CHAPTER III

FIXED POINT THEOREMS FOR MAPPINGS

INVOLVING RATIONAL EXPRESSIONS
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3.1. During the past few years a number of authors have defined contractive type mappings on a complete metric space which are generalizations of the well-known Banach contraction \([1]\), and which have the property that each such mapping has a unique fixed point. Recently, Fisher\([6]\), Kasahara \([1]\), Khan \([1]\), Pachpatte \([2]\), Das and Gupta \([1]\) etc. have established some fixed point theorems for contractive conditions involving rational expressions. Our object here is to obtain some fixed point theorems for a new class of contractive type mappings satisfying rational inequalities in complete metric space.

THEOREM 1. Let \(S\) and \(T\) be mappings of a complete metric space \((X,d)\) into itself satisfying the inequality

\[(3.1.1) \quad d(Sx,Ty) \leq q \max \left\{ \frac{d(x,Sx) \cdot d(x,Ty)}{d(x,Sx) + d(x,Ty)}, \frac{d(y,Sx) \cdot d(y,Ty)}{d(x,Sx) + d(x,Ty)} \right\},\]

where \(q\) is a positive real number.
where \( d(x, Sx) + d(x, Ty) \neq 0 \) for all \( x, y \in X \) and \( 0 < q < \frac{1}{2} \).

Then \( S \) and \( T \) have a common fixed point \( z \). Further if \( d(x, Sx) + d(x, Ty) = 0 \) implies \( d(Sx, Ty) = 0 \), then \( z \) is unique.

**Proof**: Let \( x_0 \in X \) be arbitrary and define a sequence

\[
x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0,1,2, \ldots.
\]

Suppose first of all that \( d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) = 0 \) for some \( n \). Then it follows immediately that \( x_{2n+1} \) is a common fixed point of \( S \) and \( T \). Similarly \( d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n}) = 0 \) for some \( n \) implies that \( x_{2n} \) is a common fixed point of \( S \) and \( T \).

Now suppose that \( d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) \neq 0 \) and \( d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n}) \neq 0 \) for \( n = 0,1,2, \ldots \). Then

\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})
\]

\[
< q \max \left\{ \frac{d(x_{2n}, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1})}, \frac{d(x_{2n+1}, Sx_{2n})}{d(x_{2n+1}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})} \right\}
\]
\[ q \max \left\{ \frac{d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+2})}, 0 \right\} \]

\[ \leq q \frac{d(x_{2n}, x_{2n+1})[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]}{d(x_{2n+1}, x_{2n+2})} \]

i.e.

\[ [d(x_{2n+1}, x_{2n+2})]^2 \leq q d(x_{2n+1}, x_{2n+2}) d(x_{2n}, x_{2n+1}) + \frac{1}{4} q^2 [d(x_{2n}, x_{2n+1})]^2 \]

\[ \leq \frac{1}{4} (q^2 + 4q) [d(x_{2n}, x_{2n+1})]^2 \]

which gives

\[ d(x_{2n+1}, x_{2n+2}) \leq \mu d(x_{2n}, x_{2n+1}) \]

where

\[ \mu = \frac{1}{2} (q + \sqrt{q^2 + 4q}) < 1 \quad (\therefore 0 < q < \frac{1}{2}) \]

Thus

\[ d(x_{2n+1}, x_{2n+2}) \leq \mu d(x_{2n}, x_{2n+1}) \leq \cdots \leq \mu^{2n+1} d(x_0, x_1) \]

for \( n = 0, 1, 2, \ldots \)

