CHAPTER IV

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4.1. Recently Harinath [1] obtained some results on fixed points in Pseudocompact Tichonov space which generalize some results of Fisher [3] over compact metric space. The main result of Harinath ([1], Theorems 5) is as follows:

**Theorem A.** Let $P$ be a Pseudocompact Tichonov space and $\mu$ be a non-negative real valued continuous function over $P \times P$ ($P \times P$ is Tichonov but need not be Pseudocompact) satisfying:

$$
\mu(x,x) = 0 \text{ for all } x \in P \quad \text{and} \\
\mu(x,y) \leq \mu(x,z) + \mu(y,z) \text{ for all } x,y,z \in P.
$$

If $T : P \rightarrow P$ is a continuous map satisfying:

$$
\mu(Tx,Ty) < \alpha_1 \mu(x,y) + \alpha_2 \mu(Tx,x) + \alpha_3 \mu(Tx,y) + \alpha_4 \mu(x,Ty) + \alpha_5 \mu(Ty,y)
$$

(4.4.2)
for all \( x, y \in P \) with \( x \neq y \) and \( Tx \neq Ty \),

where \( \alpha_3 \geq 0, \alpha_2 + \alpha_3 < 1 \) and \( \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 \leq 1 \), then

\( T \) has a fixed point in \( P \). This is unique, whenever

\( \alpha_1 + \alpha_3 + \alpha_4 \leq 1 \).

This theorem is further extended by Jain and Dixit [1].

The main object of the present section is to obtain

a fixed point theorem in Pseudocompact Tichonov space which

includes all the above results.

**Theorem 1.** Let \( P \) and \( \mu \) be the same as defined in Theorem A

and \( \mu \) satisfies condition (4.1.1). If \( T : P \to P \) is a

continuous map satisfying:

\[
(4.1.3) \quad \mu(Tx, Ty) < \alpha_1 \mu(x, y) + \alpha_2 \mu(Tx, x) + \alpha_3 \mu(Tx, y) \\
+ \alpha_4 \mu(x, Ty) + \alpha_5 \mu(Ty, y) + \alpha_6 \frac{\mu(Tx, x) \mu(Ty, y)}{\mu(x, y)} + \alpha_7 \frac{\mu(Tx, Ty) \mu(Ty, y)}{\mu(x, y)} + \alpha_8 \frac{\mu(Tx, x) \mu(Ty, y)}{\mu(Tx, Ty)}
\]

for all \( x, y \in P \) with \( x \neq y \) and \( Tx \neq Ty \), where \( \alpha_3 \geq 0 \),
\[ a_8 \geq 0, \ a_2 + a_3 + a_6 + a_8 < 1 \quad \text{and} \quad a_1 + a_2 + 2a_3 + a_5 + a_6 + 2a_8 \leq 1, \] 
then \( T \) has a fixed point in \( P \). This is unique, whenever \( a_1 + a_3 + a_4 + a_7 \leq 1 \).

**Proof:** We define \( \phi : P \to \mathbb{R} \) by \( \phi(p) = \mu(Tp, p) \) for all \( p \in P \), where \( \mathbb{R} \) is the set of real numbers. Clearly \( \phi \) is continuous being the composite of two continuous functions \( T \) and \( \mu \). Since \( P \) is pseudocompact Tikhonov space every real valued continuous function over \( P \) is bounded and attains its bounds. Thus there exists a point \( v \in P \) such that 
\[ \phi(v) = \inf \{ \phi(p) / p \in P \} \], where 'inf' denotes the infimum or the greatest lower bound in \( \mathbb{R} \) (note \( \phi(p) \in \mathbb{R} \)). We now affirm that \( v \) is a fixed point for \( T \). If not, let us suppose that \( Tv \neq v \). Then applying (4.1.3), we have 
\[ \phi(Tv) = \mu(T^2v, Tv) \]
\[ < a_1 \mu(Tv, v) + a_2 \mu(T^2v, Tv) + a_3 \mu(T^2v, v) \]
\[ + a_4 \mu(Tv, Tv) + a_5 \mu(Tv, v) + a_6 \frac{\mu(T^2v, Tv)}{\mu(Tv, v)} \]
\[ + a_7 \frac{\mu(Tv, Iv)}{\mu(Tv, v)} + a_8 \frac{\mu(T^2v, v)}{\mu(T^2v, Tv)} \]
or,
\[ (1 - a_2 - a_3 - a_6 - a_8) \phi(Tv) < (a_1 + a_3 + a_5 + a_8) \phi(v) \]
\[ (\therefore \ a_3 \geq 0, \ a_8 \geq 0) \]
or, $\phi(Tv) < \phi(v)$ \(\therefore \alpha_2 + \alpha_3 + \alpha_4 + \alpha_8 < 1\),

