CHAPTER VII

SOME COMMON FIXED POINT THEOREMS

FOR SET VALUED MAPPINGS

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7.1. Let $X$ be a metric space and $B(X)$ be the family of all non-empty bounded subsets of $X$. A set valued mapping $F: X \to B(X)$ is said to have a fixed point $z \in X$ if $z \in F(z)$.

Nadler [2], Kaulgud and Pai [1], have generalized a number of theorems of Kannan [2] on point valued mappings to set valued mappings. Fisher [10] proved some results on fixed point and common fixed point of set valued mappings which are generalizations of the results of Cirić [4]. If $A, B \in B(X)$ then to investigate the existence of fixed point for set valued mappings. Some mathematicians used Hausdorff metric $H(A, B)$ while other used $\mathcal{Q}(A, B) = \sup \{ d(a, b) : a \in A, b \in B \}$.

Let $(X, d)$ be a metric space and $A$ and $B$ are non-empty subsets of $X$. Then

\[
D(A, B) = \inf \left\{ d(a, b) : a \in A, b \in B \right\},
\]

\[
\mathcal{Q}(A, B) = \sup \left\{ d(a, b) : a \in A, b \in B \right\},
\]

\[
H(A, B) = \max \left\{ \sup D(a, B) : a \in A, \sup D(A, b) : b \in B \right\}.
\]

Let $2^X = \{ A : \emptyset \neq A \subseteq X \text{ and } \delta(A) < \infty \}$.
Where \( \delta(A) \) denotes the diameter of \( A \), i.e.

\[
\delta(A) = \sup \{d(a,b) : a, b \in A\}.
\]

Let \( F : X \to 2^X \), \( G : X \to 2^X \) be a point of set correspondence are the set valued or multivalued functions on \( X \).

An orbit of \((F,G)\) at a point \( x_0 \in X \) is a sequence \( \{x_n\} \) given by \( O(F,G,x_0) = \{x_n : n \in \mathbb{N}_0\} \), where \( \mathbb{N}_0 \) denotes the set of all non-negative integers and

\[
x_n = \begin{cases} 
F(x_{n-1}) & \text{when } n \text{ is odd} \\
G(x_{n-1}) & \text{when } n \text{ is even}.
\end{cases}
\]

A metric space \( X \) is said to be \((F,G)\)-orbitally complete, if every Cauchy sequence which is subsequence of orbit of \((F,G)\) at \( x_0 \) for every \( x_0 \in X \), converges to a point of \( X \).

The multivalued functions \( F,G \) are said to be \((F,G)\)-orbitally upper-semi-continuous on \( X \) if any subsequence \( \{x_{n_i}\} \) of the sequence \( \{x_n\} \), as defined in (2) converges to \( u \), then \( u \in Fu \) and \( u \in Gu \).

A metric space \((X,d)\) is said to be bounded metric space. If there exists a constant \( k > 0 \) such that \( d(x,y) \leq k \) for all \( x,y \in X \).
A family \( \mathcal{U} \) of bounded metrics on the non-empty set \( X \) is said to be comparable, if for \( d_1, d_2 \in \mathcal{U} \). There exists a constant \( c > 0 \) such that \( d_1(x, y) \leq c d_2(x, y) \) for all \( x, y \in X \).

Recently in 1990, Dhage [2] has proved the following fixed point theorem for single multivalued function.

**THEOREM A:** Let \( \mathcal{U} \) be a comparable family of metrics on a non-empty set \( M \) and \( F: M \to 2^M \) be such that \( Fx \) is closed for every \( x \in M \). Let \( M \) satisfies the following conditions:

(i) \( F \) is \( F \)-orbitally - upper-semi-continuous w.r.t. a metric \( d_1 \in \mathcal{U} \).

(ii) \( M \) is \( F \)-orbitally complete w.r.t. a metric \( d_1 \in \mathcal{U} \).

(iii) There exists a \( d \in \mathcal{U} \) and real numbers \( \beta, p \) and \( q \), such that

\[
\min \left\{ \varrho(Fx, Fy), \varrho(x, Fx), \varrho(y, Fy) \right\} + \beta \min \left\{ D(x, Fy), D(y, Fx) \right\} \leq p D(x, Fy) + q d(x, y)
\]

for all \( x, y \in M \) where \( 0 < p + q < 1 \). Then \( F \) has a fixed point.

We state the following lemma of Dhage [2], which will be used in the proof.

**LEMMA B:** Let \( d_1 \) and \( d_2 \) be two comparable metrics over a non-empty set \( X \), if multivalued functions \( F, G \) on \( X \) are \((F, G)\)-orbitally upper-semi-continuous with respect to \( d_1 \) so is w.r.t. \( d_2 \) and viceversa.
7.2. In this section we generalize the above Theorem A of Dhage [2] for a pair of set-valued mappings in two different forms.

**THEOREM 1:** Let \( \mathcal{U} \) be a comparable family of metrics on a non-empty set \( X \) and \( F : X \to 2^X \), \( G : X \to 2^X \) be such that \( Fx, Gy \) are closed for every \( x, y \in X \) respectively. Let \( X \) satisfy the following conditions:

1. (7.2.1) \( (F,G) \) are \((F,G)\)-orbitally upper-semi-continuous with respect to \( d_1 \in \mathcal{U} \).
2. (7.2.2) \( X \) is \((F,G)\)-orbitally complete w.r.t. metric \( d_1 \in \mathcal{U} \).
3. (7.2.3) There exists \( a \in \mathcal{U} \) and real numbers \( \beta, p, q \) and \( r \) such that

\[
\min \left\{ \varphi(Fx, Gy), \varphi(x, Fx), \varphi(y, Gy) \right\} + \beta \min \left\{ D(x, Gy), D(y, Fx) \right\} \leq p D(x, Fx) + q d(x, y)
\]

\[
+ r \frac{D(x, Fx)}{D(Fx, Gy)}
\]

for all \( x, y \in X \), where \( 0 < p + q + r < 1 \). Then \( F \) and \( G \) have a common fixed point. Further, if \( \frac{q}{\beta} < 1 \), then this fixed point is unique.

