CHAPTER VI

FIXED POINT THEOREMS ON 2-NORMED SPACES

THROUGH G-ITERATES AND ON 2-BANACH SPACES

( 130 - 162 )

A linear space $N$ is called 2-normed space if there is defined a non-negative real valued function denoted by $\| \cdot \|$ on $N$ satisfying the following conditions for all $a, b, c$ of $N$.

(6.1.1) $\| a, b \| = 0$ if and only if $a$ and $b$ are linearly dependent.

(6.1.2) $\| a, b \| = \| b, a \| .

(6.1.3) $\| a, \alpha b \| = |\alpha| \| a, b \|$, $\alpha$ is real.

(6.1.4) $\| a, b+c \| \leq \| a, b \| + \| a, c \| .

A sequence $\{ x_n \}$ in a 2-normed space $N$ is called a Cauchy sequence if there exists $w, z \in N$ such that $w$ and $z$ are linearly independent, the limit $\lim_{n,m \to \infty} \| x_n - x_m, w \| = 0$ and the limit $\lim_{n,m \to \infty} \| x_n - x_m, z \| = 0$. 
A sequence \( \{ x_n \} \) in a 2-normed space is called a convergent sequence if there is \( x \in \mathbb{N} \) such that

\[
\lim_{n \to \infty} \| x_n - x, z \| = 0 \text{ for all } z \in \mathbb{N}.
\]

A 2-normed space in which every Cauchy sequence is convergent sequence is called a 2-Banach space.

Clearly, 2-normed space of dimension different from one becomes a 2-metric space if a three point function is defined on it by \( d(a,b,c) = \| b-a, c-a \| \).

In 1974 Rhoades [1] introduced G-iteration scheme which is more general than that of Mann iteration.

**DEFINITION 1:** Let \( T \) be a self mapping of a Banach space \( X \). The G-iterate process associated by \( T \) is defined in the following manner.

Let \( x_0 \) be in \( X \) and set

\[
x_{n+1} = (\mu_n - \lambda_n) x_n + \lambda_n T x_n + (1 - \mu_n) T x_{n-1},
\]

for \( n \geq 0 \) where \( \{ \lambda_n \} \) and \( \{ \mu_n \} \) satisfy (i) \( \lambda_0 = \mu_0 = 1 \)

(ii) \( 0 < \lambda_n < 1, \mu_n \geq 0, n > 0 \)

(iii) \( \lim_{n \to \infty} \lambda_n = h > 0 \)

(iv) \( \lim_{n \to \infty} \mu_n = 1 \).

**NOTE:** If \( \mu_n = 1 \), the G-iterative process reduces to Mann iteration.
In 1985 Pathak [5] has generalized the concept of G-iterative process is associated by a single mapping to a pair of mappings \( T_1 \) and \( T_2 \) as given below:

**DEFINITION 2:** Let \( T_1 \) and \( T_2 \) be self mappings of a Banach space \( X \).

Then G-iterative process associated by \( T_1 \) and \( T_2 \) are defined in the following process:

For \( x_0 \in X \), set

\[
(6.1.6) \quad x_{2n+1} = \left( \mu_{2n} - \lambda_{2n} \right) x_{2n+1} + \lambda_{2n} T_1 x_{2n} + \left( 1 - \mu_{2n} \right) T_2 x_{2n-1}
\]

and

\[
x_{2n+2} = \left( \mu_{2n+1} - \lambda_{2n+1} \right) x_{2n+1} + \lambda_{2n+1} T_2 x_{2n+1} + \left( 1 - \mu_{2n+1} \right) T_1 x_{2n}, \text{ for } n \geq 0
\]

where \( \{ \lambda_n \} \) and \( \{ \mu_n \} \) satisfying (i), (ii), (iii) and (iv) of definition 1, where \( \alpha < h \leq 1 \).

Pathak [5] proved the following theorem in normed space by using the G-iterative process which was proved earlier by Pachpatte [5] in metric space using same iteration.

**THEOREM A:** Let \( X \) be a closed convex subset of a normed space \( N \).

Let \( T \) be a self mapping of \( X \) satisfying

\[
||Tx-Ty|| \leq q \max \left\{ ||x-y||, \frac{||y-Ty||[1+||x-Tx||]}{1+||x-y||} \right\},
\]

\[
\frac{1}{2} \left\{ \frac{||x-Ty||[1+||x-Tx||+||y-Tx||]}{1+||x-y||} \right\}
\]
For all \( x, y \) in \( X \) where \( 0 < q < 1 \) and \( \{x_n\} \) the sequence \( G \)-iterates associated with \( T \) be same as defined in Definition 1.

If \( \{x_n\} \) converges in \( X \), then it converges to a fixed point of \( T \).

Recently in 1990, Pathak and Dubey [1] have proved the following theorem for 2 mappings in normed space.

**THEOREM B.** Let \( X \) be a closed convex subset of a normed space \( N \). Let \( T_1 \) and \( T_2 \) be two self mappings of \( X \) satisfying:

\[
\| T_1 x - T_2 y \| \leq q \max \left\{ \| x - y \|, \| x - T_1 x \| \right\},
\]

\[
\| y - T_2 y \| \leq \| T_1 x \| + \| y - T_1 x \|. 
\]

For all \( x, y \in X \) where \( 0 < q < 1 \). Let the sequence \( \{x_n\} \) be defined in accordance with \( G \)-iterates associated with \( T_1 \) and \( T_2 \) be same as defined in Definition 2.

If \( \{x_n\} \) converges to \( z \) in \( X \), then \( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

6.2. In this section we extend the theorem A for 2-normed space and also we generalize it for a pair of mappings using \( G \)-iterative scheme.
THEOREM 1: Let $X$ be a closed convex subset of 2-normed space $N$. Let $T$ be a self mapping of $X$ satisfying:

$$(6.2.1) \quad ||Tx-Ty,a|| \leq q \max \left\{ ||x-y,a||, \frac{||y-Ty,a||}{1 + ||x-y,a||} \right\},$$

$$\frac{1}{2} \frac{||x-Ty,a||}{1 + ||x-y,a||}$$

for all $x,y,a$ in $X$, where $0 < q < 1$ and $\{x_n\}$ the sequence of $G$-iterates associated with $T$ given in Definition 1. If $\{x_n\}$ converges in $X$. Then it converges to a fixed point of $T$.