Similarly,

\[ d(x_{2n}, x_{2n+1}) = d(Sx_{2n}, T x_{2n-1}) \]

\[ \leq q \max \left\{ \frac{d(x_{2n}, Sx_{2n}) d(x_{2n}, T x_{2n-1})}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, T x_{2n-1})}, 0 \right\} \]
\[
\frac{d(x_{2n-1}, Sx_{2n})}{d(x_{2n-1}, Tx_{2n-1})} \cdot \frac{d(x_{2n-1}, Sx_{2n})}{d(x_{2n}, Sx_{2n}) + d(x_{2n}, Tx_{2n-1})} \\
= q \max 0, \frac{d(x_{2n-1}, x_{2n+1}) d(x_{2n-1}, x_{2n})}{d(x_{2n}, x_{2n+1})} \\
\leq q \frac{[d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n+1})] d(x_{2n-1}, x_{2n})}{d(x_{2n}, x_{2n+1})}
\]

i.e. \([d(x_{2n}, x_{2n+1})]^2 = q d(x_{2n-1}, x_{2n}) d(x_{2n}, x_{2n+1}) + \frac{1}{4} q^2 [d(x_{2n-1}, x_{2n})]^2\]

\[
\leq \frac{1}{4} (q^2 + 4q) [d(x_{2n-1}, x_{2n})]^2
\]

which gives \(d(x_{2n}, x_{2n+1}) \leq \mu d(x_{2n-1}, x_{2n})\),

where \(\mu = \frac{1}{2} (q + \sqrt{q^2 + 4q}) < 1\).

So \(d(x_{2n}, x_{2n+1}) \leq \mu d(x_{2n-1}, x_{2n}) \leq \cdots \leq \mu^{2n} d(x_0, x_1)\)

for \(n = 1, 2, 3, \ldots\).

Now by routine calculation for any \(k > n\), one has

\(d(x_n, x_{n+k}) \leq \frac{\mu^n}{1-\mu} d(x_0, x_1)\).
Since \( \mu < 1 \), the right hand side of the above inequality tends to zero as \( n \to \infty \). Therefore the sequence \( \{x_n\} \) is Cauchy with the limit \( z \) in \( X \). Let \( z \neq Tz \). Then

\[
d(z, Tz) = d(z, x_{2n+1}) + d(Sx_{2n}, Tz)
\]

\[
\leq d(z, x_{2n+1}) + q \max\left\{ \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, Tz)}{d(x_{2n}, x_{2n+1})+d(x_{2n}, Tz)} \right\}
\]

Letting \( n \to \infty \), we get

\[
d(z, Tz) \leq 0
\]
giving a contradiction and it follows that \( Tz = z \).

Similarly we can prove that \( Sz = z \). Thus \( z \) is a common fixed point of \( S \) and \( T \).

Now to show the uniqueness of \( z \), suppose that \( d(x, Sx) + d(x, Ty) = 0 \) implies \( d(Sx, Ty) = 0 \) and that \( T \) has a second fixed point \( w \). Then \( d(z, Sz) + d(z, Tw) = 0 \) and so \( d(z, w) = d(Sz, Tw) = 0 \). It follows that \( z = w \) and so the common fixed point \( z \) of \( S \) and \( T \) in this case is unique. This completes the proof.
In [1], Ćirić has established some interesting fixed point theorems for a family of generalized contractive type mappings. We next establish the following variants of Ćirić's results in the frame-work of the present setup of mappings.

**Theorem 2**: Let \( T_0 \) and \( \{T_n : n \in I^+\} \) \((I^+ \text{ the positive integers})\) be mappings of a non-empty complete metric space \((X, d)\) into itself satisfying the inequality

\[
(3.1.2) \quad d(T_0x, T_ny) \leq q \max \left\{ \frac{d(x, T_0x) d(x, T_ny)}{d(x, T_0x) + d(x, T_ny)} \right\} \]

\[
\frac{d(y, T_0x) d(y, T_ny)}{d(x, T_0x) + d(x, T_ny)} \}
\]

for all \( x, y \) in \( X \) and for each \( n = 1, 2, \ldots \) for which \( d(x, T_0x) + d(x, T_ny) \neq 0 \), \( 0 < q < \frac{1}{2} \), then there exists a fixed point \( z \in X \) such that \( T_nz = z \) for each \( n = 0, 1, 2, \ldots \) and for arbitrary \( x_0 \in X \) the sequence

\[
x_0, x_1 = T_0x_0, x_2 = T_1x_1, x_3 = T_0x_2, \ldots, x_{2n-1} = T_0x_{2n-2}, x_{2n} = T_nx_{2n-1}, \ldots
\]
converging to \( z \). Further, if \( d(x,T_0 x) + d(x,T_0 y) = 0 \) implies that \( d(T_0 x, T_0 y) = 0 \), then \( z \) is the unique fixed point of \( T_n \) for \( n = 0,1,2 \ldots \).