**PROOF:** We define point valued mappings \( T_1 \) and \( T_2 \) on \( X \) into itself. Let \( x_0 \in X \) be an arbitrary point and we construct a sequence \( \{x_n\} \) as follows:
\[ x_{2n+1} = T_1 x_{2n} \]
\[ x_{2n+2} = T_2 x_{2n+1} \]
\( \forall n \in \mathbb{N} \)

Let \( x_{2n+1} \in F x_{2n} \) which implies \( D(x_{2n+1}, Gx_{2n+1}) \leq \phi(Fx_{2n}, Gx_{2n+1}) \)
and \( D(x_{2n+1}, Fx_{2n}) = 0 \) and \( D(x_{2n+2}, Gx_{2n+1}) = 0 \), for all \( n \in \mathbb{N} \).

By (1), we have

\[ x_{2n+1} = T_1 x_{2n} \in F x_{2n} \]
\[ x_{2n+2} = T_2 x_{2n+1} \in Gx_{2n+1} \]

\[ d(x_{2n+1}, x_{2n+2}) = d(T_1 x_{2n}, T_2 x_{2n+1}) \leq \phi(Fx_{2n}, Gx_{2n+1}), \]
for all \( n \in \mathbb{N} \).

By applying (7.2.3), we have

\[
\min \left\{ \phi(Fx_{2n}, Gx_{2n+1}), \phi(x_{2n}, Fx_{2n}), \phi(x_{2n+1}, Gx_{2n+1}) \right\} + \beta \min \left\{ D(x_{2n}, Gx_{2n+1}), D(x_{2n+1}, Fx_{2n}) \right\}
\leq p D(x_{2n}, Fx_{2n}) + q d(x_{2n}, x_{2n+1})
\]

\[ + r \frac{dD(x_{2n}, Fx_{2n}) D(x_{2n+1}, Gx_{2n+1})}{D(Fx_{2n}, Gx_{2n+1})} \]

i.e.

\[
\min \left\{ d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \right\}
\leq p d(x_{2n}, x_{2n+1}) +qd(x_{2n}, x_{2n+1})
\]

\[ + r \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{d(x_{2n+1}, x_{2n+2})} \]
or \( \min \{ d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}) \} \leq (p+q+r)d(x_{2n}, x_{2n+1}) \Rightarrow \)

Either \( d(x_{2n}, x_{2n+1}) \leq \alpha d(x_{2n}, x_{2n+1}) \), a contradiction,

where \( \alpha = p+q+r < 1 \).

or \( d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1}) \).

By continuing above process, we get

\[
d(x_{2n+1}, x_{2n+2}) \leq \alpha d(x_{2n}, x_{2n+1}) \leq \alpha^2 d(x_{2n-1}, x_{2n}) \leq \cdots \leq \alpha^{2n-1} d(x_0, x_1)
\]

Now, for \( n < m \),

\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)
\]

\[
= \sum_{l=0}^{m-n-1} d(x_{n+l}, x_{n+l+1})
\]

\[
\leq \sum_{l=0}^{m-n-1} \alpha^{n+l} d(x_0, x_1)
\]

\[
< \frac{\alpha^n}{1-\alpha} d(x_0, x_1)
\]

Since \( d, d_1 \in \partial U \), then there exists a constant \( C > 0 \) such that

\[
d_1(x, y) \leq C d(x, y), \forall x, y \in X \ldots \ldots (2)
\]

We choose a number \( N_1 \in N \), so large that for \( C > 0 \)

\[
d(x_n, x_m) < \frac{\varepsilon}{C} \text{ for } n > N_1, \forall n, N_1 \in N \ldots (3)
\]
On putting $x = x_n$, $y = x_m$ in (2) and using (3),

$$d_1(x_n, x_m) \leq C \frac{\varepsilon}{C} = \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence w.r.t. the metric $d_1$ in $X$ and $X$ is complete metric space w.r.t. $d_1$. There exists a point $u \in X$ such that $\lim_{n \to \infty} x_n = u$.

Now, by continuity of $T_1$ and $T_2$, we have

$$u = \lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} T_1 x_{2n} = T_1 \lim_{n \to \infty} x_{2n} = T_1 u$$

and

$$u = \lim_{n \to \infty} x_{2n+2} = \lim_{n \to \infty} T_2 x_{2n+1} = T_2 \lim_{n \to \infty} x_{2n+1} = T_2 u.$$

This shows $u$ is common fixed point of $T_1$ and $T_2$.

Now using Lemma (B), we get

the multifunctions $F,G$ are $(F,G)$-orbitally semi continuous with respect to $d_1$ so that

$$u = T_1 u \in F u \text{ and } u = T_2 u \in G u.$$

$$\implies u \in F u \text{ and } u \in G u.$$ 

Thus $u$ is common fixed point of $F$ and $G$.