PROOF: Let $z \in X$ satisfying:

$$\lim_{n \to \infty} x_n = z \text{ i.e. } \lim_{n \to \infty} ||x_n - z,a|| = 0$$

since $X$ is a closed convex subset of a 2-normed space $N$ so that by triangle inequality in $X$ and using (6.1.5)

$$||z-Tz,a|| \leq ||z-x_{n+1},a|| + ||Tz-x_{n+1},a||$$
= \| z - x_{n+1},^a \| + \| x_{n+1} - Tz,^a \| \\
\leq \| z - x_{n+1},^a \| + \| (\mu_n - \lambda_n) (x_n - Tz) + \lambda_n (T x_n - Tz) \\
+ (1 - \mu_n) (T x_{n-1} - Tz),^a \| \\
\leq \| z - x_{n+1},^a \| + (\mu_n - \lambda_n) \| x_n - Tz,^a \| \\
+ \lambda_n \| T x_n - Tz,^a \| \\
+ (1 - \mu_n) \| T x_{n-1} - Tz,^a \|. \ldots (i)

By applying (6.2.1), we get

\| T x_n - T x,^a \| \leq q \max \left\{ \| x_n - z,^a \|, \frac{\| z - T z,^a \| [1 + \| x_n - T x_n,^a \|]}{1 + \| x_n - z,^a \|} \right\}

\frac{1}{2} \frac{\| x_n - T z,^a \| + [1 + \| x_n - T x_n,^a \| + \| z - T x_n,^a \|]}{1 + \| x_n - z,^a \|}

\leq q \max \left\{ \| x_n - z,^a \|, \frac{\| z - T z,^a \| [1 + \| x_n - T x_n,^a \|]}{1 + \| x_n - z,^a \|} \right\}

\frac{1}{2} \frac{\| x_n - T z,^a \| [1 + \| z - x_n,^a \| + 2 \| x_n - T x_n,^a \|]}{1 + \| x_n - z,^a \|} \ldots (ii)

by using (i) and (ii) we have

\| z - T z,^a \| \leq \| z - x_{n+1},^a \| + (\mu_n - \lambda_n) \| x_n - T z,^a \|. 
\[ + \lambda_\eta q \max \left\{ |x_n - z, a|, \frac{|z - Tz, a| |1 + |x_n - Tx_n, a||}{|1 + |x_n - z, a||} \right\} \]

\[ \frac{1}{2} \frac{|x_n - Tz, a| |1 + |z - x_n, a|| + 2 |x_n - Tx_n, a||}{|1 + |x_n - z, a||} \}

Again by using (6,1,5), we get

\[ \lambda_\eta(x_n - Tx_n) = (x_n - x_{n+1}) - (1 - \mu_n)(x_n - Tx_{n-1}) \text{ in 2-normed space.} \]

\[ |x_n - Tx_n, a|| \leq \frac{1}{\lambda_\eta} |x_n - x_{n+1}, a|| + \frac{1 - \mu_n}{\lambda_\eta} |Tx_{n-1} - x_n, a|| \ldots \text{(iv)} \]

Now,

\[ |z - Tz, a|| \leq |z - x_n, a|| + |Tx_n - x_n, a|| \]

\[ = |z - x_n, a|| + |x_n - Tx_n, a|| \]

\[ \leq |z - x_n, a|| + \frac{1}{\lambda_\eta} |x_n - x_{n+1}, a|| \]

\[ + \frac{1 - \mu_n}{\lambda_\eta} |Tx_{n-1} - x_n, a||, \text{by (iv)} \ldots \text{(v)} \]

From (iii), (iv) and (v), we have

\[ |z - Tz, a|| \leq |z - x_{n+1}, a|| + (\mu_n - \lambda_\eta)|x_n - Tz, a|| \]

\[ + \lambda_\eta q \max \left\{ |x_n - z, a|, \frac{1}{\lambda_\eta} |x_n - x_{n+1}, a|| + \frac{1 - \mu_n}{\lambda_\eta} |Tx_{n-1} - x_n, a|| \right\} \]

\[ \frac{1 + |x_n - z, a||}{1 + |x_n - z, a||} \]
\[
\frac{1}{2} \left\| x_n - Tz, a \right\| [1+3\left\| x_n - Tz, a \right\|] \\
+ \frac{2}{\lambda_n} \left\| x_n - x_{n+1}, a \right\| + 2 \frac{1-\mu_n}{\lambda_n} \left\| Tx_{n-1} - x_n, a \right\|] \\
\frac{1}{1 + \left\| x_n - z, a \right\|} \right) \\
+ (1-\mu_n) \left\| Tz_{n-1} - Tz, a \right\| \\
\]

Assuming \( n \) tends to infinity and using conditions (iii) and (iv) of Definition 1.

\[
\left\| z - Tz, a \right\| \leq \left\| z - z, a \right\| + (1-h) \left\| z - Tz, a \right\| + \eta q \max \left\{ \left\| z - z, a \right\|, \left\| z - Tz, a \right\| \right\} \\
\left\| z - Tz, a \right\| \left[ 1 + \frac{1}{\eta} \left\| z - z, a \right\| + \frac{1-1}{\eta} \left\| Tz - z, a \right\| \right] \\
1 + \left\| z - z, a \right\| \\
\frac{1}{2} \left\| z - Tz, a \right\| [1+3\left\| z - z, a \right\| + \frac{2}{\eta} \left\| z - z, a \right\| + \frac{2(1-1)}{\eta} \left\| Tz - z, a \right\|] \\
1 + \left\| z - z, a \right\| \\
\Rightarrow \left\| z - Tz, a \right\| \leq (1-h+\eta q) \left\| z - Tz, a \right\|, \text{ since } q < 1 \\
< (1-h+\eta) \left\| z - Tz, a \right\| = \left\| z - Tz, a \right\| \\
a contradiction which implies \left\| z - Tz, a \right\| = 0 \text{ or } z = Tz.

Thus \( z \) is a fixed point of \( T \).

For uniqueness of \( z \), let \( w \) is another fixed point of \( T \) such that \( z \neq w \),
\[ \| z - w, a \| = \| Tz - Tw, a \| \leq q \max \left\{ \| z - w, a \|, \frac{1}{2} \| z - w, a \| \right\} \]

\[ \frac{\| w - w, a \| \left[ 1 + \| z - w, a \| \right]}{1 + \| z - w, a \|}, \]

\[ \frac{\frac{1}{2} \| z - w, a \| \left[ 1 + \| z - z, a \| + \| z - w, a \| \right]}{1 + \| z - w, a \|}, \]

or \[ \| z - w, a \| \leq q \max \left\{ \| z - w, a \|, 0, \frac{1}{2} \| z - w, a \| \right\} \]

\[ \leq q \| z - w, a \| < \| z - w, a \|, \text{a contradiction.} \]

Thus \( z = w \).

