Some common fixed point theorems for nearly densifying maps

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Abstract

In this paper some common fixed point theorems for nearly densifying mappings of Jungck type have been obtained.

The measure of non-compactness of bounded set \( A \) in a metric space \( (X, d) \), denoted by \( a(A) \), is the infimum of all \( \epsilon > 0 \) such that \( A \) admits a finite covering by sets with diameter less than \( \epsilon \). Sastry & Naidu have introduced the concept of nearly densifying selfmaps on a metric space as follows:

A selfmapping \( T \) on a metric space \( (X, d) \) is said to be nearly densifying if \( a(T(A)) \leq a(A) \), whenever \( a(A) > 0 \), \( A \) is bounded and \( T \)-invariant.

**Theorem 1:** Let \( S \) and \( T \) be commuting, continuous and nearly densifying selfmappings of a complete metric space \( (X, d) \). Let \( F \) be symmetric and lower semi-continuous mapping of \( X \times X \times X \) into \( \mathbb{R}_+ \) satisfying the following conditions:

\[
F(x,x) = 0 \quad \forall x \in X \quad \text{...(1.1)}
\]

\[
F(x,y) \leq F(x,z) + F(z,y) \forall x,y,z \in X \quad \text{...(1.2)}
\]

\[
[F(Tx, Ty)]^n \leq c_1(F(Sx, Tx) F(Sy, Ty) + F(Sx, Ty) F(Sy, Tx)) + c_2[F(Sx, Tx) F(Sy, Ty) + F(Sx, Ty) F(Sy, Tx)] + c_3F(Sy, Sx) F(Tx, Ty) \quad \text{...(1.3)}
\]

where constants \( c_i \in \mathbb{R}_+ \) such that \( c_1 + 2c_2 + c_3 < 1 \) and for some \( x_0 \in X \) the set \( A = \{S^jTx_0 : i,j \geq 0 \} \) is bounded. Then \( S \) and \( T \) have a unique common fixed point.

**Proof:** Clearly \( A = \{x_0\} \) USAUTA. Since \( S \) and \( T \) are commuting and continuous then \( S \subseteq A \), \( T \subseteq A \). Since \( S \) and \( T \) are nearly densifying and \( (X, d) \) is complete therefore \( A \) is compact.

Let \( H = \cap \lim_{n \to \infty} (ST)^n \bar{A} \). Clearly, \( (ST)^n \bar{A} \)

is a decreasing sequence of non-empty compact subsets of \( \bar{A} \) and hence \( H \) is a non-empty compact set. Clearly \( SH \subseteq H \), \( TH \subseteq H \).

Let \( x \) be any element of \( H \) then \( x \in (ST)^n \bar{A} \) for all \( n \). Therefore there exists \( \{x_n\} \subseteq (ST)^n \bar{A} \) such that \( STx_n = x \) for all \( n \). Since \( S \) and \( T \) are continuous and \( (ST)^n \bar{A} \) is compact and closed for all \( n \) and therefore there exists a point \( p \in (ST)^n \bar{A} \) for each \( n \) and thus \( STp = x \). Therefore \( x \in SH \), \( x \in TH \). Hence \( SH = H = TH \).

Since \( F \) is lower semi-continuous, the real valued mapping \( \phi \) on \( H \) given by \( \phi(x) = F(Sx, Tx) \) is lower semi-continuous and hence attains its infimum in \( H \). Let \( \phi(u) = \inf \{F(Sx, Tx) : x \in H \} \). Since \( SH = H \) there exists \( V \in H \) such that \( u = Sv \). Suppose
there is no point \( x \) in \( X \) such that \( Sx = Tx \). By applying (1.3) and using (1.1), (1.2), we get \( [F(STv, TTv)]^2 = [F(TTv, TTv)]^2 \)
\[
\leq \{ c_i F(S^+v, TSv) + c_j F(S^+v, T^+v) + c_k F(S^+v, STv) \} F(STv, T^+v) \text{ implies } F(STv, T^+v) \leq h F(S^+v, TSv) < F(S^+v, TSv),
\]
where \( h = \frac{c_i + c_j + c_k}{1 - c_i} < 1 \).