For uniqueness of $u$,

Let $v$ is another fixed point such that $v \in F v$ and $v \in G v$.
By applying (7.2.3), we have

\[
\min \left\{ \varrho(Fu, Gv), \varrho(u, Fu), \varrho(v, Gv) \right\} + \beta \min \left\{ D(u, Gv), D(v, Fu) \right\} \leq p D(u, Fu) + q d(u, v) + r \frac{D(u, Fu) D(v, Gv)}{D(Fu, Gv)}
\]

i.e.\( \min \left\{ \varrho(u, v), \varrho(u, u), \varrho(v, v) \right\} + \beta \min \left\{ D(u, v), D(v, u) \right\} \leq p D(u, u) + q a(u, v) + o \)

or \( \beta D(u, v) \leq q d(u, v) \)

\[ \Rightarrow \quad d(u, v) \leq \frac{q}{\beta} d(u, v), \text{ since } \frac{q}{\beta} \leq 1. \]

\( < d(u, v) \), this contradiction implies \( u = v \).

Hence \( u \) is a unique fixed point of \( F \) and \( G \).

**Theorem 2:** If in Theorem 1, condition (7.2.3) is replaced by,

(7.2.4) \[
\min \left\{ \left( \varrho(Fx, Gy) \right)^2, \varrho(x, Fx) \varrho(y, Gy), \left( \varrho(y, Gy) \right)^2 \right\} + \min \left\{ D(x, Fx) D(y, Gy), D(x, Gy) D(y, Fx) \right\} \leq p D(x, Fx) D(y, Gy) + q D(y, Gy)d(x, y)
\]

for all \( x, y \in X \), where \( o < p + q < '1 \) . Then \( F \) and \( G \) have a common fixed point.

**Proof:** Let \( T_1 \) and \( T_2 \) be a single valued mappings of \( X \) into itself.
Let $x_0$ be an arbitrary point in $X$. We construct a sequence $\{x_n\}$ as defined in the same way as in Theorem 1.

On putting $x = x_{2n}$, $y = x_{2n+1}$ in (7.2.4)

$$\min \left\{ \left( \varphi(F_{x_{2n}}, G_{x_{2n+1}}) \right)^2, \varphi(x_{2n}, F_{x_{2n}}) \varphi(x_{2n+1}, G_{x_{2n+1}}), \right. \left( \varphi(x_{2n+1}, G_{x_{2n+1}}) \right)^2 \right\}$$

$$+ \beta \min \left\{ D(x_{2n}, F_{x_{2n}}), D(x_{2n+1}, G_{x_{2n+1}}) \right\}.$$ 

$$D(x_{2n}, G_{x_{2n+1}}) D(x_{2n+1}, F_{x_{2n}}) \right\}$$

$$\leq p D(x_{2n}, F_{x_{2n}}) D(x_{2n+1}, G_{x_{2n+1}}) + q D(x_{2n+1}, G_{x_{2n+1}}) d(x_{2n}, x_{2n+1})$$

i.e. \( \min \left\{ \left[ d(x_{2n+1}, x_{2n+2}) \right]^2, d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}) \right\}, \) 

$$\left[ d(x_{2n+1}, x_{2n+2}) \right]^2 \right\}$$

$$\leq p d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}) + q d(x_{2n+1}, x_{2n+2}) d(x_{2n}, x_{2n+1})$$

$$\leq k d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}), \text{ where } k = p + q < 1.$$

Suppose, \( \min \left\{ \left[ d(x_{2n+1}, x_{2n+2}) \right]^2, d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}) \right\} \)

$$= d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}).$$
Then, \(d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}) \leq kd(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})\)

\[< d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})\]

which is a contradiction.

so that, \([d(x_{2n+1}, x_{2n+2})]^2 \leq kd(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})\]

which implies \(d(x_{2n+1}, x_{2n+2}) \leq kd(x_{2n}, x_{2n+1})\)

Proceeding in this manner, we get

\[d(x_{2n+1}, x_{2n+2}) \leq kd(x_{2n}, x_{2n+1}) \leq k^2d(x_{2n-1}, x_{2n}) \leq \ldots \leq k^{2n+1}d(x_0, x_1)\]

It can be shown easily as in Theorem 1, \(\{x_n\}\) is a Cauchy sequence with respect to \(d_1\). In view of Theorem 1, \(u\) is a common fixed point of \(F\) and \(G\).

**REMARK**

(i) If \(F = G\) and \(r = 0\) in Theorem 1. We obtain Theorem A due to Dhage [2].

(ii) If \(F = G\) point valued mapping we get Theorem 1 of Dhage [1].

(iii) If taking \(F = G\) point valued mapping on \(X\), \(\beta = -1\) and \(p = r = 0\), we get theorem of Ciric[3].

(iv) If \(F = G\) in Theorem 2 we obtain Theorem 2 due to Dhage [2]
7.3. In this section we have extended the Theorem C for a pair of set-valued mappings.

Recently in 1990, Murthy and Pathak [1] proved the following for a pair of point valued mappings for periodic fixed point.

**THEOREM C**: Let $E$ and $F$ be two selfmappings of a complete metric space $(X,d)$ such that there exists positive integers $p(x)$ and $q(y)$ such that for each $x,y \in X$.

$$d(E^{p(x)}x,F^{q(y)}y) \leq \frac{\alpha d(x,E^{p(x)}x) \ d(y,F^{q(y)}y)}{[d(x,F^{q(y)}y)+d(y,E^{p(x)}x)+d(x,y)]} + \beta d(x,y)$$

where $\alpha, \beta$ are non negative real numbers such that $\alpha + \beta < 1$.

Then $E,F$ have a unique common fixed point.