We generalise the above result for two mappings.

**Theorem 2**: Let \( X \) be a closed convex subset of a 2-normed space \( N \). Let \( T_1 \) and \( T_2 \) be two self mappings of \( X \) satisfying:

\[ \| I_1 x - T_2 y, a \| \leq q \max \left\{ \| x - y, a \|, \right\} \]

\[ \frac{\| y - T_2 y, a \| \left[ 1 + \| x - T_1 x, a \| \right]}{1 + \| x - y, a \|}, \]

\[ \frac{\frac{1}{2} \| x - T_2 y, a \| \left[ 1 + \| x - T_1 x, a \| + \| y - T_1 x, a \| \right]}{1 + \| x - y, a \|}, \]

For all \( x, y, a \in X \) with \( 0 < q < 1 \). The sequence \( \{x_n\} \) be defined in accordance with the \( G \)-iterates associated with \( T_1 \) and \( T_2 \) satisfying the same as given in Definition 2.
If \( \{x_n\} \) converges to \( z \) in \( X \). Then \( z \) is the common fixed point of \( T_1 \) and \( T_2 \).

**Proof:** Let \( z \in X \) such that

\[
\lim_{n \to \infty} x_n = z \quad \text{i.e.} \quad \lim_{n \to \infty} ||x_n - z, a|| = 0.
\]

Since \( X \) is a closed convex subset of a 2-normed space \( N \) so that by triangle inequality in \( X \) and using (6.1.6), we get

\[
||z - T_2 z, a|| \leq ||z - x_{2n+1}, a|| + ||T_2 z - x_{2n+1}, a||
\]

\[
= ||z - x_{2n+1}, a|| + ||x_{2n+1} - T_2 z, a||
\]

\[
\leq ||z - x_{2n+1}, a|| + ||(\mu_{2n} - \lambda_{2n})(x_{2n} - T_2 z) + \lambda_{2n}(T_1 x_{2n} - T_2 z) + (1 - \mu_{2n})(T_2 x_{2n+1} - T_2 z), a||
\]

\[
\leq ||z - x_{2n+1}, a|| + (\mu_{2n} - \lambda_{2n}) ||x_{2n} - T_2 z, a||
\]

\[
+ \lambda_{2n} ||T_1 x_{2n} - T_2 z, a|| + (1 - \mu_{2n}) ||T_2 x_{2n+1} - T_2 z, a||
\]

\[
\leq ||T_1 x_{2n} - T_2 z, a|| \leq q \max \left\{ ||x_{2n} - z, a||, \frac{||z - T_2 z, a||[1 + ||x_{2n} - T_1 x_{2n}, a||]}{1 + ||x_{2n} - z, a||} \right\}
\]

\[
\leq \frac{1}{2} \frac{||x_{2n} - T_2 z, a||[1 + ||x_{2n} - T_1 x_{2n}, a||] + ||z - T_1 x_{2n}, a||}{1 + ||x_{2n} - z, a||}
\]

By applying (6.2.2), we have
\[ \begin{align*}
&\leq q \max \left\{ \left| \left| x_{2n} - z, a \right| \right|, \left| \left| z - T_2 z, a \right| \right| \left[ 1 + \left| x_{2n} - T_1 x_{2n}, a \right| \right] \right\} \\
&\quad \cdot \frac{1}{1 + \left| x_{2n} - z, a \right|} \\
&\quad \cdot \frac{\frac{1}{2} \left| \left| x_{2n} - T_2 z, a \right| \right| \left[ 1 + \left| x_{2n} - z, a \right| + 2 \left| z - T_1 x_{2n}, a \right| \right]}{1 + \left| x_{2n} - z, a \right|} \\
&\quad \cdot \ldots \ldots (2)
\end{align*} \]

From (1) and (2), we have

\[ \left| \left| z - T_2 z, a \right| \right| \leq \left| \left| z - x_{2n+1}, a \right| \right| + \left( \mu_{2n} - \lambda_{2n} \right) \left| \left| x_{2n} - T_2 z, a \right| \right| \]

\[ + \lambda_{2n} q \max \left\{ \left| \left| x_{2n} - z, a \right| \right|, \left| \left| z - T_2 z, a \right| \right| \left[ 1 + \left| x_{2n} - T_1 x_{2n}, a \right| \right] \right\} \cdot \frac{1}{1 + \left| x_{2n} - z, a \right|} , \]

\[ \left| \left| z - T_2 z, a \right| \right| \left[ 1 + \left| x_{2n} - z, a \right| + \left| z - T_1 x_{2n}, a \right| \right] \]

\[ = \frac{\frac{1}{2} \left| \left| x_{2n} - T_2 z, a \right| \right| \left[ 1 + \left| x_{2n} - z, a \right| + \left| z - T_1 x_{2n}, a \right| \right]}{1 + \left| x_{2n} - z, a \right|} \]

\[ + \left( 1 - \lambda_{2n} \right) \left| \left| T_2 x_{2n-1} - T_2 z, a \right| \right| \ldots \ldots (3) \]

Again by using (6.1.6)

\[ \lambda_{2n} \left( x_{2n} - T_1 x_{2n} \right) = \left( x_{2n} - x_{2n+1} \right) - \left( 1 - \mu_{2n} \right) \left( x_{2n} - T_2 x_{2n-1} \right) , \]

in 2-normed space, we get

\[ \left| \left| x_{2n} - T_1 x_{2n-1}, a \right| \right| \leq \frac{1}{\lambda_{2n}} \left| \left| x_{2n} - x_{2n+1}, a \right| \right| \]

\[ + \frac{1 - \mu_{2n}}{\lambda_{2n}} \left| \left| x_{2n} - T_2 x_{2n-1}, a \right| \right| \ldots \ldots (4) \]

and
\[ ||z - T_1 x_{2n}, a|| \leq ||z - x_{2n}, a|| + ||T_1 x_{2n} - x_{2n}, a|| \]

\[ = ||z - x_{2n}, a|| + ||x_{2n} - T_1 x_{2n}, a|| \]

\[ \leq ||z - x_{2n}, a|| + \frac{1}{\lambda_{2n}} ||x_{2n} - x_{2n+1}, a|| \]

\[ + \frac{1 - \mu_{2n}}{\lambda_{2n}} ||x_{2n} - T_1 x_{2n-1}, a|| \] by using (3) ...(5)