Thus \( \phi(Tv) < \phi(v) = \phi(u) \), a contradiction since \( \phi(u) \) is the infimum. Therefore there exists \( z \in H \) such that \( Sz = Tz \) and hence \( S^2z = STz = TSz \). Let \( S^2z \neq Sz \) then by applying (1.3) and using (1.1) we get \( [F(S^2z, Sz)]^2 = [F(TSz, Tz)]^2 \)
\[
\leq (c_i + c_j)[F(S^2z, Sz)]^2 < [F(S^2z, Sz)]^2
\]
a contradiction, which proves \( Sz = S^2z = TSz \) and thus \( Sz \) is a common fixed point of \( S \) and \( T \).

To prove uniqueness, let \( \omega \) be another common fixed point of \( S \) and \( T \). Then by applying (1.3) we get easily \( [F(Sz, \omega)]^2 = [F(TSz, Tw)]^2 \leq (c_i + c_j)[F(Sz, \omega)]^2 < [F(Sz, \omega)]^2 \).

This contradiction proves that \( S \) and \( T \) have unique common fixed point.

**Theorem 2:** Let \( S \) and \( T \) be commuting, continuous and nearly densifying selfmappings of a complete metric space \((X, d)\) satisfying

\[
F(Tx, Ty) < \max \left\{ \frac{F(Sx, Tx)}{F(Sx, Sy)}, \frac{F(Sy, Tx)}{F(Sy, Ty)}, \frac{F(Sx, Sy)}{F(Tx, Ty)} \right\}
\]
(2.1)

for \( Sx \neq Sy \), \( Tx \neq Ty \) where \( F \) be symmetric and lower semi-continuous mapping of \( X \times X \) into \( R^+ \) such that \( F(x, x) = 0 \) \( \forall x \in X \) and for some \( x_0 \) in \( X \) the set \( A = \{ S^iTx_0 : i, j \geq 0 \} \) is bounded. Then \( S \) and \( T \) have a unique common fixed point.

**Proof:** The set \( H \) is defined in the same manner and we can prove \( SH = H = TH \) as in the proof of Theorem 1. The real valued mapping \( \phi \) on \( H \) given by \( \phi(x) = F(Sx, Tx) \) is lower semi-continuous and hence attains its infimum at \( u \in H \). There exists \( v \in H \) such that \( u = Sv \). Suppose there is no point \( x \) in \( X \) such that \( Sx = Tx \). Then by applying (2.1) we have \( F(STv, TTv) = F(TSz, TTv) \)
\[
< \max \left\{ \frac{F(S^+v, TSv) F(STv, T^+v)}{F(S^+v, STv)}, \frac{F(STv, TSv) F(S^+v, T^+v)}{F(TTv, TTv)} \right\}
\]
implies \( F(STv, T^+v) < F(S^+v, STv) \), thus \( \phi(TTv) < \phi(Sv) = \phi(u) \), a contradiction. Hence, there exists \( z \in H \) such that \( Sz = Tz \) and therefore \( S^2z = STz = TSz \). Let \( S^2z \neq Sz \) then by applying (2.1) we get
\[
F(S^2z, Sz) = F(TSz, Tz)
\]

\[
< \max \left\{ \frac{F(S^2z, TSz) F(Sz, Sz)}{F(TSz, Tz)} \right\}
\]
implies \( F(S'z, Sz) < F(S'z, Sz) \). This contradiction proves that \( Sz = S'z = Tsz \). Thus \( Sz \) is a common fixed point of \( S \) and \( T \). Uniqueness of the common fixed point can be easily seen as in the proof of Theorem 1.

**Theorem 3:** If in Theorem 2, contractive condition (2.1) is replaced by

\[
\frac{\max \{ F(Sx, Tx), F(Sy, Ty) \}}{\max \{ F(Sx, Sy), F(Sx, Ty), F(Sy, Tx) \}} < \frac{F(Tx, Ty)}{F(Tx, Ty)}
\]

then \( S \) and \( T \) have a unique common fixed point.

**Proof:** Follows easily.

**References**


Some fixed point theorems for two and three mappings

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(Acceptance Date 1 Nov. 93)

Abstract

The main object of this paper is to generalize the results of fixed points of Fisher¹, Fisher and Khan² and Murthy and Pathak³ for two and three mappings on metric spaces.