**THEOREM 3**: Let $E$ and $F$ be two mappings of a complete metric space $(X,d)$ into $B(X)$ and for each $x,y$ of $X$, there exist positive integers $p(x), q(y)$ satisfying the following inequality:

$$\xi(E^{p(x)}x,F^{q(y)}y) \leq \frac{\alpha \xi(x,E^{p(x)}x) \ \xi(y,F^{q(y)}y)}{[ \xi(x,F^{q(y)}y)+\xi(y,E^{p(x)}x)+d(x,y)]} + \beta \ d(x,y)$$

\( \forall \ x,y \in X \) where $\alpha, \beta \geq 0$, such that $\alpha + \beta < 1$.

Then $E,F$ have a unique common fixed point.
PROOF: Let \( x_0 \) be any point in \( X \). Now, we consider the sequence \( \{ x_n \} \) by putting \( X_1 = E^{p(x_0)}x_0 \) and take a point \( x_1 \in X_1 \) and \( X_2 = F^{q(x_1)}x_1 \) and take a point \( x_2 \in X_2 \).

In general,

\[
X_{2n+1} = E^{p(x_{2n})}x_{2n} \quad \text{and take points} \quad x_{2n+1} \in X_{2n+1}
\]

and

\[
X_{2n+2} = F^{q(x_{2n+1})}x_{2n+1} \quad \text{and take points} \quad x_{2n+2} \in X_{2n+2}
\]

\( \forall n \in \mathbb{N}_0 \) where \( \mathbb{N}_0 \) is set of non-negative integers.

If \( X_{2n+1} = X_{2n+2} \). Then \( \{ x_n \} \) is a Cauchy sequence. Therefore, we can suppose \( X_{2n+1} \neq X_{2n+2} \), by applying (7.3.1), we have,

\[
\sigma(x_{2n+1}, x_{2n+2}) = \sigma(E^{p(x_{2n})}x_{2n}, F^{q(x_{2n+1})}x_{2n+1})
\]

\[
\leq \alpha \frac{\sigma(x_{2n}, E^{p(x_{2n})}x_{2n}) \cdot \sigma(x_{2n+1}, F^{q(x_{2n+1})}x_{2n+1})}{[\sigma(x_{2n}, F^{q(x_{2n+1})}x_{2n+1}) + \sigma(x_{2n+1}, E^{p(x_{2n})}x_{2n}) + d(x_{2n}, x_{2n+1})]} + \beta d(x_{2n}, x_{2n+1})
\]

\[
\leq \alpha \frac{\sigma(x_{2n}, x_{2n+1}) \cdot \sigma(x_{2n+1}, x_{2n+2})}{[\sigma(x_{2n}, x_{2n+2}) + \sigma(x_{2n+1}, x_{2n+2}) + \sigma(x_{2n}, x_{2n+1}) + \beta \sigma(x_{2n}, x_{2n+1})]} + \beta \sigma(x_{2n}, x_{2n+1})
\]
Now we have to show that first term of R.H.S. is less than \( \zeta(x_{2n}, x_{2n+1}) \)

suppose

\[
\frac{\zeta(x_{2n}, x_{2n+1}) \zeta(x_{2n+1}, x_{2n+2})}{\zeta(x_{2n}, x_{2n+2}) + \zeta(x_{2n}, x_{2n+1})} < \zeta(x_{2n}, x_{2n+1})
\]

i.e. \( \zeta(x_{2n}, x_{2n+1}) \frac{\zeta(x_{2n+1}, x_{2n+2})}{\zeta(x_{2n}, x_{2n+2})} \)

\[
< [\zeta(x_{2n}, x_{2n+1})][2 \zeta(x_{2n}, x_{2n+1}) + \zeta(x_{2n+1}, x_{2n+2})]
\]

or \( 0 < 2[\zeta(x_{2n}, x_{2n+1})]^2 \) which is true and hence,

\[
\zeta(x_{2n+1}, x_{2n+2}) \leq (\alpha + \beta) \zeta(x_{2n}, x_{2n+1})
\]

\[
= q \zeta(x_{2n}, x_{2n+1}), \text{ where } q = \alpha + \beta < 1.
\]

By proceeding in the same manner, we get

\[
\zeta(x_{2n+1}, x_{2n+2}) \leq q \zeta(x_{2n}, x_{2n+1}) \leq q^2 \zeta(x_{2n-1}, x_{2n}) \leq \ldots \leq q^{2n+1} \zeta(x_0, x_1)
\]

for \( m > n \),

\[
\zeta(x_n, x_m) \leq \zeta(x_n, x_{n+1}) + \ldots + \zeta(x_{m-1}, x_m)
\]

\[
\leq (q^n + q^{n+1} + \ldots + q^{m-1}) \zeta(x_0, x_1)
\]

\[
\leq \frac{q^n}{1-q} \zeta(x_0, x_1)
\]
Since \( q < 1 \), there exists \( \varepsilon > 0 \), for \( m > n \).

\[
d(x_n, x_m) \leq \varepsilon(x_n, x_m) < \varepsilon.
\]

Thus it follows that \( \{x_n\} \) is a Cauchy sequence in complete metric space \((X, d)\). Therefore, it converges to some \( u \in X \).

On putting \( x = x_{2n} \) and \( y = u \) in (7.3.1), we get

\[
\xi(x_{2n+1}, F^nu) = \xi(E^{x_{2n}}, F^nu) = \xi(u, F^nu) \leq \alpha \frac{\xi(x_{2n}, x_{2n+1}) \xi(u, F^{\xi(u)}u)}{[\xi(x_{2n}, F^{\xi(u)}u) + \xi(u, x_{2n+1}) + \xi(x_{2n}, u)]} + \beta \xi(x_{2n}, u)
\]

Letting \( n \) tends to infinity, we have

\[
\xi(u, F^nu) \leq \frac{\xi(u, u) \xi(u, F^{\xi(u)}u)}{[\xi(u, F^{\xi(u)}u) + \xi(u, u) + \xi(u, u)]} + \beta \xi(u, u)
\]

or \( \xi(u, F^nu) \leq 0 \) which implies \( F^nu = \{u\} \).

