CHAPTER I

INTRODUCTION

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1.1 Due to its wide applications the fixed point theory has occupied very important position in analysis. Fixed point theory plays a significant role not only in solving various differential equations, integral equations, partial differential equations, random differential equations but also in solving the boundary value problems, non-linear problems and eigen value problems etc. It is also being used successfully in the study of theory of games, mechanics, topological dynamics and in the characterization of the completeness of metric spaces.

A point which remains invariant under a transformation is called a fixed point i.e. a self-mapping $T$ defined on the set $X$ is said to have the fixed point $u$ in $X$ if $Tu = u$. Bronwer, perhaps the first mathematician who in 1912 placed the foundation stone of fixed point theory by introducing the theorem which states that if $C$ be a unit ball in $\mathbb{R}^n$ Euclidean $n$-dimensional space and $T : C \to C$ be a continuous mapping then equation $Tx = x$ has a solution in $C$ i.e. $T$ has a fixed point in $C$. The
particular case of this theorem on the real line can be stated as 'let $T: [0,1] \to [0,1]$ be a continuous mapping, then $T$ has a fixed point'.

In 1927, Schauder [1] generalized the Brouwer's theorem for infinite dimensional spaces by proving that a continuous mapping of a compact convex set of a Banach space to itself has at least one fixed point. Schauder in 1930 [2] improved his result by relaxing the condition of compactness and proved that a closed, bounded, convex set of a Banach space has a fixed point under a self compact mapping. This theorem is of great importance in the numerical treatment of equations in analysis.

S. Banach 'The father of the fixed point theory' in 1922 [1] introduced and studied the basic and very fruitful concept of contraction mappings. Let $(X, d)$ be a metric space. A mapping $T: X \to X$ is called contraction mapping if

$$(1.1.1) \quad d(Tx, Ty) \leq K d(x, y), \forall x, y \in X, \quad 0 \leq K < 1.$$ 

It is clear that in a contraction mapping the distance between the images of any two points is always less than the distance between the two points and hence
this mapping is used to contract the distance between the two points. Every contraction mapping is continuous but the converse may not be true, since every translation mapping \( T: \mathbb{R} \to \mathbb{R} \) given by \( Tx = x + h, \ h > 0 \) is continuous but not contraction.

Banach, established a theorem popularly known as Banach's contraction theorem which states that every self-contraction mapping of a complete metric space has a unique fixed point. A large number of mathematicians got inspiration from Banach contraction principle since then and they enriched the field of fixed point theory by introducing a variety of contraction mappings. Most of them adopted the same technique as used by Banach: place contractive condition on maps so that suitable iterations give Cauchy sequences; introduce hypothesis of completeness in the range containing those sequences. The contractive conditions on maps have two roles: first, they assure that certain iterations are Cauchy; and second, they assure that uniqueness of the fixed point.

In 1962, Rakotch [1] proved the fixed point theorem by replacing the Lipschitz constant \( K \) in \((1.1.1)\) by a monotonic decreasing function \( \alpha:(0,\infty) \to [0,1]\). Reich [1], generalized the Rakotch's result by taking monotonically
decreasing functions $a, b, c$ from $(0, \infty)$ into $[0, 1)$ satisfying $a(t) + b(t) + c(t) < 1$, such that for each $x, y \in X$, $x \neq y$

\begin{equation}
(1.1.2) \quad d(Tx, Ty) \leq a(d(x, y)) d(x, Tx) + b(d(x, y)) d(y, Ty) + c(d(x, y)) d(x, y)
\end{equation}

Ciric [1] proved the fixed point theorem by taking non-negative functions $q, r, s, t$ satisfying:

\[ \sup_{x, y \in X} \left\{ q(x, y) + r(x, y) + s(x, y) + 2t(x, y) \right\} \leq \lambda < 1 \]

such that, for each $x, y \in X$.

\begin{equation}
(1.1.3) \quad d(Tx, Ty) \leq q(x, y) d(x, y) + r(x, y) d(x, Tx) + s(x, y) d(y, Ty) + t(x, y) \left\{ d(x, Ty) + d(y, Tx) \right\}
\end{equation}

Chu and Diaz [1] proved that a self-mapping $T$ defined on a complete metric space has a unique fixed point if for some positive integer $n$, $T^n$ is a contraction.