By inequalities (3), (4) and (5), we get

\[ ||z - T_2 z, a|| \leq ||z - x_{2n+1}, a|| + (\mu_{2n} - \lambda_{2n}) ||x_{2n} - T_2 z, a|| \]

\[ + \lambda_{2n} \max \{ ||x_{2n} - z, a|| \} , \]

\[ ||z - T_2 z, a|| \left[ 1 + \frac{1}{\lambda_{2n}} ||x_{2n} - x_{2n+1}, a|| \right] \]

\[ + \frac{1 - \mu_{2n}}{\lambda_{2n}} ||x_{2n} - T_2 x_{2n-1}, a|| \]

\[ - \frac{1}{2} \left[ ||x_{2n} - T_2 z, a|| \right] \left( 1 + 3 ||z - x_{2n}, a|| + \frac{2}{\lambda_{2n}} ||x_{2n} - x_{2n+1}, a|| \right) \]

\[ + 2 \frac{1 - \mu_{2n}}{\lambda_{2n}} ||x_{2n} - T_2 x_{2n-1}, a|| \]

\[ - \frac{1}{2} ||x_{2n} - T_2 z, a|| \left( ||x_{2n} - x_{2n+1}, a|| \right) \]

\[ + (1 - \lambda_{2n}) ||T_2 x_{2n-1} - T_2 z, a|| \]
On letting $n$ tend to infinity and using (iii) and (iv) of Definition 1.

$$||z-T_2z, a|| \leq ||z, a|| + (1-h)||z-T_2z, a|| + hq \max \{||z, a||,$$

$$\frac{||z-T_2z, a||}{1+||z-z, a||} \left[ 1 + \frac{1}{h} ||z-z, a|| + \frac{1-1}{h} ||z-T_2z, a|| \right] \right},$$

$$\frac{1}{2} \left\{ ||z-T_2z, a|| \left[ 1 + 3 ||z-z, a|| + \frac{2}{h} ||z-z, a|| + 2 \frac{1-1}{h} ||z-T_2z, a|| \right] \right\}\right),$$

$$+ (1-h)||T_2z-T_2z, a||$$

$$\leq 0 + (1-h)||z-T_2z, a|| + o + hq \max \{0, ||z-T_2z, a||,}$$

$$\frac{1}{2} ||z-T_2z, a|| \} + o$$

$$\leq (1-h+hq) ||z-T_2z, a|| < (1-h)||z-T_2z, a||,$$

since $q < 1$, a contradiction which implies $||z-T_2z, a|| = 0$

i.e. $z = T_2z$.

Similarly it can be shown that $T_1z = z$.

Thus $z$ is a common fixed point of $T_1$ and $T_2$.

To prove uniqueness, suppose $w$ is another fixed point $T_1$ and $T_2$ such that $z \neq w, ||T_1w - w, a|| = 0$ and $||T_2w - w, a|| = 0$, $\forall a \in X$. 
By applying (6.2.2), we have

\[ ||z-w, a|| = || T_1 z - T_2 w, a|| \]

\[ \leq q \max \{ ||z-w, a||, o, \frac{1}{2} ||z-w, a|| \} \]

\[ \leq q ||z-w, a|| < ||z-w, a|| . \]

It follows \( z = w \).

Thus \( z \) is unique common fixed point of \( T_1 \) and \( T_2 \).

**Remark:** (i) \( T_1 = T_2 \) in Theorem 2, we have Theorem 1.

(ii) If \( T_1 = T_2 \) and normed space in place of 2-normed space we get Theorem A.

6.3. This section is completely devoted to extend the Theorem B. For normed space of section 6.1, for three mappings on 2-normed space, using G-iterative scheme.

**Theorem 3.** Let \( X \) be a closed convex subset of a 2-Normed space \( N \) and Let \( T_1, T_2 \) and \( P \) be three self mappings of \( X \) satisfying:

\[ (6.3.1) \quad ||T_1 Px - T_2 Py, a|| \leq q \max \{ ||x-y, a||, ||x-T_1 Px, a|| , \]

\[ ||y-T_2 Py, a||, ||x-T_2 Py, a|| + ||y-T_1 Px, a|| \} \]

for all \( x, y, a \in X \) where \( 0 < q < 1 \). Let the sequence \( \{ x_n \} \) be defined in accordance with \( G \) iterates associated with \( T_1 P \) and \( T_2 P \) as given below:
For $x_0 \in X$,

$$
(6.3.2) \quad x_{2n+1} = (\mu_{2n} - \lambda_{2n})x_{2n} + \lambda_{2n}T_1^px_{2n}
+ (1-\mu_{2n})T_2^px_{2n-1}
$$

$$
(6.3.3) \quad x_{2n+2} = (\mu_{2n+1} - \lambda_{2n+1})x_{2n+1} + \lambda_{2n+1}T_1^px_{2n+1}
+ (1-\mu_{2n+1})T_1^px_{2n}
$$

for $n \geq 0$, where $\{\lambda_n\}$ and $\{\mu_n\}$ satisfying conditions (i), (ii), (iii) and (iv) of Definition 1. If $\{x_n\}$ converges to $z$ in $X$. Then $z$ is a common fixed point of $T_1$ and $T_2$. Further, if $P$ commutes with $T_1$ or $T_2$ and $0 < q < 1/2$, then $T_1$ and $T_2$ and $P$ have a unique common fixed point.

**PROOF**: First, we have to show that

$$
||T_1^px_{2n} - x_{2n+1}, a|| \text{ and } ||T_2^px_{2n-1} - x_{2n}, a||
$$

are bounded by using the triangle inequality in 2-normed

$$
||T_1^px_{2n} - x_{2n+1}, a|| \leq ||T_1^px_{2n} - x_{2n}, a|| + ||x_{2n+1} - x_{2n}, a||
$$

$$
= ||T_1^px_{2n} - x_{2n}, a|| + ||x_{2n} - x_{2n+1}, a||
$$
Let \( \lim_{n \to \infty} x_n = z \), where \( z \in X \). Then \( \| x_{2n} - x_{2n+1}, a \| \to 0 \) as \( n \to \infty \), so that

\[
\lim_{n \to \infty} \| T_1^{P} x_{2n} - x_{2n+1}, a \| \leq \lim_{n \to \infty} \| T_1^{P} x_{2n} - x_{2n}, a \| \quad \ldots (1)
\]

From condition (6.3.2) and using the concept of 2-mormed space

\[
\lambda_{2n} \| T_1^{P} x_{2n} - x_{2n}, a \| = \| (x_{2n+1} - x_{2n}) - (1 - \mu_{2n})(T_2^{P} x_{2n} - x_{2n}), a \|
\]

\[\leq \| x_{2n+1} - x_{2n}, a \| + (1 - \mu_{2n}) \| T_2^{P} x_{2n} - x_{2n}, a \| \]

Assuming \( n \) tends to infinity and using (iii) and (iv) of Definition 1, we have

\[
\lim_{n \to \infty} \lambda_{2n} \| T_1^{P} x_{2n} - x_{2n}, a \| \leq 0 \quad \ldots (2)
\]

By inequalities (1) and (2), we get

\[
\lim_{n \to \infty} \| T_1^{P} x_{2n} - x_{2n+1}, a \| \leq \lim_{n \to \infty} \| T_2^{P} x_{2n} - x_{2n}, a \| = 0
\]

Thus \( \| T_1^{P} x_{2n} - x_{2n+1}, a \| \) is bounded.