Similarly, by applying (7.3.1) for \( x = u, y = x_{2n+1} \), we get

\[
\xi(u, F^{\xi(u)}u) \leq \alpha, \text{ which gives } E^u = \{u\}.
\]

Thus \( u \) is a periodic point of \( E \) and \( F \), i.e., \( u = E^p u = F^q u \).

Let \( v \) is another periodic point of \( E \) and \( F \), i.e.,

\[
E^p(v) = F^q(v) = \{v\}
\]
On putting $x = u$, $y = v$ in (7.3.1), we get

$$\zeta(u, v) = \zeta(E^{p(u)}u, F^{q(v)}v)$$

$$\leq \frac{\alpha \zeta(u, E^{p(u)}u) \zeta(v, F^{q(v)}v)}{\zeta(u, F^{p(u)}v) + \zeta(v, E^{p(u)}u) + d(u, v)} + \beta d(u, v)$$

$$\leq \frac{\alpha \zeta(u, u) \zeta(v, v)}{\zeta(u, v) + \zeta(u, v) + \zeta(v, u)} + \beta \zeta(u, v)$$

$$= \beta \zeta(u, v) < \zeta(u, v), \text{ since } \alpha + \beta < 1$$

is a contradiction and hence $u = v$.

Thus $u$ is unique periodic point of $E^{p(u)}u$ and $F^{q(u)}$.

Now $\{u\} = E^{p(u)}u \Rightarrow Eu = E^{p(u)}u$ implies $Eu$ is a periodic point of $E^{p(u)}u$. Since $u$ is unique. Therefore $\{u\} = Eu$.

Similarly $\{u\} = F^{q(u)}u \Rightarrow Fu = F^{q(u)}u$.

Therefore $Fu$ is a periodic point of $F^{q(u)}u$.

But unicity of $u$, $\{u\} = F^{q(u)}u$,

Which implies $\{u\} = Fu$. Hence $u$ is a common fixed point of $E$ and $F$.

For uniqueness of fixed point of $u$, it can be easily seen that $u$ is unique fixed point of $E$ and $F$. 
7.4. Caristi [1] introduced an important fixed point theorem where the function is neither of contraction type nor continuous. He proved for a selfmapping $T$ of complete metric space $(X,d)$ and a lower semi continuous mapping $\phi : X \rightarrow [0,\infty)$ satisfying $d(x,Tx) \leq \phi(x) - \phi(Tx)$. Then $T$ has a unique fixed point.

Let $(X,d)$ be a metric space, then

(i) $\text{CB}(X) = \{ E : E \text{ is a non empty closed and bounded subset of } X \}$

(ii) $\text{N}(\epsilon, A) = \{ x \in X \mid d(x,y) < \epsilon \text{ for some } y \in A \}$, $\epsilon > 0$ and $A \in \text{CB}(X)$.

(iii) $H(A,B) = \inf \{ \text{E} \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A) \}$,

$\epsilon > 0$ and $A,B \in \text{CB}(X)$.

The function $H$ is a metric for $\text{CB}(X)$ called the Hausdorff metric for $\text{CB}(X)$.

Let $(X,d_1)$ and $(Y,d_2)$ be two metric spaces. A function $F : X \rightarrow \text{CB}(Y)$ is said to be a multivalued contraction mapping of $X$ into $Y$ if,

$H(F(x), F(y)) \leq \alpha d_1(x,y)$, $\forall x,y \in X$, $0 \leq \alpha < 1$

**Lemma D:** If $A,B \in \text{CB}(X)$ and $x \in A$, then, for each positive number $\eta$, there exists a $y \in B$ such that...
d(\overline{x}, y) \leq H(A, B) + \eta.

**Lemma F:** Let \( \{ A_n \} \) be a sequence of sets in \( CB(X) \) and suppose \( \lim H(A_n, A_0) = 0 \), where \( A_0 \in CB(X) \). Then, if \( x_n \in A_n \), \( n = 1, 2, 3 \ldots \), and \( \lim x_n = x_0 \), it follows that \( x_0 \in A_0 \).

Chandel [1] proved the following theorem for pair of multivalued mappings under Caristi-Kirk type condition.

**Theorem F:** Let \((X, d)\) be a complete metric space and \( F_1, F_2 \) be two continuous multivalued mappings on \( X \). Let there be two mappings \( \emptyset \) and \( \psi \) on \( CB(X) \to [0, \infty) \) such that

\[
H(F_1(x), F_2(y)) \leq \emptyset(x) - \emptyset(F_1(x)) + \psi(y) - \psi(F_2(y)),
\]

for all \( x, y \in X \).

\[(7.4.2) \quad \emptyset(A) \leq \emptyset(B) \text{ and } \psi(A) \leq \psi(B), \quad A, B \text{ in } CB(X) \text{ and } A \subset B. \] The \( F_1 \) and \( F_2 \) have a common fixed point, further, if any of the multivalued mappings is one-one, then this common fixed point is a unique common fixed point of the one-one mapping.

We extend above theorem for three setvalued mappings and obtained their unique common fixed point.

