}(t)
\]
\[ = H_{\alpha}(\bar{\eta}) N_{\alpha}(\bar{\eta}) \lim_{t \to 0} H_{\alpha}(t) N_{\alpha}(t) + \]
\[ + \int_0^\infty H_{\alpha}(t) N'_{\alpha}(t) \, dt
\]
\[ = (\int_0^\eta + \int_\eta^\infty) H_{\alpha}(t) N_{\alpha}(t) \, dt, \text{ by Lemma 2.}
\]
\[ = \int_0^\eta o(p(1-t)) \frac{t^k}{r^{k+1}} \, dt + \int_\eta^\infty o(p(1-t)) \frac{t}{r^2} \, dt \text{ by (3.5.3)}
\]
\[ = o(p(x)) + o(p(x))
\]
(3.6.4)

\[ = o(p(x))
\]

Collection of (3.6.1), (3.6.2), (3.6.3) and (3.6.4) together with Theorem 1 of Khan \(^1\) completes the proof of Theorem 2.

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1. Khan, F.M. (K) P.14