CHAPTER IX

S. N. SUMMABILITY OF

FOURIER SERIES

( 84 - 91 )
In 1949 Meyer-König introduced the so-called $S_\alpha$-method of summability which is one of a family of transformations including the Euler, Borel and Taylor (Circle methods) methods. Later in 1959 Jamimovski introduced the $\left[ F, \alpha_n \right]$ transformation which includes the Euler method $(E, q)$, Karamata method $(K)$ and Lototsky method as particular cases.

For the first time Meir and Sharma introduced generalization of the $S_\alpha$-method and called it $\left[ S, \alpha_n \right]$ method. They obtained sufficient conditions for the regularity of this method. They have also critically examined the behaviour of its Lebesgue constant.

Let $\{s_j\}_0^\infty$ be a given sequence of real or complex numbers we shall say that $\{s_j\}$ is the $\left[ S, \alpha_n \right]$ transform of $\{s_j\}$, i.e. the sequence of partial sums of the series $\sum a_n$ if

$$\sigma_n = \sum_{k=0}^{\infty} c_{n,k} S_k \quad (n = 0, 1, 2, \ldots)$$

converges, where $(c_{n,k})$ is given by the identity

$$\sum_{j=0}^{n} \frac{1 - \alpha_j}{1 - \alpha_j \theta} = \sum_{k=0}^{n} c_{n,k} \theta^k$$

The sequence $\{s_j\}$ is said to be $\left[ S, \alpha_n \right]$ summable to $\sigma$ if

$$\lim_{n \to \infty} \sigma_n = \sigma$$

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1. Meyer König (M)
2. Jakimovski, A. (J)
3. Meir, A and Sharma, A (M.S.)
4. Ibid
Let \( f(x) \in L_2(0,2\pi) \) and be periodic with period \( 2\pi \)
outside this range. Let the Fourier series associated with this function be

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \sum_{n=0}^{\infty} A_n (x)
\]

and as usual let us denote

\[
\phi(t) = \phi_x(t) = \frac{1}{2} \left\{ f(x + t) + f(x-t) - 2s \right\}
\]

\( s \) is a constant.

Also,

\[
V_N = 1 + 2 \sum_{j=0}^{N} \frac{\alpha_j}{1-\alpha_j}
\]

\[
T_N = 2 \sum_{j=0}^{N} \frac{\alpha_j}{(1-\alpha_j)^2}
\]

Meir and Sharma\(^1\) proved following theorem regarding regularity of \([S, \alpha_n]\) method

**Theorem A** Suppose the sequence \( \{\alpha_j\} \) satisfied

\[
|\alpha_j| < 1 \quad (j = 0, 1, 2 \ldots)
\]

\[
H = \prod_{j=0}^{\infty} \frac{|1 - \alpha_j|}{|1 - |\alpha_j||} < \infty
\]

\[
\sum_{j=0}^{\infty} |\alpha_j| = \infty
\]

Then the \([S, \alpha_n]\) transformation is regular.

Meir and Sharma\(^2\) while studying Lebesque constant established that when \( V_N \) and \( T_N \) are bounded the \([S, \alpha_n]\) method sums only convergent Fourier series and so hereafter we shall assume \( T_N \rightarrow \infty \) and \( V_N \rightarrow \infty \) with \( N \)

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1. Meir, A and Sharma, A. (M.S.)
2. Ibid.
Lebesque gave the convergence criteria for the Fourier series. He proved following theorem

**Theorem** Fourier series (9.2.1) converges to the value \( f(x) \) at every point \( x \) at which

(9.2.8) \[ \Phi(t) = o(t) \quad \text{as} \quad t \to +0 \]

and

(9.2.9) \[ \int_{\eta}^{\infty} \frac{\Phi(t) - \Phi(t+\eta)}{t} \, dt \to 0 \quad \text{as} \quad \eta \to 0 \]

Voluminous work, on this convergence criteria relating to different summability methods namely (C,1), (H,1), (N,pn), (E), (E,1), \([F,q_n]\), ...

... Taylor (circle method) etc. had been done during the last fifty years. In this chapter we have studied \([s, \alpha_n]\) method of summability for Fourier series. In fact we shall prove the following theorem

**Theorem** If

(9.2.10) \[ \int_{a}^{t} \Phi(t) \, dt = o(t) \quad \text{as} \quad t \to +0 \]

and

(9.2.11) \[ \lim_{n \to \infty} \int \Phi(t) - \Phi(t + 2\pi/\sqrt{n}) \, \exp \left( -\frac{t^2}{n^2} \right) \, dt \to 0 \]

where \( \xi \) is positive constant then the Fourier series of \( f \) is \([s, \alpha_n]\) summable to \( s \) at the point \( x \).