**Theorem 4:** Let \((X, d)\) be a complete metric space and \( S, T \) and \( P \) be three continuous multivalued mappings on \( X \). Let there be two mappings \( \emptyset \) and \( \psi \) on \( CB(X) \to [0, \infty) \) such that for each \( x, y \) of \( X \).
(7.4.3) \[ H(SP(x), \, TP(y)) \leq \varnothing(x) - \varnothing(SP(x)) + \psi(y) - \psi(TP(y)) \]

(7.4.4) \[ \varnothing(A) \leq \varnothing(B) \text{ and } \psi(A) \leq \psi(B) ; \, A, B \text{ in } CB(X) \text{ and } A \subseteq B . \]

Then SP and TP have a common fixed point. Further, if one of the multivalued mapping is one-one and P commutes with S or T, then this common fixed point is a unique fixed point of S, T and P.

**Proof:** Let \( x_0, \, y_0 \in X \) and \( x_1 \in SP(x_0) \) then for \( \eta = \frac{1}{2} \nolinebreak \\
\text{by lemma D, there exists a } y_1 \in TP(y_0) \text{ such that} \]
\[
d(x_1, y_1) < H(SP(x_0), TP(y_0)) + \frac{1}{2} \\
\leq \varnothing(x_0) - \varnothing(SP(x_0)) + \psi(y_0) - \psi(SP(y_0)) + \frac{1}{2} , \]
\text{by (7.4.3).} \ldots (1)

Since \( x_1 \in SP(x_0) \) i.e. \( \{x_1\} \subseteq SP(x_0) \), by (7.4.4), we have
\[
\varnothing(x_1) \leq \varnothing(SP(x_0)) \ldots \ldots \ldots \ldots (2)
\]
Similarly if for \( y_1 \in TP(y_0) \) i.e. \( \{y_1\} \subseteq TP(y_0) \)
\[
\implies \psi(y_1) \leq \psi(TP(y_0)) \ldots \ldots \ldots (3)
\]
By using inequalities (1), (2) and (3), we get
\[
d(x_1, y_1) < \varnothing(x_0) - \varnothing(x_1) + \psi(y_0) - \psi(y_1) + \frac{1}{2} . \]
Again for $y_1 \in TP(y_0)$, there exists an element $x_2$ such that,

$$d(x_2, y_1) < H(SP(x_1), TP(y_0)) + \frac{1}{2}$$

$$\leq \emptyset(x_1) - \emptyset(SP(x_1)) + \psi(y_0) - \psi(TP(y_0)) + \frac{1}{2}$$

$$\leq \emptyset(x_1) - \emptyset(x_2) + \psi(y_0) - \psi(y_1) + \frac{1}{2}$$

since $x_2 \in SP(x_1)$, there exists a $y_2 \in TP(y_1)$ such that

$$d(x_2, y_2) < \emptyset(x_1) - \emptyset(x_2) + \psi(y_1) - \psi(y_2) + \frac{1}{2^2}$$

By repeating the process in similar manner, we obtain the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \in SP(x_{n-1}), \ y_n \in TP(y_{n-1}) \ and$$

$$d(x_{n+1}, y_n) < \emptyset(x_n) - \emptyset(x_{n+1}) + \psi(y_{n-1}) - \psi(y_n) + \frac{1}{2^n} \ldots (4)$$

Now,

$$\sum_{r=1}^{n} d(x_r, y_r) < \sum_{r=1}^{n} [\emptyset(x_{r-1}) - \emptyset(x_r) + \psi(y_{r-1}) - \psi(y_r)]$$

$$+ \sum_{r=1}^{n} \frac{1}{2^r} \ldots$$

$$\leq \emptyset(x_0) - \emptyset(x_n) + \psi(y_0) - \psi(y_n) + 1$$

$$\leq \emptyset(x_0) + \psi(y_0) + 1, \ [\text{since } \emptyset(x_n) \geq 0, \ and \ \psi(y_n) \geq 0]$$

$$\ldots \ldots \ldots \ldots (5)$$

Now, $d(x_r, x_{r+1}) \leq d(x_r, y_r) + d(y_r, x_{r+1})$

$$\Rightarrow \sum_{r=1}^{n} d(x_r, x_{r+1}) \leq \sum_{r=1}^{n} [d(x_r, y_r) + d(y_r, x_{r+1})]$$
\[ \langle \emptyset(x_0) + \psi(y_0) + 1 \rangle + \langle \emptyset(x_1) + \psi(y_1) + 1 \rangle, \text{using (5)} \]

\[ = \emptyset(x_0) + \emptyset(x_1) + 2 \psi(y_0) + 2 \]

Since \( \emptyset \) and \( \psi \) are defined on \( CB(X) \rightarrow [0, \infty) \), R.H.S is finite
so that \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) \) is convergent, which implies
\( d(x_n, x_{n+1}) \rightarrow 0 \) as \( n \) tends to infinity.

Hence \( \{x_n\} \) is a Cauchy sequence and \((X, d)\) is complete,
therefore it converges to some \( z_1 \in X \) i.e. \( \lim_{n \to \infty} x_n = z_1 \).

Since \( S \) and \( P \) are continuous and \( SP \) being the composite of
continuous functions of \( S \) and \( P \) and \( x_n \in SP(x_{n-1}) \), for \( n = 1, 2, \ldots \),
by using Lemma E, \( z_1 \) is a fixed point of \( SP \) i.e. \( z_1 \in SP(z_1) \)
or \( \{z_1\} = SP(z_1) \).