Similarly,

\[
\| T_2^{P} x_{2n} - x_{2n}, a \| \leq \| T_2^{P} x_{2n} - x_{2n-1}, a \| + \| x_{2n-1} - x_{2n}, a \|
\]

on letting \( n \to \infty \)

\[
\lim_{n \to \infty} \| T_2^{P} x_{2n} - x_{2n-1}, a \| = 0 \quad \ldots (3)
\]
Using condition (6.3.3), we have

\[ x_{2n+1} \leq |T_2^P x_{2n+1} - x_{2n+1}, a| \leq |T_2^P x_{2n} - x_{2n+1}, a| + (1 - \mu_{2n+1}) |T_2^P x_{2n} - x_{2n+1}, a| \]

Letting \( n \) tend to infinity and using (iii) and (iv) of Definition 1, we have

\[ \lim_{n \to \infty} \left( |T_2^P x_{2n+1} - x_{2n+1}, a| \right) \leq |z - z, a| \]

\[ + (1 - \lim_{n \to \infty} |T_2^P x_{2n} - x_{2n+1}, a| \]

i.e. \( \lim_{n \to \infty} |T_2^P x_{2n+1} - x_{2n+1}, a| \leq 0 \).

Now replacing \( n \) by \( n - 1 \)

\[ \lim_{n \to \infty} |T_2^P x_{2n-1} - x_{2n-1}, a| \leq 0 \quad \cdots \quad (4) \]

By relation (3) and (4), we get

\[ \lim_{h \to \infty} |T_2^P x_{2n+1} - x_{2n+1}, a| \leq 0 \]

Let \( z \notin x \), such that \( \lim_{n \to \infty} x_n = z \) i.e. \( \lim_{n \to \infty} |x_n - z, a| = 0 \).

Now we show that \( z \) is common fixed point of \( T_1^P \) and \( T_2^P \).
If we consider $z \neq T_2Pz$, then

$$||z-T_2Pz, a|| \leq ||z-x_{2n+1}, a|| + ||x_{2n+1} - T_2Pz, a||$$

$$+ (\mu_{2n} - \lambda_{2n}) ||x_{2n} - T_2Pz, a||$$

$$+ \lambda_{2n} ||T_1Px_{2n} - T_2Pz, a||$$

, by (6.3.2) ... (5)

By applying (6.3.1),

$$||T_1Px_{2n} - T_2Pz, a|| \leq q \max \left\{ ||x_{2n} - z, a||, ||x_{2n} - T_1Px_{2n}, a|| \right\}$$

$$||z - T_2Pz, a||, \left[ ||x_{2n} - T_2Pz, a|| + ||T_1Px_{2n}, a|| \right] \right\}$$

$$\leq q \max \left\{ ||x_{2n} - z, a||, ||x_{2n} - T_1Px_{2n}, a||, ||z - T_2Pz, a|| \right\}$$

$$\left[ ||x_{2n} - T_2Pz, a|| + ||z - x_{2n}, a|| + ||T_1Px_{2n} - x_{2n}, a|| \right] \right\}$$

... (6)

From inequalities (5) and (6),

$$||z-T_2Pz, a|| \leq ||z-x_{2n+1}, a|| + (\mu_{2n} - \lambda_{2n}) ||x_{2n} - T_2Pz, a||$$

$$+ (1-\mu_{2n}) ||T_2Px_{2n-1} - T_2Pz, a|| + \lambda_{2n} q \max$$

$$\left\{ ||x_{2n} - z, a||, ||x_{2n} - T_1Px_{2n}, a||, ||z - T_2Pz, a||, \right\}$$

$$\left[ ||x_{2n} - T_2Pz, a|| + ||z - x_{2n}, a|| + ||T_1Px_{2n} - x_{2n}, a|| \right]$$

....... .... (7)
By condition (6.3.3), we get

\[ \|x_{2n} - T_{2}P_{2n}x_{2n}, a\| \leq \frac{1}{\lambda_{2n}} \|x_{2n+1} - x_{2n}, a\| \]

\[ + \frac{(1-\mu_{2n})}{\lambda_{2n}} \|T_{2}P_{2n-1}x_{2n}, a\| \] \dots (8)

By inequalities (7) and (8)

\[ \|z-T_{2}P_{z}, a\| \leq \|z-x_{2n+1}, a\| + (\mu_{2n} - \lambda_{2n}) \|x_{2n} - T_{2}P_{z}, a\| \]

\[ + (1 - \mu_{2n}) \|T_{2}P_{2n-1} - T_{2}P_{z}, a\| \]

\[ + \lambda_{2n} \cdot \max \{ \|x_{2}, a\|, \frac{1}{\lambda_{2n}} \|x_{2n} - x_{2n+1}, a\| \]

\[ + \frac{1-\mu_{2n}}{\lambda_{2n}} \|T_{2}P_{2n-1} - x_{2n}, a\| \}, \|z-T_{2}P_{z}, a\|, \]

\[ \|x_{2n} - T_{2}P_{z}, a\| + \|z-x_{2n}, a\| + \frac{1}{\lambda_{2n}} \|x_{2n} - x_{2n+1}, a\| \]

\[ + \frac{1-\mu_{2n}}{\lambda_{2n}} \|T_{2}P_{2n-1} - x_{2n}, a\| \} \}

Assuming n tends to infinity and using (iii) and (iv) of Definition 1.

\[ \|z-T_{2}P_{z}, a\| \leq \|z-z, a\| + (1-h) \|z-T_{2}P_{z}, a\| + (1-1) \|T_{2}P_{z} - T_{2}P_{z}, a\| \]

\[ + h \cdot \max \{ \|z-z, a\|, \frac{1}{h} \|z-z, a\| + \frac{1-1}{h} \|T_{2}P_{z} - z, a\| \}

\[ , \|z-T_{2}P_{z}, a\| \}, \|z-T_{2}P_{z}, a\| + \|z-z, a\| + \frac{1}{h} \|z-z, a\| \]

\[ + \frac{1-1}{h} \|T_{2}P_{z} - z, a\| \} \}
\[ \leq \alpha + (1-h) ||z-T_2Pz, a|| + \alpha + hq \max \left\{ \frac{\alpha}{q}, ||z-T_2Pz, a|| \right\} \]

\[ \leq (1-h-hq)||z-T_2Pz, a|| < (1-hh) ||z-T_2Pz, a|| \]

a contradiction which implies \( ||z-T_2Pz, a|| = \alpha \) gives \( z = T_2Pz \).