Introduction

Recently in 1990, Murthy and Pathak⁴ proved the following fixed point theorem for two selfmappings defined on a metric space.

**Theorem 1:** Let \((X,d)\) be a metric space. \(T_1, T_2\) be selfmaps of \(X\) such that
\[
d(x, T_1x) \leq \frac{a_1 d(x, T_2x) d(x, T_2y)}{d(x, T_2y)}
\]

for all \(x, y \in X, x \neq y\), where \(r, s > 0\) are integers and \(a, b\) are non-negative real numbers such that \(a + b < 1\). If for some \(x \in X\) the sequence \(\{x_n\}\) consisting of points \(x_{n+1} = T_1x_{2n}, x_{2n+2} = T_2x_{2n+1}\) has a convergent subsequence \(\{x_{n_k}\}\) converging to a point \(p\), then \(T_1\) and \(T_2\) have a unique common fixed point \(p\) in \(X\).

Now, we prove the following theorems.

**Theorem 2:** Let \(S\) and \(T\) be selfmappings of a metric space \((X,d)\) satisfying
\[
d(Sx, Ty) \leq \frac{a_1 d(x, Sx) d(x, Ty)}{d(x, Ty)} + a_2 d(y, Ty) d(y, Sx)
\]

for all \(x, y \in X; p, q > 0\) are integers; \(a_i\) \((1 \leq i \leq 4)\) are non-negative real numbers such that \(a_1 + a_2 < 1\). If for some \(x_0 \in X\), the sequence \(\{x_n\}\) given by \(Sx_{2n} = x_{2n+1}\), \(T_{2n+1} = x_{2n+2}\) has a convergent subsequence \(\{x_{n_k}\}\) in \(X\) and \(d(x, Ty) + d(y, Sx) > 0\), otherwise \(d(Sx, Ty) = 0\) then \(S\) and \(T\) have unique common fixed point.

**Proof:** By applying condition (i), we get
\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, T_{2n+1})
\]

for all \(x \in X\), \(Sx_{2n} \neq x_{2n+1}\) and \(T_{2n+1} \neq x_{2n+2}\). Hence, \(x_{2n+1}, x_{2n+2}\) are convergent subsequences of \(\{x_n\}\).
implies, \( d(x_{2n+1}, x_{2n+2}) \leq a_1 d(x_{2n}, x_{2n+1}) + a_2 d(x_{2n}, x_{2n+2}) \leq (a_1 + a_2) d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+2}) \)

i.e. \( d(x_{2n+1}, x_{2n+2}) \leq k_1 d(x_{2n}, x_{2n+1}) \) where

\[
k_1 = a_1 + a_3 \leq a_3 < 1
\]

Again by applying condition (i), we can easily get

\[
d(x_{2n+1}, x_{2n}) = d(S^p x_{2n}, T^q x_{2n-1}) \leq k_2 d(x_{2n-1}, x_{2n})
\]

where \( k_2 = \frac{1}{1 - a_3} < 1 \)

For \( k = \max \{k_1, k_2\} \) we have \( d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n-1}, x_{2n}) \).

On proceeding in the same manner we have

\[
d(x_{2n+1}, x_{2n+r}) \leq k d(x_{2n}, x_{2n+1}) \leq k^2 d(x_{2n-1}, x_{2n}) \leq \ldots \leq k^{n+1} d(x_0, x_1),
\]

showing that \( \{x_n\} \) is a monotonically decreasing sequence of positive real numbers which converges to some \( z \) of \( X \) and therefore its subsequence \( \{x_{nk}\} \) will also converge to the same limit \( z \).

Now, we show that \( z \) is a common fixed point of \( S^p \) and \( T^q \). Let \( S^p z \neq z \). Then we have

\[
\begin{align*}
d(S^p z, x_{2nk+1}) &= d(S^p z, T^q x_{2nk}) \\
&= a_1 d(z, S^p z) d(z, x_{2nk+1}) + a_2 d(x_{2nk}, x_{2nk+1}) d(z, x_{2nk}) + d(x_{2nk}, S^p z) + d(z, x_{2nk+1}) + d(x_{2nk}, S^p z)
\end{align*}
\]

Taking limit as \( n \to \infty \), we get

\[
\begin{align*}
a_1 d(z, S^p z) d(z, z) + a_2 d(z, z) d(z, S^p z) + a_3 d(z, z) + a_4 d(z, S^p z) d(z, z)
\end{align*}
\]

\[
\begin{align*}
d(z, z) + d(z, S^p z) + a_3 d(z, z) + a_4 d(z, S^p z)
\end{align*}
\]

\[
\begin{align*}
&\leq a_1 d(z, S^p z) d(z, z) + a_2 d(z, z) d(z, S^p z) + a_3 d(z, z) + a_4 d(z, S^p z),
\end{align*}
\]

a contradiction which shows that \( S^p z = z \).