Sehgal [1] in 1969 proved that for a continuous self-mapping $T$ of a complete metric space $(X, d)$ has a unique fixed point if for each $x \in X$ there exists a positive integer $n = n(x)$ such that for all $y \in X$. 
(1.1.4) \( d(T^n x, T^n y) \leq K d(x, y), \ 0 \leq K < 1 \)

Holmes [1] improved the Sehgal's result by taking the positive integer \( n \) which depends upon both \( x \) and \( y \).

Banach's contraction principle has many fruitful applications but it has one disadvantage - the definition requires that \( T \) be continuous throughout the space \( X \).

In 1968, Kannan [2] gave an important result that does not require the continuity of mapping \( T \).

(1.1.5) \( d(Tx, Ty) \leq \alpha \left[ d(x, Tx) + d(y, Ty) \right] \)

for all \( x, y \) of \( X \), \( 0 < \alpha < \frac{1}{2} \). Then \( T \) has a unique fixed point.

Chatterjea [3] proved that a selfmapping \( T \) defined on the complete metric space \( (X, d) \) has unique fixed point if condition (1.1.5) is replaced by

(1.1.6) \( d(Tx, Ty) \leq \alpha \left[ d(x, Ty) + d(y, Tx) \right] \)

In 1973, Hardy and Rogers [4] have obtained the more general theorem by combining the conditions of Banach, Kannan and Chatterjea which states that a selfmapping \( T \) of complete metric space \( (X, d) \) has a unique fixed point if
there exist constants \( a_1 \geq 0 \), \( \sum_{i=1}^{5} a_i < 1 \) satisfying for all \( x, y \) of \( X \)

\[
(1.1.7) \quad d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)
\]

Due to symmetry, condition (1.1.7) can be replaced by

\[
(1.1.8) \quad d(Tx, Ty) \leq a d(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)]
\]

for all \( x, y \) of \( X \), \( a, b, c \geq 0 \) such that \( a + 2b + 2c < 1 \).

Edelstein [1] introduced and studied the concept of contractive mappings which are more general than the Banach's contraction mappings. A mapping \( T \) defined on the metric space \( (X, d) \) to itself is called contractive if

\[
(1.1.9) \quad d(Tx, Ty) < d(x, y), \forall x \neq y \in X.
\]

If a contractive mapping has a fixed point then clearly it will be the unique. Also, every contractive mapping is continuous. Unlike contraction mappings, a contractive mapping defined on a complete metric space may not have a fixed point as shown by an example given
by Khan and Khan [1]. Edelstein proved that if $T$ is a contractive self-mapping on a compact metric space then $T$ has a unique fixed point. From the above it follows that completeness of the space is not enough for the existence of a fixed point. Fixed point theorems for contractive mappings, therefore, require further restrictions on the mappings, on the space or on its range.

Sehgal [2] extended the Edelstein result by replacing the condition (1.1.9) by

$$(1.1.10) \ d(Tx, Ty) < \max \ \{d(x, Tx), d(y, Ty), d(x, y)\}$$

In a compact space each sequence has a convergent subsequence and therefore a contractive mapping on a compact metric space to itself has a unique fixed point. Most of the fixed point theorems for continuous self-mappings of compact metric spaces are proved by using the fundamental principle that continuous mapping $F(x) = d(x, Tx)$ attains its minimum over a compact space being the composite of continuous mappings $d$ and $T$ with the given contractive condition. Jungck [1] proved a very useful theorem on compact metric spaces 'Let $T$ be a continuous self-map of a compact metric space $(X,d)$.
Then $T$ has a fixed point if for any $x, y \in X$, $Tx \neq Ty$ there is a mapping $S$ which commutes with $T$ such that $d(Sx, Sy) < d(Tx, Ty)$.