\[\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 + 2\alpha_8 \leq 1\]

leading to a contradiction and hence $v \in P$ is a fixed point for $T$.

To show the uniqueness of $v$, if possible, let $w \in P$ be another fixed point for $T$, i.e. $Tw = w$ and $w \neq v$ in $P$.

Then the application of (4.1.3) gives

\[\mu(v, w) = \mu(Tv, Tw)\]

\[\leq (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7) \mu(v, w)\]

\[\leq \mu(v, w) (\therefore \alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leq 1)\]

again leading to a contradiction which shows that $v \in P$ is unique. This completes the proof of the theorem.

Every metric space is a Hausdorff space. Hence as a particular case of our theorem, we have the following result on a compact metric space.
COROLLARY. Let \((M,d)\) be a compact metric space and 
\(T : M \rightarrow M\) be a continuous map satisfying:
\[
d(Tx,Ty) < a_1 d(x,y) + a_2 d(Tx,x) + a_3 d(Tx,y) \\
+ a_4 d(x,Ty) + a_5 d(Ty,y) + a_6 \frac{d(Tx,x) d(Ty,y)}{d(x,y)} \\
+ a_7 \frac{d(x,Ty) d(Tx,y)}{d(x,y)} + a_3 \frac{d(Tx,x) d(Tx,y)}{d(Tx,Ty)}
\]
for all \(x,y \in M\) with \(x \neq y\) and \(Tx \neq Ty\), where \(a_3 \geq 0\),
\(a_8 \geq 0\), \(a_2+a_3+a_6+a_8 < 1\) and \(a_1 + a_2 + 2a_3 + a_5 + a_6 + 2a_8 \leq 1\),
then \(T\) has a fixed point in \(M\). This is unique, when ever
\(a_1 + a_3 + a_4 + a_7 \leq 1\).

REMARKS:

(1) If \(a_6 = a_7 = a_8 = 0\), Theorem 1 reduces to
Theorem A.

(2) If \(a_8 = 0\), Theorem 1 reduces to a result of Jain
and Dixit ([1], Theorem 1).

(3) In the corollary, if we take
(1) \(a_1 = 1\) and \(a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0\),
we get Fisher's Theorem 1 [3].
(ii) \( a_1 = a_3 = a_4 = a_6 = a_7 = a_8 = 0 \) and \( a_2 = a_5 = \frac{1}{2} \), we get Fisher's Theorem 2 [3].

(iii) \( a_1 = a_2 = a_5 = a_6 = a_7 = a_8 = 0 \) and \( a_3 = a_4 = \frac{1}{2} \), we get Fisher's Theorem 3 [3].

(iv) \( a_2 = a_5 \), \( a_3 = a_4 \) and \( a_6 = a_7 = a_8 = 0 \), we obtain Fisher's Theorem 4 [3].