Similarly we can prove \( T^q z = z \). Let \( z' \) be another common fixed point of \( S^p \) and \( T^q \). Then by applying (i), we get easily

\[
d(z, z') = d(S^p z, T^q z')
\]

\[
\begin{align*}
a_1 d(z, z') d(z, z') + a_2 d(z', z') d(z', z)
\end{align*}
\]

\[
\begin{align*}
&\leq a_1 d(z, z') + d(z', z) + a_2 d(z', z') + a_3 d(z', z') + a_4 d(z', z)
\end{align*}
\]

\[
\begin{align*}
&\leq a_1 d(z, z') + d(z', z)
\end{align*}
\]

\[
\begin{align*}
&\leq \frac{a_3 + a_4}{2} d(z, z') < d(z, z').
\end{align*}
\]

This contradiction proves that \( z = z' \).

If \( d(x, T^q y) + d(y, S^p x) = 0 \) implies \( d(S^p x, T^q y) = 0 \), then \( d(S^p z, T^q z') = 0 \) i.e. \( d(z, z') = 0 \) showing that \( z = z' \). Hence in each case, we see that \( z \) is a unique common fixed point of \( S^p \) and \( T^q \).

Now, \( S^p(Sz) = S^{p+1}z = S(S^p z) = Sz \), implies that \( Sz \) is a fixed point of \( S^p \). But \( S^p \) has unique fixed point \( z \), therefore \( Sz = z \). Similarly \( T^q z = z \).

Let \( w \) be another common fixed point of \( S \) and \( T \). Then by using (i) we get

\[
d(z, w) = d(Sz, Tw) = d(S^p z, T^q w) \leq \frac{a_3 + a_4}{2}
\]

\[
\begin{align*}
d(z, w) < d(z, w)
\end{align*}
\]

a contradiction which proves that \( S \) and \( T \) have unique common fixed point.
Remark 1: For \( p = q = 1 \), we have the following particular cases:

(a) For \( a_1 = a_2 = k \) and \( a_3 = a_4 = 0 \) we get the result of Fisher and Khan\(^1\).

(b) For \( a_1 = a_2 = 0 \) we get the result of Fisher\(^2\).

Theorem 3: Let \( T_1, T_2 \) and \( P \) be three self mappings of a metric space \((X, d)\) satisfying

\[
d((T_1P)^s x, (T_2P)^s y) \leq a \frac{d(x, (T_1P)^s x)}{d(x, (T_2P)^s y)} + \beta d(x, y) \quad \text{...(ii)}
\]

for all \( x \neq y \in X; s > 0 \) are integers; \( a, \beta \) are non-negative real numbers such that \( a + \beta < 1 \). If for any \( x_0 \in X \), the sequence \( \{x_n\} \) consisting of points \( x_{2n+1} = (T_1P)^s x_{2n}, \ x_{2n+2} = (T_2P)^s x_{2n+1} \) has a convergent subsequence \( \{x_{nk}\} \) in \( X \), then \( (T_1P)^s \), \( (T_2P)^s \) have a common fixed point. Further if \( T_1 \) or \( T_2 \) commutes with \( P \) then \( T_1, T_2 \) and \( P \) have a unique common fixed point.