Motivated by the fact that a fixed point of a selfmapping $T$ of the space $X$ is a common fixed point of $T$ and the identity mapping $I_X$ of $X$, Jungck [1] replaced the identity mapping with a continuous mapping in order to generalize the Banach contraction principle in the following manner. Let $T$ be a continuous selfmapping of a complete metric space $(X, d)$. If there exists a mapping $S : X \to X$ and a constant $0 \leq K < 1$ such that

\begin{align*}
(1.1.11) \quad & (i) \quad T \text{ and } S \text{ commute.} \\
& (ii) \quad S(X) \subset T(X) \quad \text{and} \\
& (iii) \quad d(Sx, Sy) \leq Kd(Tx, Ty)
\end{align*}

for all $x, y$ of $X$, then $S$ and $T$ have a unique common fixed point.

Meir and Keeler [1] gave a new type of condition to generalize Banach's contraction principle. They proved that if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

\begin{align*}
(1.1.12) \quad & \varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \varepsilon
\end{align*}
then $T$ has a unique fixed point.

Park and Bae [1] combined the results of Jungck and Meir and Keeler and proved 'Let $T$ be a continuous selfmap of a complete metric space $(X,d)$ and $S : X \to X$ be a mapping commuting with $T$ such that given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $d(Sx, Sy) < \varepsilon$, $S(x) \subseteq T(x)$ and $\varepsilon \leq d(Tx, Ty) < \varepsilon + \delta$ for all $x, y$ of $X$. Then $S$ and $T$ have a unique common fixed point in $X$. Obviously $T$ is contractive. Maiti and Pal [1], Yen and Chung [1], Ciric [5] and others have obtained the various fixed point theorems by introducing more general conditions than Meir and Keeler.

Caristi [1] introduced an important fixed point theorem where the mapping is not of contraction type and even not continuous which states that $T$ is a selfmapping of complete metric space $(X,d)$, $\phi : X \to [0, \infty)$ is lower semi continuous. If for each $x$ of $X$

\[(1.1.13) \quad d(x, Tx) \leq \phi(x) - \phi(Tx)\]

Then $T$ has a fixed point.

If the contractive definition is strong enough then there is exactly one fixed point and it can be obtained by taking successive iterates of the mapping, starting with
any point in the space. Some of the weaker or more
general contractive conditions produce more than one
fixed point. For example identity mapping is non-expansive
and every point is a fixed point.

1.2. **FIXED POINT THEOREMS FOR
NON-EXPANSIVE MAPPINGS**:

A non-expansive mapping is a natural generali-
ization of Banach contractive mapping. A mapping $T : X \to X$
is called non-expansive if $d(Tx, Ty) \leq d(x, y)$ for each
$x, y$ of $X$. It is clear that class of non-expansive mappings
contains the classes of contraction and contractive
mappings and therefore this class possesses many interesting
results.

Some important properties of the contractive
mappings are not satisfied by non-expansive mappings. The
existence of a fixed point does not necessarily imply its
uniqueness. The identity mapping which is non-expansive
on a metric space has every point as a fixed point. If $T^n$
has a fixed point for some positive integer $n$, then it
does not imply that $T$ has a fixed point and the sequence
of iterates $\{T^n\}$ may not converge even in a compact space.
Belluce and Kirk [1] proved that a non-expansive selfmap of a metric space with diminishing orbital diameters has a fixed point whenever an orbit has a cluster point. By using monotone operator theory Browder [1] showed that every non-expansive selfmapping of a closed converse bounded subset of a Hilbert space has a fixed point. Gohde [1] obtained the same result on uniformly convex Banach spaces.

1.3. **Fixed Point Theorems on Hausdorff Spaces:**

A space is called Hausdorff if every pair of distinct points of it have disjoint neighbourhoods. Every metric space is a Hausdorff space and every convergent sequence in a Hausdorff space has unique limit point. The set of fixed points on a Hausdorff space is closed.