Similarly it can be shown easily that \( T_1P = z \). Hence \( z \) is common fixed point of \( T_1P \) and \( T_2P \).

For uniqueness of \( z \), suppose \( z' \) is another fixed point of \( T_1P \) and \( T_2P \) such that \( z \neq z' \), by applying (6.3.1)

\[ ||z-z', a|| = ||T_1Pz - T_2Pz, a|| \]

\[ \leq q \max \left\{ ||z-z', a||, ||z-T_1Pz, a|| \right\} \]

\[ ||z-T_2Pz', a|| \leq q \left\{ ||z-T_2Pz', a|| + ||z-T_2Pz, a|| \right\} \]

\[ \leq q \max \left\{ ||z-z', a||, ||z-z, a||, ||z'-z', a|| \right\} \]

\[ \leq 2q ||z-z', a|| < ||z-z', a|| \left( \cdot \cdot \cdot \alpha < q < \frac{1}{2} \right) \]

a contradiction which shows \( z = z' \).

Suppose \( P \) commutes with \( T_1 \) i.e. \( T_1P = PT_1 \). By applying (6.3.1)

\[ ||Pz-z, a|| = ||PT_1Pz - T_2Pz, a|| = ||T_1PPz - T_2Pz, a|| \]
\[
\leq q \max \left\{ ||Pz-z, a||, ||Pz-Pz, a||, o ||z-Pz, a|| \right\}
\]
or \(1-q) \ ||Pz-z, a|| \leq o \). Since \(1-q \neq o\) implies \(Pz = z\). Similarly, on considering \(T_2 P = PT_2\), we can show that \(Pz = z\).

Now \(T_1 Pz = z, T_2 Pz = z\) and \(Pz = z \implies T_1 z = T_2 z = z\)

thus \(z\) is a common fixed point of \(T_1, T_2\) and \(P\).

Uniqueness of \(z\) follows easily from (6.3.1).

**Remark:** (i) If \(P = I\), where \(I\) is identity mapping and normed space in place of 2-normed space, we get theorem A.

(ii) If \(T_1 = T_2 = I\) and \(P = I\), we have corollary of Pathak and Maity [1] in Banach space.

(iii) If \(P = I\), we get a parallel result of Theorem B for 2-normed spaces.

6.4. Sharma and Bajaj [1] have proved the following theorem for Banach space.

**Theorem C.** Let \(K\) be a closed and convex subset of a Banach space \(X\). Let \(F : K \rightarrow K, G : K \rightarrow K\) satisfying the following conditions:

(a) \(F\) and \(G\) commutes
(b) \( F^2 = I \) and \( G^2 = I \), where \( I \) denotes the Identity mapping.

(c) \[ ||F x - F y|| \leq \alpha ||G x - G y|| + \beta(||G x - F x|| + ||G y - F y||). \]

For every \( x, y \in K \) and \( \alpha, \beta \) and \( \alpha + 4\beta < 2 \). Then there exists at least one fixed point \( x_0 \in K \) such that \( F(x_0) = G(x_0) \). Further if \( \alpha \leq 1 \), then \( x_0 \) is the unique fixed point of \( F \) and \( G \).

In this section we generalize above theorem for 2-Banach space and obtained unique common fixed point theorems for two and three mappings.

**THEOREM 4**: Let \( F \) and \( G \) be two mappings of 2-Banach space into itself satisfying the following conditions:

1. \( F \) and \( G \) commutes i.e. \( FG = GF \).
2. \( F^2 = I \) and \( G^2 = I \) where \( I \) denotes the Identity mapping.
3. \[ ||F x - F y, a|| \leq \alpha ||G x - G y, a|| + \beta(||G x - F x, a|| + ||G y - F y, a||) \]
   \[ + \gamma(||G x - F, a|| + ||G y - F, a||) \]
for every \( x, y, a \in X \) and \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 4\beta + 2\gamma < 2 \).
and \( a + 2\gamma < 1 \). Then \( F \) and \( G \) have a unique common fixed point.

**Proof:** From the conditions (6.4.1) and (6.4.2), \((FG)^2 = I\).

Let \( x_0 \) be an arbitrary point in \( X \). Suppose \( H = 1/2 \) (\( I + FG \)), put \( y = Hx \), \( z = FGy \) and \( u = 2\gamma - z \).

By applying (6.4.3) and using (6.4.1) and (6.4.2), we have,

\[
||z-x,a|| = ||FGy-x,a|| = ||FGy - F^2x,a|| = ||FGy - FFx, a||
\]

\[
\leq \alpha \left( ||G^2y - GFx,a|| + \beta[||G^2y - FGy,a|| + ||GFx - F^2x,a||] \right)
\]

\[
+ \gamma \left( ||G^2y - F^2x,a|| + ||GFx - FGy,a|| \right)
\]

\[
\leq \alpha \left( ||y-GFx,a|| + \beta[||y-FGy,a|| + ||GFx-x,a||] \right)
\]

\[
+ \gamma \left( ||y-x,a|| + ||GFx-FGy,a|| \right)
\]

\[
= ||\frac{1}{2}(x+FGx)-FGx,a|| + \beta[||y-FGy,a|| + ||x-FGx,a||]
\]

\[
+ \gamma \left[ \frac{1}{2} (x + FGx) - x,a || + ||FGx - \frac{1}{2}FG(x+FGx),a || \right]
\]

or \( ||z-x,a|| \leq \alpha/2 ||x-FGx,a|| + \beta[||y-FGy,a|| + ||x-FGx,a||]
\]

\[
+ \gamma[||x-FGx,a||] \quad \ldots \quad (1)
\]