4.2. In 1981, Ćirić [8] established some interesting fixed point theorems in compact metric space. We now extend some of his results by proving:

THEOREM 2. Let \( P \) and \( \mu \) be the same as defined in Theorem A and \( \mu \) satisfies condition (4.1.1). If \( T: P \rightarrow P \) is a continuous map satisfying:

\[
(4.2.1) \quad \mu(T^n x, T^n y) < \max \left\{ \mu(x, y), \mu(x, Tx), \mu(y, Ty), \frac{1}{2}[\mu(x, Ty) + \mu(y, Tx)] \right\}
\]

for all distinct \( x, y \in P \), where \( n = n(x, y) \) is a positive integer, then \( T \) has a unique fixed point in \( P \).
**Proof:** Let the mapping $\phi$ and the point $v$ be defined as in the proof of Theorem 1. We now affirm that $v$ is a fixed point of $T$. If not, let us suppose that $\phi(v) = (Tv, v) > 0$. Then by (4.2.1) for $n = n(Tv, v)$ we have

$$\phi(T^n v) = \mu(T^nTv, T^n v) = \mu(T^nTv, T^n v)$$

$$< \max \{ \mu(Tv, v)\mu(Tv, T^2v), \mu(v, Tv) \}$$

$$= \frac{1}{2}\left[ \mu(Tv, Tv) + \mu(v, T^2v) \right] \}$$

$$\leq \max \{ \phi(v), \phi(Tv), \frac{1}{2} \left[ \phi(v) + \phi(Tv) \right] \}$$

Since $\max \{ \phi(v), \phi(Tv), \frac{1}{2} \left[ \phi(v) + \phi(Tv) \right] \} = \phi(v)$, we have $\phi(T^n v) < \phi(v)$, which is a contradiction.

Therefore $v$ is a fixed point of $T$.

The uniqueness of the fixed point follows in the following way:

Let $w$ be a fixed point of $T$ different from $v$.

Then

$$d(T^n v, T^n w) < \max \{ d(v, w), d(v, Tv), d(w, Tw)$$

$$= \frac{1}{2} \left[ d(v, Tw) + d(w, Tv) \right] \}$$

for some $n = n(v, w)$.
But this is impossible since $Tv = v = T^n v$ and $Tw = w = T^n w$. This completes the proof.

**Theorem 3.** Let $P$ and $\mu$ be the same as defined in Theorem A and $\mu$ satisfies condition (4.1.1). If $T : P \to P$ is a continuous map satisfying:

$$
(4.2.2) \quad \mu(T^n x, T^n y) < \max_{0 \leq i,j,k,l,m \leq n} \left\{ \mu(T^i x, T^j y), \mu(T^j x, T^{j+1} x), \mu(T^k y, T^{k+1} y), \frac{1}{2}[\mu(T^0 x, T^l + 1 y) + \mu(T^m y, T^{m+1} y)] \right\}
$$

for some positive integer $n = n(x, y)$ and $x, y \in P$ for which the right hand side of the inequality is positive, then $T$ has a unique fixed point in $P$.

**Proof:** Let the mapping $\phi$ and the point $v$ be defined as in the proof of Theorem 1 and let $n = n(Tv, v)$. Then by (4.2.2) we have

$$
\phi(T^n v) = \mu(T^n Tv, T^n v) < \max_{0 \leq i,j,k,l,m \leq n} \left\{ \mu(T^i v, T^j v), \mu(T^{j+1} v, T^{j+1} v), \mu(T^k v, T^k v), \frac{1}{2}[\mu(T^m v, T^{m+1} v) + 0] \right\}.
$$
Using the triangle inequality and (4.2.2) we obtain
\[ \phi(T^n v) < \phi(v), \] which is a contradiction. Therefore right hand side is zero for \( x = T v \) and \( y = v \).

The uniqueness of the fixed point follows easily.
This completes the proof.

**Remark:** We may assume that the righthand side of
inequality (4.2.2) is positive for each \( x, y \in P \). For
if it is not positive, then \( x = y = Tx \), which means that \( T \)
has a fixed point.

As particular cases of our theorems, we have the
following results of Ćirić [8] on compact metric space.