Singh and Zorzitto [1] proved the following theorem:

Let $f$ be a continuous selfmapping of Hausdorff space $X$. Let $F : X \times X \rightarrow [0, \infty)$ be a continuous mapping such that

$$(1.3.1) \quad f(fx, fy) \leq F(x, y) \quad \forall x, y \in X.$$ 

If there exists $x_0 \in X$ such that $\{f^n x_0\}$ has a convergent subsequence then $f$ has a fixed point.
Kiventidis [1] extended the above result by replacing the conditions (1.3.1) by

(1.3.2) (i) $F(x, y) \neq o, \forall x \neq y$

(ii) $F(fx, fy) \leq F(x, y) - \omega(F(x, y)) \forall x, y \in X$

where $\omega : R^+ \rightarrow R^+$ is a continuous function with

$o < \omega(r) < r$, for all $r \in R^+ - \{0\}$. If for some $x_o \in X$

the sequence $\{f^n x_o\}$ has a convergent subsequence, then $f$

has a unique fixed point.

Popa [1] has generalized the results of Ray [1],
[2] and Jaggi [1] on Hausdorff spaces and proved the
following important fixed point theorem.

Let $T$ be a continuous selfmapping of a Hausdorff
space $X$ and $f$ be a non-negative real valued continuous
mapping of $X \times X$ satisfying the conditions

(1.3.3) (i) $f(x, y) \neq o, \forall x \neq y$

(ii) $f(Tx, Ty) \leq \frac{a f(x, Tx) f(y, Ty)}{f(x, y)} + b f(x, y),$

$\forall x \neq y, a, b \in R^+$ and $a + b < 1$

(iii) $f^2(x, y) \geq f(x, x) f(y, y), \forall x \neq y.$
If for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a convergent subsequence then $T$ has a unique fixed point.

1.4. **FIXED POINT THEOREMS FOR DENSIFYING MAPPINGS:**

The concept of 'Measure of non-compactness of bounded set $A$' of metric space $X$ denoted by $\alpha(A)$ introduced by C. Kuratowski [2], is the infimum of all $\varepsilon > 0$ such that $A$ admits a finite covering by sets with diameter less than $\varepsilon$. The concept of densifying mapping was introduced and studied by Furi and Vignoli [1]. A self mapping $T$ defined on a metric space $X$ is called densifying if for every bounded subset $A$ of $X$ with $\alpha(A) > 0$ we have $\alpha(T(A)) < \alpha(A)$. They proved that a contractive continuous densifying selfmapping of a bounded complete metric space has unique fixed point.

K. Iseki [1] has proved that if $T$ be a continuous densifying selfmapping of a bounded complete metric space then for every $x \neq y \in X$, $y \neq Ty$.

(1.4.1) $d(Tx, Ty) < a d(x, y) + b \left\{ d(x, Tx) + d(y, Ty) \right\}$,

where $a, b > 0$ such that $a + 2b = 1$ then $T$ has a fixed point.

Ray and Fisher [1] have obtained common fixed point theorem for two densifying mappings. Let $S$ and $T$ be continuous,
densifying self mappings of a complete metric space \((X,d)\) satisfying:

\[
(1.4.2) \quad d(Sx,Ty) < \max \left\{ d(x,y), d(x,Sx), d(y,Ty), \right. \\
\left. \frac{1}{2} \left[ d(x,Ty) + d(y,Sx) \right] \right\}
\]

for all \(x \neq y \in X\). If for some \(x_0\) in \(X\), the sequence \(\{x_n\}\) defined by \(Sx_{2n} = x_{2n+1}, Tx_{2n+1} = x_{2n+2}\) is bounded then either \(S\) or \(T\) has a fixed point \(z\). Further if \(z\) is a common fixed point of \(S\) and \(T\), then it is the unique fixed point of \(S\) and \(T\).

The renowned mathematician B.E. Rhoades [4],[5], [6], [7] has done a very difficult task to compare and summarize most of the contractive type mappings. This work of Rhoades is very valuable and essential to study the fixed point theory.