**9.3** For the proof of our theorem we shall need following estimates

Let

\[ k_n(t) = e^{i \theta} \prod_{j=0}^{n} \frac{1 - \alpha_j}{1 - \alpha_j e^{2it}} \]

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1 - Lebesque, H (L)
See also Zygmund (2), p.29.
Then for \( 0 < t < \infty \)

\[
(9,3.1) \quad I_m \left| K_n(t) \right| = 0 \left( \frac{1}{t} \sqrt{T_n} \right)
\]

(See Meir and Sharma \(^1\))

Now by the regularity condition of \( \left[ S_n, \alpha_n \right] \)

method, we have \( \sum \alpha_j = \infty \) which implies that \( V_n \) and \( T_n \) tend to

infinity as \( n \to \infty \). Now choose \( \xi_n = \xi_n(n) \) such that \( T_n \xi_n^3 = o(1) \)

and \( \frac{T_n \xi_n^2}{S_{n+1}} = o(1) \) as \( n \to \infty \) and it is obvious that

\( V_n^2 / T_n \to \gamma \) as \( n \to \infty \)

So, for \( t \) sufficiently small

\[
\frac{1 - \alpha_j}{1 - \alpha_j e^{xt}} = \left[ \frac{1 - \alpha_j (e^{xt} - 1)}{1 - \alpha_j} \right]^{-1}
\]

\[
= 1 + \alpha_j \left( \frac{2it - 2t^2 + o(t^3)}{1 - \alpha_j} \right) + \left( \frac{\alpha_j}{1 - \alpha_j} \right)^2 \left( -4t^2 + o(t^3) \right)
\]

\[
= 1 + \frac{2\alpha_j it}{1 - \alpha_j} - \frac{2\alpha_j (1+\alpha_j)}{(1-\alpha_j)^2} t^2 + O(\alpha_j^3 t^3)
\]

\[
= e^{\exp \left[ \frac{2\alpha_j it}{1 - \alpha_j} - \frac{2\alpha_j}{(1-\alpha_j)^2} t^2 + O(\alpha_j t^3) \right]}
\]

Therefore for \( t \) to be very small

\[
\prod_{j=0}^{n} \frac{1 - \alpha_j}{1 - \alpha_j e^{xt}} = e^{\exp \left\{ (it \sum_{j=0}^{n} \frac{2\alpha_j}{1 - \alpha_j}) - t^2 \left( \sum_{j=0}^{n} \frac{2\alpha_j}{(1 - \alpha_j)^2} \right) + t^3 \sum_{j=0}^{n} \alpha_j \right\}}
\]

\( 1 = \text{Meir, A and Sharma, A (MS)} \)
\[ = \exp \left[ V_n t - T_n t^2 + o(1) \right] \]

Therefore for \( t \) to be very small \( o(t) \approx 0 \)

\[ |K_n(t)| = e^{-T_n t^2} \exp \left\{ (\nu_n + 1) i t \right\} \]

(9.3.2) \( I_m |K_n(t)| = 0 \left( e^{-T_n t^2} \sin (\nu_n t) \right) \)

9.4 **Proof of the Theorem**

It is well known that

\[ \delta_k - s = \frac{2}{\lambda} \int_0^\lambda \frac{\phi(t) \sin \left( \nu + 1/2 \right) t dt}{t} + o(1) \]

Then

\[ e^{-\delta_k} o(1) = \frac{2}{\lambda} \int_0^\lambda \frac{\phi(t)}{t} t \left\{ \sum_{k=0}^{\nu_n} c_{n,k} \sin(k+1/2) t dt \right\} + o(1) \]

(9.4.1)...

\[ = I_1 + I_2 + I_3, \text{ say} \]

where \( 1/3 \ll \ll 1/2 \)

Now first of all consider \( I_3 \)

\[ |I_3| = o(1) \int_{T_n^{-x}}^\lambda \frac{\phi(t)}{t} t \sqrt{T_n} \ dt \text{ by (9.3.1)} \]

\[ = o \left( \frac{1}{\sqrt{T_n}} \right) \left[ o(T_n^{-x}) \right], \text{ On integration by parts} \]

(9.4.2)...

\[ = o(1) \text{ as } n \rightarrow \infty \because \alpha \ll 1/2 \]

Next

\[ |I_1| = o(1) \int_0^\lambda \frac{\phi(t)}{t} e^{-T_n t^2/4} \sin (\nu_n + 1) t/2 dt \text{ by (9.3.2)} \]

\[ = o \left( \frac{1}{\nu_n} \right) o \left( \frac{1}{\sqrt{T_n}} \right) \]

(9.4.3)...

\[ = o(1) \text{ by hypothesis (9.2.10)} \]