Proof: By applying conditions (ii), we have

\[
d(x_{2n+1}, x_{2n+2}) \leq \frac{a}{d(x_{2n}, x_{2n+1})} + \frac{\beta}{1 - \alpha} d(x_{2n}, x_{2n+1})
\]

\[
\leq k d(x_{2n}, x_{2n+1}) \quad \text{where} \quad k = \frac{a + \beta}{1 - \alpha} < 1.
\]

Again,

\[
d(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n}) = \frac{d((T_1P)^s x_{2n}, (T_2P)^s x_{2n+1})}{d(x_{2n}, x_{2n+1})} \leq k d(x_{2n-1}, x_{2n})
\]

On continuing the process, we get

\[
d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n}, x_{2n+1}) \leq k^2 d(x_{2n-1}, x_{2n}) \leq \ldots \leq k^{n+1} d(x_0, x_1)
\]

Thus \( \{x_n\} \) is a monotonic decreasing sequence of positive real numbers converges to some \( z \in X \). Since \( \{x_n\} \) has a convergent subsequence \( \{x_{nk}\} \) therefore it will also converge to same \( z \).

Now, we show that \( z \) is a common fixed point of \( (T_1P)^s \) and \( (T_2P)^s \).

Let \( (T_1P)^s z \neq z \). By applying (ii), we get

\[
d((T_1P)^s z, x_{2nk}) = d((T_1P)^s z, (T_2P)^s x_{2nk-1}) \leq \frac{a}{d(z, (T_1P)^s z)} + \beta d(z, x_{2nk-1}) + d(z, x_{2nk})
\]

on taking limit \( n \to \infty \), we get

\[
d((T_1P)^s z, z) \leq \frac{d(z, z) + d(z, (T_1P)^s z) + d(z, z)}{1 - \alpha} \leq \frac{d(z, (T_1P)^s z)}{1 - \alpha}
\]

implies \( d((T_1P)^s z, z) \leq 0 \) which proves \( (T_1P)^s z = z \). Similarly we can prove \( (T_2P)^s z = z \).

To prove uniqueness, let \( z' \) be another common fixed point of \((T_1P)^s \) and \((T_2P)^s \). Then

\[
d(z, z') = d((T_1P)^s z, (T_2P)^s z') \leq \frac{a}{d(z, z')} + \beta d(z, z') \leq \beta d(z, z') = d(z, z') \leq \beta d(z, z') < d(z, z')
\]

a contradiction, which shows that \( z = z' \).

Now, \( (T_1P)^s z = (T_1P)^s z = (T_1P)^s z = (T_1P)^s z \) shows that \( (T_1P)^s z \) is a fixed point of \((T_1P)^s \). But \((T_1P)^s z \) has unique fixed point \( z \). Therefore \( T_1Pz = z \). Similarly \( T_2Pz = z \). Let \( T_1 \) commutes with \( p \).
Now, we show that $z$ is a common fixed point of $T_1$, $T_2$ and $p$. Let $p z \neq z$. Then

\[ d(pz, z) = d(p T_1pz, T_2pz) = d(T_1pz, T_2pz) = d((T_1p)pz, (T_2p)pz) \]

\[ \leq a \frac{d(pz, (T_1p)pz) + d(z, (T_2p)pz) + d(pz, z) + \beta d(pz, z)}{d(pz, (T_1p)pz) + d(z, (T_1p)pz) + d(pz, z) + \beta d(pz, z)} \]

\[ \leq \beta d(pz, z) < d(pz, z), \text{ a contradiction which shows } pz = z. \]

Thus $T_1pz = T_2pz = z$ implies $T_1z = T_2z = z = pz$.

Similarly, if $T_2$ commutes with $p$ we can prove $T_1z = T_2z = px = z$.

To prove uniqueness, let $w$ be another common fixed point of $T_1$, $T_2$ and $p$.

Then,

\[ d(z, w) = d(T_1pz, T_2pw) = d(T_1pz, (T_2p)pw) \]

\[ \leq a \frac{d(z, T_1pz) + d(w, T_2pw) + d(z, w) + \beta d(z, w)}{d(z, T_1pz) + d(w, T_2pw) + d(z, w) + \beta d(z, w)} \]

\[ \leq \beta d(z, w) < d(z, w), \text{ a contradiction which proves that } T_1, T_2 \text{ and } p \text{ have a unique common fixed point.} \]

**Remark 2**: (a) For $p = I$, we get Theorem 1.

(b) For $p = I_1$; $r = s = 1$; $T_1 = T_2 = f$ we get the result of Jaggi and Das.