**Corollary 2.** Let \( T \) be a continuous mapping on the compact
metric space \( M \) into itself satisfying
\[
d(T^n x, T^n y) < \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \right. \\
\left. \frac{1}{2} \left[ d(x, Ty) + d(y, Tx) \right] \right\}
\]
for all distinct \( x, y \in M \), where \( n = n(x, y) \) is a positive
integer. Then \( T \) has a unique fixed point in \( M \).
COROLLARY 3. Let $T$ be a continuous mapping on the compact metric space $M$ into itself satisfying

$$d(T^nx, T^ny) < \max_{0 \leq i, j, k, l, m \leq n} \{d(T^ix, T^jy), d(T^ix, T^{i+1}x),$$

$$d(T^ky, T^{k+1}y), \frac{1}{2}[d(T^lx, T^{l+1}y) + d(T^my, T^{m+1}x)]\}$$

for some positive integer $n = n(x, y)$ and $x, y$ in $M$ for which the right hand side of inequality is positive. Then $T$ has a unique fixed point in $M$.

4.3. Finally in this section we present some more results on fixed points in Pseudocompact Tichonov space which include some results of Fisher [1] established in compact space.

THEOREM 4. Let $P$ and $\mu$ be the same as defined in Theorem A and $\mu$ satisfies condition (4.1.1). If $T: P \rightarrow P$ is a continuous map satisfying:

$$(4.3.1) \quad \{\mu(Tx, Ty)\}^2 < \max \{\mu(x, Tx)\mu(y, Ty), c\mu(x, Ty)\mu(y, Tx)\}$$

for all distinct $x, y \in P$, where $c > 0$. Then $T$ has a fixed point in $P$ which is unique whenever $c \leq 1$. 

**Proof:** Let the mapping $\phi$ and the point $v$ be defined as in the proof of Theorem 1 and let us suppose that $Tv \neq v$. Then by (4.3.1), we have

$$\left\{ \phi(Tv) \right\}^2 = \left\{ \mu(T^2v, Tv) \right\}^2$$

$$= \max \{ \mu(Tv, T^2v) \mu(v, Tv), c_\rho(Tv, Tv) \mu(v, T^2v) \}$$

$$= \mu(Tv, T^2v) \mu(v, Tv)$$

which, since $\mu(T^2v, Tv) \geq 0$, implies

$$\phi(Tv) < \phi(v)$$

leading to a contradiction and therefore $Tv = v$, i.e. $v \in P$ is a fixed point for $T$.

To prove the uniqueness of $v$, if possible, let $w \in P$ be another fixed point for $T$, i.e. $Tw = w$ and $w \neq v$. Then using (4.3.1) we have

$$\left\{ \mu(v, w) \right\}^2 = \left\{ \mu(Tv, Tw) \right\}^2$$

$$< \max \{ \mu(v, T_tv) \mu(w, Tw), c_\rho(v, Tw) \mu(w, T^2v) \}$$

$$= c\left\{ \mu(v, w) \right\}^2 \leq \left\{ \mu(v, w) \right\}^2 (\therefore c \leq 1)$$
again leading to a contradiction which prove that \( v \in P \) is unique. This completes the proof of the theorem. An easy consequence of this theorem yields the following result.

**Corollary 4.** (Fisher [4], Theorem 6). Let \( T \) be a continuous self-map of a compact metric space \((M, d)\) satisfying

\[
\left\{ d(Tx, Ty) \right\}^2 < \max \left\{ d(x, Tx) d(y, Ty), c d(x, Ty) d(y, Tx) \right\}
\]

for all distinct \( x, y \in M \), where \( c \geq 0 \). Then \( T \) has a fixed point in \( M \). If \( c \leq 1 \), then the fixed point is unique.

**Theorem 5.** Let \( P \) and \( \mu \) be the same as defined in Theorem A. Let \( T : P \to P \) be a continuous map satisfying

\[
(4.3.2) \quad \left\{ \mu(Tx, Ty) \right\}^2 < \frac{1}{2} \left\{ \mu(x, Tx) \mu(x, Ty) + \mu(y, Ty) \mu(y, Tx) \right\}
\]

for all distinct \( x, y \in P \). Then \( T \) has a unique fixed point in \( P \).