Similarly,
\[ \sum_{r=1}^{n} d(y_r, y_{r+1}) \leq \sum_{r=1}^{n} [d(y_r, x_{r+1}) + d(x_{r+1}, y_{r+1})] \]

\[ = \sum_{r=1}^{n} [d(x_{r+1}, y_r) + d(y_{r+1}, y_{r+1})] \]

\[ < \langle \emptyset(x_1) + \psi(y_0) + 1 \rangle + \langle \emptyset(x_1) + \psi(y_1) + 1 \rangle \]

\[ = 2 \emptyset(x_1) + \psi(y_0) + \psi(y_1) + 2 \]

Since R.H.S. is finite, therefore series \( \sum_{n=1}^{\infty} d(y_n, y_{n+1}) \) is
convergent which implies \( d(y_n, y_{n+1}) \rightarrow 0 \) as \( n \) tends to
infinity.
Therefore \( \{ y_n \} \) is a Cauchy sequence in complete metric space \((X, d)\), it converges to \( z_2 \in X \), i.e. \( \lim_{n \to \infty} y_n = z_2 \), by Lemma E, \( z_2 \) is a fixed point of \( TP \).

Now, \( d(z_1, z_2) \leq d(z_1, x_n) + d(x_n, y_n) + d(y_n, z_2) \)

Letting \( n \to \infty \), we get \( d(z_1, z_2) \leq d(z_1, z_2) \), a contradiction which implies \( z_1 = z_2 \) .... (4)

Let \( SP \) is one-one and \( w \) be second fixed point of \( SP \).
Suppose \( z_1 = z_2 = z \). Then by applying (7.4.3), we get

\[
H(SP(w), TP(z)) \leq \phi(w) - \phi(SP(w)) + \psi(z) - \psi(TP(z)).
\]

[since \( w \in SP(w) \) and \( z \in TP(z) \), by (7.4.4),

We have \( \phi(w) \leq \phi(SP(w)) \) and \( \psi(z) \leq \psi(TP(z)) \), respectively.

Hence \( H(SP(w), TP(z)) \leq \phi(w) - \phi(w) + \psi(z) - \psi(z) = 0 \),

so that \( SP(w) = TP(z) \).

Since this relation is true for any fixed point \( w \) of \( SP \) and also \( z \) is a fixed point of \( SP \), we get

\( SP(w) = TP(z) \).

Thus \( z \) is a unique fixed point of \( SP \). Similarly, if \( TP \) is one-one, this common fixed point \( z \) would be unique fixed point of \( SP \) and \( TP \).
If $P$ commutes with $S$, i.e., $SP = PS$, on putting $x = P(z)$, $y = z$ in (7.4.3), we have

$$d(P(z), z) = d(P(SP(z)), TP(z)) < H(PS(P(z)), TP(z))$$

$$= H(SP(P(z)), TP(z)).$$

$$\leq \varnothing(P(z) - \varnothing(SP(P(z))) + \psi(z) - \psi(TP(z))$$

$$\leq \varnothing(P(z)) - \varnothing(SP(P(z))) + \psi(z) - \psi(TP(z))$$

$$\leq \varnothing(P(z)) - \varnothing(P(z)) + \psi(z) - \psi(z)$$

$$= 0,$$

which implies $z$ is a fixed point of $P$, if we consider $TP = PT$ it can be shown that $z$ is a fixed point of $P$ i.e. $z \in P(z)$.

Since $S$, $T$ and $P$ are continuous, so that

$$SP(z) = z = P(z)$$

and

$$TP(z) = z = P(z)$$

imply

$$S(z) = T(z) = P(z) = \{z\}$$

Thus $z$ is common fixed point $S$, $T$ and $P$.

Finally, we have to show that $z$ is unique. Let $w$ is another fixed point of $S$, $T$ and $P$ such that, $\{w\} = S(w) = T(w) = P(w)$, by applying (7.4.3), we have

$$d(w, z) = d(SP(w), TP(z)) < H(SP(w), TP(z))$$

$$\leq \varnothing(w) - \varnothing(SP(w)) + \psi(z) - \psi(TP(z))$$
\[ \phi(w) - \phi(w) + \psi(z) - \psi(z) = 0 \]

which implies \( z = w \). If any one of the multivalued mappings \( S, T \) and \( P \) is one-one which implies \( z = S(z) = T(z) = P(z) \)
i.e. \( z \) is unique fixed point of \( S, T \) and \( P \).

REMARK : (i) If \( P = I \) (Identity mapping) we have
Theorem \( F \) due to Chandel [1].

(ii) If \( P = I \) and on putting \( S = In, T = g, x = y, \)
we get proposition of Felix Browder [2].

COROLLARY 1: Let \((X,d)\) be a complete metric space and
\( T = \{ F_\lambda : \lambda \in \Lambda \} \) be a family of continuous multivalued
mappings on \( X \). Suppose that for each \( \lambda \in \Lambda \), there exists
\( \phi_\lambda \) on \( CB(X) \to [0,\infty) \) satisfying :

\[(7.2.5) \quad \phi_\lambda(A) \leq \phi_\lambda(B) \forall A, B(x) \text{ with } A \subseteq B.\]

\[(7.2.6) \quad \text{Let there exists a mapping } F_\alpha P \in \Gamma \text{ which is}
one-one and for another } F_\beta P \in \Gamma \text{ with } (\beta \neq \alpha), \text{ we have}
\[ H(F_\alpha P(x), F_\beta P(y)) \leq \phi_\alpha(x) - \phi_\alpha(F_\alpha P(x)) \\
+ \phi_\beta(y) - \phi_\beta(F_\beta P(y)) \]

for all \( x, y \) in \( X \) and \( P \) commutes with \( F_\alpha \) or \( F_\beta \). Then the
mapping of \( T \) has a common fixed point which is the unique
fixed point of \( F_\lambda P \).
**THEOREM 5**: Let \((X,d)\) be a metric space. Let \(F\) and \(G\) be continuous multivalued mappings on \(X\) and \(\emptyset\) be a mapping on \(\mathcal{CB}(X) \to [0,\infty)\) such that

\[(7.4.7)\quad \emptyset(A) \leq \emptyset(B) \forall A, B \in \mathcal{CB}(X) \text{ with } A \subset B\]

\[(7.4.8)\quad H(FG(x), z) \leq \emptyset(x) - \emptyset(FG(x)), \forall x \in X.\]

Then, \(z\) is a fixed point of \(FG\). If \(FG = GF\) and \(FG\) is one one, then \(z\) is unique common fixed point of \(F\) and \(G\).