**References**


Some common fixed point theorems for asymptotically regular mappings

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(Acceptance Date 16 June 94)

Abstract

The main purpose of this paper is to generalize the result of Jaggi and to get some more results of common fixed points for asymptotically regular mappings.

For three selfmappings $P, Q$ and $T$ of metric space $(X, d)$, the sequence $\{x_n\}$ is given by $T_{x_{2n+1}} = P_{x_{2n}}, T_{x_{2n+2}} = Q_{x_{2n+1}}$, for $n = 0, 1, 2, \ldots$. The set $O(P, Q, T, x_0) = \{T_{x_n}, n = 1, 2, 3, \ldots\}$ is called the orbit of $(P, Q, T)$ at $x_0$. $T$ is said to be orbitally continuous at $x_0$ if it is continuous on $O(P, Q, T, x_0)$. $X$ is called orbitally complete at $x_0$ if every Cauchy sequence in $O(P, Q, T, x_0)$ converges in $X$. $(P, Q)$ is said to be asymptotically regular with respect to $T$ at $x_0$ if

$$\lim_{n \to \infty} d(T_{x_n}, T_{x_{n+1}}) = 0$$

Our following theorem generalizes the result of Jaggi.

Theorem 1: Let $P, Q$ and $T$ be three selfmappings of metric space $(X, d)$ satisfying:

$$d(Px, Qy) \leq \frac{\alpha d(Tx, Px) + d(Ty, Qy)}{d(Tx, Ty)} + \beta$$

for all $x \neq y$ of $X$, $\alpha \leq \alpha < \beta < 1$, $(P, Q)$ is asymptotically regular with respect to $T$ at some $x_0$ of $X$. $X$ is orbitally complete at $x_0$ and $T$ is orbitally continuous at $x_0$. If $T$ commutes with $P$ or $Q$, then $P, Q$ and $T$ have a unique common fixed point.

Proof: Since $(P, Q)$ is asymptotically regular with respect to $T$ at $x_0$ then $\lim_{n \to \infty} d(Tx_n, T_{x_{n+1}}) = 0$.

$$\text{(1.2)}$$

where sequence $\{T_{x_n}\}$ is given by $T_{x_{2n+1}} = P_{x_{2n}}, T_{x_{2n+2}} = Q_{x_{2n+1}}$. If $\{T_{x_0}\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ and strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k < n_k$ and $d(T_{x_{m_k}}, T_{x_{n_k}}) \geq \epsilon$, $d(T_{x_{m_k}}, T_{x_{n_k-1}}) < \epsilon$ for all positive integers $k$. Clearly, we have $d(T_{x_{m_k}}, T_{x_{n_k}}) \to \epsilon$, as $k \to \infty$, by (1.2)

Let $B_1 = \{k : m_k$ is even and $n_k$ is odd$, B_2 = \{k : m_k$ is even and $n_k$ is even$, B_3 = \{k : m_k$ is odd and $n_k$ is even$\}$, and $B_4 = \{k : m_k$ is odd and $n_k$ is odd$. Definitely, at least one of $B_1$ is infinite.
implies \( d(T_{x_n}, T_{x_{n+1}}) \leq k \ d(T_{x_n-1}, T_{x_n}) \).

Thus we have \( d(T_{x_n}, T_{x_{n+1}}) \leq k \ d(T_{x_n-1}, T_{x_n}) \), for all \( n \).

Thus \( \{d(T_{x_n}, T_{x_{n+1}})\} \) is a decreasing sequence of non-negative real numbers and therefore will converge to a real number say \( \omega \).

Letting \( n \to \infty \) in (1.7), we get \( \omega \leq k \omega \) which gives \( \omega = 0 \). Thus \( \lim_{n \to \infty} d(T_{x_n}, T_{x_{n-1}}) = 0 \)

and therefore by Theorem 1, it follows that \( P, Q \) and \( T \) have a unique common fixed point.

**Theorem 2:** If in Theorem 1, contractive condition (1.1) is replaced by \( d(Px, Qy) \leq q \max \{d(Tx, Ty), d(Tx, Px), d(Ty, Qy)\} \)

\( d(T_{x_n}, T_{x_{n+1}}) \).

Thus \( \{d(T_{x_n}, T_{x_{n+1}})\} \) is a decreasing sequence of non-negative real numbers and therefore will converge to a real number say \( \omega \).