Again by applying (6.4.3) and using (6.4.1) and (6.4.2)

\[
||u-x,a|| = ||2\gamma - z,a|| = 2[\frac{1}{2} (x+FGx) - FGy-x,a|| = ||FGx-FGy,a||
\]
\[ \leq \alpha \left[ |G^2 x-G^2 y, a| + \beta \left[ |G^2 x-FGx, a| + |G^2 y-FGy, a| \right] \right] \\
+ \gamma \left[ |G^2 x-FGx, a| + |G^2 y-FGx, a| \right] \]

or

\[ |u-x, a| \leq \alpha/2 \left| \left| x-FGx, a \right| + \beta \left| \left| x-FGx, a \right| + \left| y-FGy, a \right| \right| \]

\[ + \gamma \left| \left| x-FGx, a \right| \right| \ldots \quad (2) \]

by (1) and (2) and using concept of 2-Banach space

\[ |z-u, a| \leq |z-x, a| + |u-x, a| \]

\[ = \alpha \left| \left| x-FGx, a \right| + 2\beta \left| \left| x-FGx, a \right| + \left| y-FGy, a \right| \right| \]

\[ + 2\gamma \left| \left| x-FGx, a \right| \right| \ldots \quad (3) \]

Now,

\[ |z-u, a| = |z-(2y-z), a| = 2|z-y, a| = 2\left| FGy-y, a \right| \ldots (4) \]

by using (3) and (4), we have

\[ 2\left| y-FGy, a \right| \leq \alpha \left| \left| x-FGx, a \right| + 2\beta \left| \left| x-FGx, a \right| + \left| y-FGy, a \right| \right| \]

\[ + 2\gamma \left| \left| x-FGx, a \right| \right| \]
\[ \| y-FGy, a \| \leq \left( \frac{\alpha+2\beta+2\gamma}{2(1-\beta)} \right) \| x - FGx, a \| \]

= \| x - FGx, a \| \ldots (5)

where \( q = \frac{\alpha+2\beta+2\gamma}{2(1-\beta)} < 1 \)

\[ \Rightarrow \alpha+4\beta+2\gamma < 2 \]

We have to show that \( \{ H^n x \} \) is a Cauchy sequence

\[ \| H^2 x - Hx, a \| = \| Hy-y, a \| = \| 1/2(I+FG)y-y, a \| = 1/2 \| FGy-y, a \| \]

\[ \leq 1/2 q \| x - FGx, a \| = q \| Hx-x, a \| \]

or \[ \| H^2 x - Hx, a \| \leq q \| Hx-x, a \| , \]

Therefore \( \{ H^n x \} \) is a Cauchy sequence and it converges to some \( x \in X \) i.e. \( \lim_{n \to \infty} H^n x = x \), which implies \( Hx = x \) or \( x \)

is a fixed point of \( H \).

But \( H = 1/2 (I + FG) \) so that \( Hx = 1/2 x + 1/2 FGx \)

this implies \( FGx = x \) \[ \ldots (7) \]

\( \Rightarrow Fx = F(FGx) = F^2 Gx = Gx \), using (7) \[ \ldots (8) \]

Again, by applying (6.4.3) and using (7),(8), we get,

\[ \| x - Fx, a \| = \| F^2 x - Fx, a \| \]
\[ \leq \alpha \left| \left| \text{GF} x_o - G x_o, a \right| + \beta \left( \left| \left| \text{GF} x_o - F^2 x_o, a \right| + \left| G x_o - F x_o, a \right| \right) \right| \\
+ \gamma \left( \left| \left| \text{GF} x_o - F x_o, a \right| + \left| G x_o - F^2 x_o, a \right| \right) \right| \]

or \[ \left| \left| x_o - F x_o, a \right| \right| \leq (\alpha + 2\gamma) \left| \left| x_o - F x_o, a \right| \right| < \left| \left| x_o - F x_o, a \right| \right| \]

This contradiction implies \( x_o = F x_o \) and hence

\[ x_o = F x_o = G x_o \ldots \ldots \quad (9) \]

Finally to show that \( x_o \) is unique, suppose \( y_o \) is another common fixed point of \( F \) and \( G \) such that \( x_o \neq y_o \).

by applying (6.4.3) and using (1), (2), (8) and (9), we have

\[ \left| \left| x_o - y_o, a \right| \right| = \left| \left| F^2 x_o - F G x_o, a \right| \right| \]

\[ \leq \alpha \left| \left| \text{GF} x_o - G^2 y_o, a \right| \right| \]

\[ + \beta \left( \left| \left| \text{GF} x_o - F^2 x_o, a \right| + \left| G^2 y_o - F G y_o, a \right| \right) \right| \]

\[ + \gamma \left( \left| \left| \text{GF} x_o - F G y_o, a \right| + \left| G^2 y_o - F^2 x_o, a \right| \right| \right) \]

\[ \leq \alpha \left| \left| x_o - y_o, a \right| \right| + \beta \left( \left| \left| x_o - x_o, a \right| + \left| y_o - y_o, a \right| \right| \right) \]

\[ + \gamma \left( \left| \left| x_o - y_o, a \right| + \left| y_o - x_o, a \right| \right| \right) \]
\( \leq (\alpha + 2\gamma) \|x_o - y_o, a\| < \|x_o - y_o, a\| \), a contradiction which gives \( x_o = y_o \). Hence \( x_o \) is a unique common fixed point of \( F \) and \( G \).

**Remark**: (i) On taking \( \gamma = 0 \) and Banach space in place of 2-Banach space in Theorem 4 we get Theorem A of Sharma and Bajaj [1].

(ii) If \( G = I \) and \( \gamma = 0 \) and Banach space in place of 2-Banach space in Theorem 4 we have Theorem of Iseki [1].

(iii) If \( \beta = \gamma = 0 \) and \( X \) is Banach space we get Theorem of Khan [3].

**Theorem 5**: Let \( X \) be a 2-Banach space and \( E, F \) and \( G \) are self maps of \( X \) satisfying the following conditions:

(6.4.4) \( E^2 = I, F^2 = I, G^2 = I \), where \( I \) is an Identity mapping.

(6.4.5) \( EF = FE, FG = GF, EG = GE \).

(6.4.6) \( \|Ex - Ey, a\| \leq \alpha \|FGx - FGy, a\| \)

\[ + \beta (\|FGx - Ex, a\| + \|FGy - Ey, a\|) \]

\[ + \gamma (\|FGx - Ey, a\| + \|FGy - Ex, a\|) \]

\[ + \delta \frac{\|FGx - Ex, a\| \cdot \|FGy - Ey, a\|}{\|FGx - FGy, a\|} \]
for every \( x, y, a \in X \); \( \alpha, \beta, \gamma, \delta \geq 0 \), \( \alpha + 4\beta + 2\gamma + 4\delta < 2 \) and \( \alpha + 2\gamma < 1 \). Then \( E, F \) and \( G \) have a unique common fixed point.