**Theorem 3:** Let \( F = \{ T_\lambda : \lambda \in \Lambda \} \) be a family of functions which maps a nonempty complete metric space \((X,d)\) into itself. If there exists some \( T_{\lambda_0} \in F \) such that for each \( T_{\lambda} \in F (\lambda \neq \lambda_0) \) there are positive integers \( i_{\lambda} \) and \( j_{\lambda} \) such that

\[
(3.1.3) \quad d(T_{\lambda_0} x, T_{\lambda} y) \leq q \max \left\{ \frac{d(x,T_{\lambda} x)d(x,T_{\lambda} y)}{d(x,T_{\lambda_0} x) + d(x,T_{\lambda_0} y)} \right\}
\]

\[
\frac{d(y,T_{\lambda_0} x) d(y,T_{\lambda} y)}{d(x,T_{\lambda_0} x) + d(x,T_{\lambda} y)}
\]

for all \( x,y \) in \( X \) for which \( d(x,T_{\lambda_0} x) + d(x,T_{\lambda} y) \neq 0 \), \( 0 < q < \frac{1}{2} \), then every \( T_{\lambda} \in F \) has a fixed point \( z \) in \( X \).

Further, if \( d(x,T_{\lambda_0} x) + d(x,T_{\lambda} x) = 0 \) implies that \( d(T_{\lambda_0} x, T_{\lambda} y) = 0 \), then \( z \) is the unique common fixed point for \( F \).
3.2. Let $X$ be a non-empty set, $d_1$ and $d_2$ two metrics on $X$ and $f : X \rightarrow X$. For such mappings, Maia [1] proved a fixed point theorem which was generalized in many directions by S.P. Iseki [2], Singh [1] and others. We next establish the following variant of Maia's result in the present set up of mappings.

**Theorem 4**: Let $X$ be a metric space with two metrics $d_1$ and $d_2$. If $X$ satisfies the following conditions:

1. $d_1(x, y) \leq d_2(x, y)$ for every $x, y$ in $X$,
2. $X$ is complete with respect to $d_1$,
3. two mappings $S, T : X \rightarrow X$ are continuous with respect to the metric $d_1$ and

$$d_2(Sx, Ty) \leq q \max \left\{ \frac{d_2(x, Sx)d_2(x, Ty)}{d_2(x, Sx) + d_2(x, Ty)}, \frac{d_2(y, Sx)d_2(y, Ty)}{d_2(y, Sx) + d_2(y, Ty)} \right\}$$

for all $x, y$ in $X$ for which $d_2(x, Sx) + d_2(x, Ty) \neq 0$ where $0 < q < \frac{1}{2}$, then $S$ and $T$ have a common fixed point $z$.

Further if $d_2(x, Sx) + d_2(x, Ty) = 0$ implies that $d_2(Sx, Ty) = 0$ then $z$ is unique.
**PROOF:** Let \( x_0 \in X \) be arbitrary and define a sequence

\[
x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, \ldots .
\]

Then by following exactly the same steps as in the proof of Theorem 1 with suitable modifications we have for any \( k > n, \)

\[
d_2 (x_n, x_{n+k}) \leq \frac{\mu^n}{1-\mu} d_2 (x_0, x_1), \text{ where } \\
\mu = \frac{1}{2} \left( q + \sqrt{q^2 + 4q} \right) < 1 \quad (\therefore 0 < q < \frac{1}{2})
\]

Now using \( d_1 \leq d_2, \) where

\[
d_1 (x_n, x_{n+k}) \leq \frac{\mu^n}{1-\mu} d_2 (x_0, x_1).
\]