Letting \( n \to \infty \) in (1.7), we get \( \omega \leq k \omega \) which gives \( \omega = 0 \). Thus \( \lim_{n \to \infty} d(T_{x_n}, T_{x_{n-1}}) = 0 \)

and therefore by Theorem 1, it follows that \( P, Q \) and \( T \) have a unique common fixed point.

**Theorem 2:** If in Theorem 1, contractive condition (1.1) is replaced by \( d(Px, Qy) \leq q \max \{d(Tx, Ty), d(Tx, Px), d(Ty, Qy)\} \)

\( d(T_{x_n}, T_{x_{n+1}}) \).

Thus \( \{d(T_{x_n}, T_{x_{n+1}})\} \) is a decreasing sequence of non-negative real numbers and therefore will converge to a real number say \( \omega \).

Letting \( n \to \infty \) in (1.7), we get \( \omega \leq k \omega \) which gives \( \omega = 0 \). Thus \( \lim_{n \to \infty} d(T_{x_n}, T_{x_{n-1}}) = 0 \)

and therefore by Theorem 1, it follows that \( P, Q \) and \( T \) have a unique common fixed point.

**F.L.T.:** As in Theorem 1, we can prove \( d(x, x_{n+1}, T_{x_{n+1}}) \to 0 \) as \( n \to \infty \) for the set \( S \).

Let \( B_1 \) is infinite then for all \( k \in B_1 \), we have on applying (2.1)

\( d(T_{x_{mk+1}}, T_{x_{mk+2}}) \leq d(P_{x_{mk+1}}, Q_{x_{mk+2}}) \leq q \max \{d(T_{x_{mk}}, T_{x_{mk+1}}), d(T_{x_{mk+1}}, T_{x_{mk+2}})\} \)

\( d(T_{x_{mk}}, T_{x_{mk+1}}) \).

Thus \( \{d(T_{x_{mk}}, T_{x_{mk+1}})\} \) is a decreasing sequence of non-negative real numbers and therefore will converge to a real number say \( \omega \).

Letting \( n \to \infty \) in (1.7), we get \( \omega \leq k \omega \) which gives \( \omega = 0 \). Thus \( \lim_{n \to \infty} d(T_{x_{mk}}, T_{x_{mk+1}}) = 0 \)

and therefore by Theorem 1, it follows that \( P, Q \) and \( T \) have a unique common fixed point.

**Corollary 4:** Let \( P, Q \) and \( T \) be three self-mappings of a metric space \( (X, d) \) satisfying conditions (2.1) of Theorem 2 and rest of the conditions of Corollary 1 then \( P, Q \) and \( T \) have a unique common fixed point.
Proof: If $n$ is odd, then
\[
d(T_{x_0}, T_{x_{n+1}}) = d(P_{x_{n-1}}, Q_{x_n}) \leq q \max \left\{ \frac{d(T_{x_{n-1}}, T_{x_0})}{d(T_{x_{n-1}}, T_{x_0})}, \frac{d(T_{x_{n-1}}, T_{x_0})}{d(T_{x_{n-1}}, T_{x_0})}, \ldots, \frac{d(T_{x_{n-1}}, T_{x_0})}{d(T_{x_{n-1}}, T_{x_0})} \right\}
\]
implies $d(T_{x_0}, T_{x_{n+1}}) \leq q \cdot d(T_{x_{n-1}}, T_{x_0})$.

If $n$ is even, then
\[
d(T_{x_n}, T_{x_{n+1}}) = d(Q_{x_{n-1}}, P_{x_n}) = d(P_{x_n}, Q_{x_{n-1}}) \leq q \max \left\{ \frac{d(T_{x_n}, T_{x_{n-1}})}{d(T_{x_n}, T_{x_{n-1}})}, \frac{d(T_{x_n}, T_{x_{n-1}})}{d(T_{x_n}, T_{x_{n-1}})}, \ldots, \frac{d(T_{x_n}, T_{x_{n-1}})}{d(T_{x_n}, T_{x_{n-1}})} \right\}
\]
implies $d(T_{x_0}, T_{x_{n+1}}) \leq q \cdot d(T_{x_{n-1}}, T_{x_0})$.

Rest of the proof is similar to that of Corollary 1.

References

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