**Proof**: Let \( \phi \) and \( v \) be as in the proof of Theorem 1. If \( v \in P \) is not a fixed point of \( T \), then applying (4.3.2) we have

\[
\left\{ \phi(Tv) \right\}^2 = \left\{ \mu \left( TV, TV \right) \right\}^2 < \frac{1}{2} \left\{ \mu(TV, TV) \mu(TV, TV) \right\}
\]
\[ + \mu(v, Tv) \mu(v, T^2v) \]
\[ = \frac{1}{2} \mu(v, T^2v) \mu(v, Tv) \]
\[ \leq \frac{1}{2} \mu(v, Tv) \left[ \mu(v, Tv) + \mu(Tv, T^2v) \right] \]

which, since \( \mu(v, Tv) > 0 \), implies

\[ \phi(Tv) < \phi(v) \]

leading to a contradiction and hence \( Tv = v \), i.e. \( v \in P \)
is a fixed point for \( T \).

Uniqueness of \( v \) follows easily as in Theorem 4.

**Corollary 2.** (Fisher [6], Theorem 4). Let \( T \) be a
continuous self-map of a compact metric space \((M, d)\)
satisfying
\[ \left\{ d(Tx, Ty) \right\}^2 < \frac{1}{2} \left\{ d(x, Tx) d(x, Ty) + d(y, Ty) d(y, Tx) \right\} \]

for all distinct \( x, y \in M \). Then \( T \) has a unique fixed point
in \( M \).

**Theorem 6.** Let \( P \) and \( \mu \) be the same as defined in Theorem A
Let \( T : P \rightarrow P \) be a continuous map satisfying
\[ (4.3.3) \ \mu(Tx, Ty) < \left[ \left\{ \mu(x, Tx) \right\}^2 + \left\{ \mu(y, Ty) \right\}^2 \right]/\left[ \mu(x, Tx) + \mu(y, Ty) \right] \]
for all \( x, y \in P \) for which \( \mu(x, Tx) + \mu(y, Ty) = 0 \). Then

\( T \) has a fixed point in \( P \). Further, if

\[
\mu(x, Tx) + \mu(y, Ty) = 0 \quad \text{implies} \quad \mu(Tx, Ty) = 0,
\]

then the fixed point is unique.

**Proof:** Let \( \phi \) and \( v \) be as in the proof of Theorem 1. If \( v \in P \) is not a fixed point of \( T \), then applying (4.3.3) we have

\[
\phi(Tv) = \mu(T^2v, Tv)
\]

\[
< \frac{\left( \mu(Tv, T^2v) \right)^2 + \left( \mu(v, Tv) \right)^2}{\mu(Tv, T^2v) + \mu(v, Tv)}
\]

\[
= \frac{\left( \phi(Tv) \right)^2 + \left( \phi(v) \right)^2}{\phi(Tv) + \phi(v)}
\]

i.e.

\[
\left( \phi(Tv) \right)^2 + \phi(Tv) \phi(v) < \left( \phi(Tv) \right)^2 + \left( \phi(v) \right)^2
\]

which, since \( \phi(v) > 0 \), implies

\[
\phi(Tv) < \phi(v)
\]

leading to a contradiction and hence \( Tv = v \).

Uniqueness of \( v \) follows from the stated condition.
**COROLLARY 6.** (Fisher [6], corollary, p. 34) Let \( T \) be a continuous self-map of a compact metric space \((M, d)\) satisfying:

\[
d(Tx, Ty) < \frac{\left\{d(x, Tx)\right\}^2 + \left\{d(y, Ty)\right\}^2}{d(x, Tx) + d(y, Ty)}
\]

for all \( x, y \in M \) for which \( d(x, Tx) + d(y, Ty) \neq 0 \). Then \( T \) has a fixed point in \( M \). Further, if

\[
d(x, Tx) + d(y, Ty) = 0 \quad \text{implies} \quad d(Tx, Ty) = 0,
\]

then the fixed point is unique.