This shows that the sequence \( \{x_n\} \) is a Cauchy sequence with respect to \( d_1. \) Since \( X \) is complete with respect to \( d_1, \) the sequence \( \{x_n\} \) has a limit \( z \) in \( X. \) Hence by the continuity of \( S \) with respect to the metric \( d_1, \)

\[
z = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} Sx_{2n} = S \lim_{n \to \infty} x_{2n} = Sz.
\]

Similarly we have \( z = Tz. \) Therefore \( z \) is a common fixed point of \( S \) and \( T. \) The uniqueness of \( z \) follows as in the proof of Theorem 1.
3.3. Motivated by the work of Nadler [1], we study the stability of fixed points in metric space for a sequence of functions in the present section. We prove the following:

**THEOREM 5.** Let $T_n$ be a selfmap defined on a metric space $(X,d)$ with $u_n$ as a fixed point for $n = 1, 2, ...,$. If $T_n$ satisfies the conditions:

\[(3.3.1)\] for some $\alpha_i \in [0,1)$ \((i = 1, 2, 3, 4)\) with 
\[\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1,
\]
\[d(T_n(x), T_n(y)) \leq \alpha_1 \frac{d(x, T_n(y))d(y, T_n(x))}{d(x, y)} \alpha_2 d(x, y) + \alpha_3 d(x, T_n(y)) + \alpha_4 d(y, T_n(x))
\]

for each $n$ and $x, y \in X$ with $x \neq y$ and

\[(3.1.2)\] $T_n$ converges pointwise to $T$.

Then $u_n \to u$ if and only if $u$ is a fixed point of $T$.

**PROOF:** If $u = u_n$ for some $n$, the assertion follows easily.

Hence $u \neq u_n$ for any $n$. Let $u_n \to u$, then

\[d(u, T(u)) \leq d(u, u_n) + d(u_n, T_n(u)) + d(T_n(u), T(u))
\]

\[= d(u, u_n) + d(T_n(u_n), T_n(u)) + d(T_n(u), T(u))
\]
\[ \leq d(u, u_n) + \alpha_1 \frac{d(u_n, T_n(u)) \cdot d(u, T_n(u))}{d(u_n, u)} + \alpha_2 d(u_n, u) + \alpha_3 d(u_n, T(u)) + \alpha_4 d(u, T_n(u_n)) + d(T_n(u), T(u)) \]

\[ = (1 + \alpha_2 + \alpha_4) d(u, u_n) + (\alpha_1 + \alpha_3) d(u_n, T_n(u)) + d(T_n(u), T(u)) \]

\[ \leq (1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \cdot d(u, u_n) + (\alpha_1 + \alpha_3) d(u, T(u)) + (1 + \alpha_1 + \alpha_3) \cdot d(T_n(u), T(u)) \]

which implies that

\[ d(u, T(u)) \leq \frac{(1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}{(1 - \alpha_1 - \alpha_3)} \cdot d(u, u_n) + \frac{(1 + \alpha_1 + \alpha_3)}{(1 - \alpha_1 - \alpha_3)} \cdot d(T_n(u), T(u)) \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \text{ ( } \because \alpha_1 + \alpha_3 < 1 \text{ )} \]

and hence \( T(u) = u \).
Conversely, we assume that $T(u) = u$, then

$$d(u_n, u) = d(T_n(u_n), T(u)) \leq d(T_n(u_n), T(u)) + d(T(u), T(u))$$

$$\leq \alpha_1 \frac{d(u_n, T(u)) d(u, T(u))}{d(u_n, u)} + \alpha_2 d(u_n, u) + \alpha_3 d(u_n, T(u)) + \alpha_4 d(u, T(u)) + d(T(u), T(u))$$

$$= (\alpha_1 + \alpha_3) d(u_n, T(u)) + (\alpha_2 + \alpha_4) d(u_n, u) + d(T(u), T(u))$$

$$\leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(u_n, u) + (1 + \alpha_1 + \alpha_3) d(T(u), T(u))$$

which implies that

$$d(u_n, u) \leq \frac{(1 + \alpha_1 + \alpha_3)}{(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)} d(T(u), T(u))$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty \quad (\because \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1)$$

Therefore $u_n \rightarrow u$. This completes the proof of the theorem.

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