**PROOF**: Let \(x_0\) be arbitrary point in \(X\), then by using Lemma D, there exists an \(x_1 \in FGx_0\) i.e. \(\{x_1\} \subset FGx_0\) such that by applying \((7.4.8)\), we have

\[
d(x_1, z) < H(FG(x_0), z) + \frac{1}{2} \leq \emptyset(x_0) - \emptyset(FG(x_0)) + \frac{1}{2}\]

\[
\leq \emptyset(x_0) - \emptyset(x_1) + \frac{1}{2}, \text{ [since } \emptyset(x_1) \leq \emptyset(FG(x_0)) \text{]}\]

Similarly, there exists an \(x_2 \in FG(x_1)\) such that

\[
d(x_2, z) < H(FG(x_1), z) + \frac{1}{2^n}, \text{ (by Lemma D) }\]

\[
\leq \emptyset(x_1) - \emptyset(FG(x_1)) + \frac{1}{2^n} \text{ (by } 7.4.8)\]

\[
\leq \emptyset(x_1) - \emptyset(x_2) + \frac{1}{2^n} \text{ (by } 7.4.7)\]

By continuing above process, we have a sequence \(\{x_n\}\) such that \(x_n \in F(x_{n-1})\) and

\[
d(x_n, z) \leq H(FG(x_{n-1}), z) + \frac{1}{2^n} \leq \emptyset(x_{n-1}) - \emptyset(x_n) + \frac{1}{2^n}, \quad n=1,2,\ldots\]
Now, 
\[ \sum_{i=1}^{n} d(x_i, z) \leq \sum_{i=1}^{n} \left[ \varphi(x_{i-1}) - \varphi(x_i) + \frac{1}{2^i} \right] \]
\[ \leq \varphi(x_0) - \varphi(x_n) + 1 \]
\[ < \varphi(x_0) + 1, \text{ since } \varphi(x_n) \geq 0 \]

R.H.S. is finite quantity, therefore the series \( \sum_{i=1}^{\infty} d(x_i, z) \)
is convergent. This implies \( d(x_n, z) \to 0 \) as \( n \to \infty \).

Hence \( x_n \to z \) as \( n \to \infty \).

Since \( F \) and \( G \) are continuous so that composite
mapping \( FG \) is continuous and \( x_n \in FG(x_{n-1}), n = 1, 2, \ldots \), by
using Lemma E, \( z \) is a fixed point of \( FG \).

Let \( FG \) is one-one and \( z \) is a fixed point of \( FG \),
by applying (7.4.8)

\[ H(FG(z_1), z) \leq \varphi(z_1) - \varphi(FG(z)) \]
\[ \leq \varphi(z_1) - \varphi(z_1) = 0 \quad [ \text{since } \varphi(z_1) \leq \varphi(FG(z)) ] , \]
so that \( H(FG(z_1), z) \leq 0 \), which implies \( z \in FG(z_1) \) i.e. \( \{z_1\} = FG(z_1) \).

This being these of any fixed point of \( FG \). Since \( z \) is a
fixed point of \( FG \). Thus \( FG(z) = \{z\} = FG(z_1) \), \( FG \) is one-one
which shows \( z \) is a unique fixed point \( FG \).

Let \( FG = GF \), again by applying (7.4.8)
\[ d(Gz, z) \leq H(Gz, z) = H(GF(G(z)), z) = H(FG(G(z)), z) \]
\[ \leq \emptyset(z) - \emptyset(FG(G(z))) \]
\[ = \emptyset(z) - \emptyset(G(FG(z))) \quad [\text{since } FG(z) = \{z\}] \]
\[ \leq \emptyset(z) - \emptyset(G(z)) \]
\[ \leq \emptyset(z) - \emptyset(z) = 0 \]

\[ d(Gz, z) \leq 0. \] Thus \( z \) is a fixed point \( G \).

Now, since \( F \) and \( G \) are continuous and \( G(z) = \{z\} \)

\[ \implies FG(z) = \{z\} = G(z) \quad \] or \( F(z) = \{z\} \)

so that \( z \) is a common fixed point of \( F \) and \( G \).

For uniqueness of \( z \), let \( w \) is another fixed point of \( F \) and \( G \).

i.e. \( \{w\} = Fw = Gw \). By applying (7.4.8) and using (7.4.7)

\[ d(w, z) \leq H(F(w), z) = H(FG(w), z) \leq \emptyset(w) - \emptyset(FG(w)) \]
\[ \leq \emptyset(w) - \emptyset(w) = 0 \]

\[ \implies d(w, z) \leq 0 \] which gives \( w = z \).

Hence \( z \) is a unique fixed point of \( F \) and \( G \).

**Remark**: (i) If \( G = I \) (Identity mapping) in Theorem 5,

we have Theorem (2) of Chandel [1].

(ii) \( P = I \), in Corollary 1, we get Corollary of Chandel [1].