Lastly

\[ I_2 = o(1) \int_{S_{\nu_n}}^\lambda \frac{\phi(t)}{t} e^{-T_n t^2/4} \sin \frac{\nu_n t}{2} dt \]

\[ \text{by hypothesis (9.2.10)} \]
\[
= \int_{\frac{2\pi}{V_n}}^{\frac{2\pi}{V_n}} \frac{\phi(t) - \phi(t+\frac{2\pi}{V_n})}{t} \exp\left(-\frac{T_n}{4} t^2\right) \sin \frac{V_n t}{n^2} \, dt \\
+ \int_{\frac{2\pi}{V_n}}^{\frac{2\pi}{V_n}} \phi(t+\frac{2\pi}{V_n}) \left\{ \exp\left(-\frac{T_n}{4} t^2\right) - \exp\left(-\frac{T_n}{4} \left(t+\frac{2\pi}{V_n}\right)^2\right) \right\} \sin \frac{V_n t}{n^2} \, dt \\
= \int_{0}^{\frac{2\pi}{V_n}} \frac{\phi(t+\frac{2\pi}{V_n})}{t+\frac{2\pi}{V_n}} \exp\left(-\frac{T_n}{4} \left(t+\frac{2\pi}{V_n}\right)^2\right) \left\{ \frac{1}{t+\frac{2\pi}{V_n}} - \frac{1}{t+\frac{2\pi}{V_n}} \right\} \sin \frac{V_n t}{n^2} \, dt \\
+ \int_{\frac{2\pi}{V_n}}^{\frac{2\pi}{V_n}} \frac{\phi(t+\frac{2\pi}{V_n})}{t+\frac{2\pi}{V_n}} \exp\left(-\frac{T_n}{4} \left(t+\frac{2\pi}{V_n}\right)^2\right) \sin \frac{V_n t}{n^2} \, dt \\
\text{(9.4.4)} \Rightarrow \sum_{\tau=1}^{N} I_2 (\tau). \\
\text{Take} \\
I_2 (1) = o(1) \text{ by hypothesis (9.2.11) of our theorem} \\
\text{Consider,} \\
I_2 (2) = \int_{\frac{2\pi}{V_n}}^{\frac{2\pi}{V_n}} \frac{\phi(t+\frac{2\pi}{V_n})}{t} \exp\left(-\frac{T_n}{4} t^2\right) \sin \frac{V_n t}{n^2} \, dt \\
\text{Now by the application of mean value theorem of differential calculus to expression under \[ \] sign, we have}
\[ |I_2(2)| \leq O(1) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t + 2\pi/V_n) \right| \exp\left(-\frac{T_n \delta^2}{4} \right) (T_n^2) V_n(t) \, dt. \]

(for \( t < \delta < t + 2\pi/V_n < 2t \))

\[ = O\left(\frac{T_n \delta}{V_n} \right) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t + 2\pi/V_n) \right| \exp\left(-\frac{T_n}{4} \right) \frac{1}{t(t + 2\pi/V_n)} \, dt. \]

\[ I_2(3) = O\left(\frac{1}{V_n^{\alpha}} \right) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t + 2\pi/V_n) \right| \exp\left(-\frac{T_n}{4} \right) \frac{1}{t(t + 2\pi/V_n)} \, dt. \]

\[ = O\left(\frac{1}{V_n^{\alpha}} \right) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t + 2\pi/V_n) \right| \exp\left(-\frac{T_n}{4} \right) \frac{1}{t(t + 2\pi/V_n)} \, dt. \]

Now integrate by parts and use the fact that

\[ \frac{T_n}{V_n \sqrt{2}} \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ with hypothesis of our theorem.} \]

\[ I_2(4) = o(1) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t) \right| V_n t \, dt. \]

\[ I_2(5) = o(1) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t) \right| \, dt. \]

\[ = o(T_n^\alpha) \int \frac{T_n^{-\alpha}}{t} \left| \phi(t) \right| \, dt. \]

\[ I_2(6) = o(1) \]

Collection of (9.4.4) (9.4.5) ... (9.4.8) gives

\[ I_2 = o(1). \]

Collection of (9.4.1) (9.4.2) (9.4.3) & (9.4.9)

completes the proof of the theorem.
9.5 REMARKS

It is interesting to note that if $\alpha_j = r < 1$ for all $j$ then $\sum_{n} s_n \alpha_n \sum_{n} \alpha_n$ method reduces to the well known Taylor method (or circle method of summability), so if we put $\alpha_j = r < 1$ for all $j$ our theorem takes the following form

If

$$
(9.5.1) \quad \int_{0}^{t} \phi(t) \ dt = o(t) \ as \ t \to +c, \ and
$$

$$
(9.5.2) \quad \lim_{n \to \infty} \left( \int_{0}^{t} \phi(t) - \hat{\phi}(t+1/n) \right) \ exp \left( \frac{-nrt^2}{t} \right) \ exp \left( \frac{-nrt^2}{2(1-r)^2} \right) \ dt \to 0
$$

Where $\xi$ is +ve constant, then the Fourier series is $\sum_{n} s_n \alpha_n$ summable to $S$ at $t = x$.

This indicates that the result of Holland Sahney and Tzimbalario\(^1\) on Taylor summability of Fourier series is the particular case of our theorem.

It is also desirable to note that $r = 0$ leads to the ordinary convergence therefore this value is excluded as far the summability $s_n \alpha_n$ method is concerned but in that case our theorem coincides with well known Lebesgue convergence criteria for the Fourier series.

\(0 = 0 = 0 = 0\)

\(^1\) Holland A.S.D., Sahney, B.N. Tzimbalario, J (HST)