1.5. APPLICATIONS OF FIXED POINT THEORY:

Fixed point theorem have very fruitful applications in solving the various problems of differential and integral equations, partial differential equations, random differential equations. They are also widely used in solving the boundary value, eigen value and non-linear problems. To make the detail study of the various applications of fixed point theory one may study the work of Czerwik Stefan [1], Smart [1] Kolmogorov and Fomin [1] and Swaminathan [1], Hu [1] used the fixed point theorem to see the metric completeness. He proved that a metric space is complete if and only if any Banach contraction on closed subsets has a fixed point. Subrahmanyam [1], Sullivan [1], Weston [1] and Taskovic [1] and some others who characterized completeness of metric spaces by using different fixed point theorems.

In Chapter II, we have obtained some fixed point theorems on Quasi-metric spaces and two metric spaces. In section 2.2 we have generalized the fixed point theorem of Chikkala and Baisnab [1] on Quasi-metric spaces. In section 2.4 we have generalized the related fixed point theorem on two complete metric spaces of Fisher [8] and obtained its analogous results on compact metric spaces.

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Chapter III is concerned with the common fixed point theorems for nearly densifying mappings. In section 3.2 and 3.3 we have proved some common fixed point theorems for Jungck type [1] for nearly densifying mappings and common fixed point theorems for a pair of nearly densifying mappings on complete metric spaces. Most of the result of this Chapter have been published in the Journal Ultra Scientist of Phyl. Sciences Vol.5, No.2, 220-222, (1993) and in the journal Mathematics Education, Siwan Vol. No. XXIX No. 2 June 1995. (accepted: please see Appendix)

In Chapter IV, we have obtained some fixed point theorems for asymptotically regular mappings. In section 4.2 we have modified the fixed point theorem of Jaggi and Das [1] by using the concept of asymptotic regularity of mapping at a point. In section 4.3 we have proved some results for the common fixed point of three self mapping by simply using the contractive condition of Jaggi [1] for single mapping. In section 4.4, we have generalized the result of Sastry, Naidu, Rao and Rao [1] for four mappings under asymptotic regularity of mappings. Most of the results of this Chapter have been published in Ultra Scientist of Physical Sciences Vol.6(2), 1994 and in Acta Ciencia India, Vol. No. 22, 1996 (accepted: Please see Appendix)
Chapter V is completely devoted to the study of common fixed point theorems for two and three mappings on metric and 2-Bimetric spaces. In section 5.2, we have generalized the results of fixed points of Fisher [6], Fisher and Khan [1] and Murty and Pathak [1] for two and three mappings of metric spaces. We have introduced the concept of 2-Bimetric spaces in section 5.3 and we have generalized the fixed point theorem of Pathak and Dubey [1] of single mapping to a pair of mappings to get common fixed point of Ciric type along with other results in section 5.4. Most of the results of this chapter have been published in Vol.6(1) 1994 of Journal Ultra Scientist of Physical Sciences.

In Chapter VI, we have proved some fixed point theorem on 2-normed spaces through $G$-iterates and on 2-Banach spaces. In section 6.2, we have extended the theorem of Pathak [5] and generalized it for a pair
of mappings on 2-normed space using the $G$-iterative scheme. Section 6.3 is concerned with to extend the fixed point theorem of two mappings in normed space of Pathak and Dubey [1] to three mappings defined on 2-normed space using $G$-iterative scheme. Result of Sharma and Bajaj [1] on Banach space for two mappings have generalized by us for two and three mappings defined on 2-Banach space in section 6.4.

Chapter VII is completely devoted to the study of common fixed point theorems for set valued mappings. In section 7.2, we have generalized the result of Dhage [2] set valued mapping for a pair of set valued mappings in two different forms. The result of Murthy and Pathak [1] for a pair of point valued mappings for periodic fixed point has been extended by us for pair of valued mappings in section 7.3.
In section 7.4 the fixed point theorem of Chandel [1] for a pair of multi valued mappings have extended by us for three set valued mappings under Caristi-Kirk type conditions.