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \). Let \( H = \frac{1}{2}(I + EFG) \), put \( y = Hx \), \( z = EFGy \) and \( u = 2y - z \), and by using (6.4.4) and (6.4.5),

\[
(EFG)^2 = I = (FG)^2 = (GE)^2 = (EF)^2 \quad \ldots \quad (1)
\]

Applying (6.4.6) and using (6.4.4), (6.4.5) and (1) we have

\[
||z - x, a|| = ||EFGy - x, a|| + ||EFGy - E^2x, a|| + ||EFGy - EEx, a||
\]

\[
\leq \alpha ||(FG)^2y - FGEx, a|| + \\
+ \beta [||EFGy, a|| + ||FGEx - E^2x, a||] \\
+ \gamma [||FGEx - EFGy, a|| + ||FGEx - EEx, a||] \\
+ \delta \frac{||FGy - EFGy, a|| + ||FGEx - E^2x, a||}{||(FG)^2y - FGEx, a||}
\]

\[
\leq \alpha ||y - EFGx, a|| + \beta [||y - EFGy, a|| + ||EFGy - x, a||] \\
+ \gamma [||y - x, a|| + ||EFGy - EFGx, a||] \\
+ \delta \frac{||y - EFGy, a|| + ||FGEx - x, a||}{||y - EFGx, a||}
\]
\[= \alpha \| \frac{1}{2} (x + 2F \cdot x) - 2F \cdot x, a \| + \beta \| y - 2F \cdot y, a \| + \| x - 2F \cdot x, a \| \]

\[+ \gamma \| \frac{1}{2} (x + 2F \cdot x) - x, a \| + \| 2F \cdot x - 2F \cdot x, a \| \]

\[+ \delta \| y - 2F \cdot y, a \| \cdot \| x - 2F \cdot x, a \| \]

After solving, we get

\[\| z - x, a \| \leq \alpha/2 \| x - 2F \cdot x, a \| + \beta \| y - 2F \cdot y, a \| + \| x - 2F \cdot x, a \| \]

\[+ \gamma \| x - 2F \cdot x, a \| + 2\delta \| y - 2F \cdot y, a \|.\]

\[\leq (\alpha/2 + \beta + \gamma) \| x - 2F \cdot x, a \| + (\beta + 2\delta) \| y - 2F \cdot y, a \|.\]

Similarly, by applying (6.4.6) and using (6.4.4), (6.4.5) and (1), we have

\[\| u - x, a \| = \| 2y - z - x, a \| = \| 2Hx - 2F \cdot y, a \| = \| 2F \cdot x - 2F \cdot x, a \|\]

\[\leq \alpha \| (F \cdot G)^2 x - (F \cdot G)^2 y, a \|\]

\[+ \beta \| (F \cdot G)^2 x - 2F \cdot x, a \| + \| (F \cdot G)^2 y - 2F \cdot y, a \|\]

\[+ \gamma \| (F \cdot G)^2 x - EFG \cdot y, a \| + \| (F \cdot G)^2 y - EFG \cdot x, a \|\]

\[+ \delta \| (F \cdot G)^2 x - EFG \cdot x, a \| \cdot \| (F \cdot G)^2 y - EFG \cdot y, a \| \]

\[\| (F \cdot G)^2 x - (F \cdot G)^2 y, a \| \]
\[ \leq \alpha \| x-y, a \| + \beta \left[ \| x-EFGx, a \| + \| y-EFGy, a \| \right] + \gamma \left[ \| x-EFGy, a \| + \| y-EFGx, a \| \right] + \delta \frac{\| x-EFGx, a \| \cdot \| y-EFGy, a \|}{\| x-y, a \|} \]

or \[ \| u-x, a \| \leq (\alpha/2 + \beta + \gamma) \| x-EFGx, a \| + (\beta + 2\delta) \| y-EFGy, a \| \ldots (3) \]

Now, \[ \| z-u, a \| \leq \| z-x, a \| + \| u-x, a \| \], so that by using (2) and (3), we have

\[ \| z-u, a \| \leq 2(\alpha/2 + \beta + \gamma) \| x-EFGx, a \| + 2(\beta + 2\delta) \| y-EFGy, a \| \ldots (4) \]

Again, \[ \| z-u, a \| = 2\| z-y, a \| = 2\| y-EFGy, a \| \] \ldots \ldots \ldots (5)

By inequalities (4) and (5), we get

\[ \| y-EFGy, a \| \leq (\alpha/2 + \beta + \gamma) \| x-EFGx, a \| + (\beta + 2\delta) \| y-EFGy, a \| \]

or \[ \| y-EFGy, a \| \leq q \| x-EFGx, a \| \] \ldots \ldots \ldots (6)

where \[ q = \frac{\alpha/2 + \beta + \gamma}{1 - \beta - 2\delta} < 1 \]

\[ \Rightarrow \alpha + 4\beta + 2\gamma + 4\delta < 2 \]

To show that \( \{ H^n x \} \) is a Cauchy sequence in \( X \).

Now, \[ \| H^2x-Hx, a \| = \| H(y-a) \| = \frac{1}{2} \| y-EFGy, a \| \]

\[ \leq q/2 \| x-EFGx, a \| = q \| Hx-x, a \| \]
or \[ ||H^2x - Hx, a|| \leq q ||Hx - x, a|| \]

Therefore \( \{H^n x\} \) is a Cauchy sequence in \( X \) and since \( X \) is a Banach space, it converges to some point \( x_0 \) in \( X \). i.e.

\[ \lim_{n \to \infty} H^n x = x_0 \]

which implies \( Hx_0 = x_0 \) or \( x_0 \) is a fixed point of \( H \).

Since \( H = 1/2 (I + EFGx) \), so that \( EFGx_0 = x_0 \) which gives \( x_0 \) is a fixed point of \( EFG \). ... ... (7)

Now, \( FG(EFGx_0) = FGx_0 \) and \( G(EFGx_0) = EFx_0 \)

\[ \implies Ex_0 = FGx_0 \text{ and } EFx_0 = Gx_0 \] respectively ... ... (8)

By applying (6.4.6) and using (6.4.4), (6.4.5), (7) and (8), we have

\[ ||Gx_0 - x_0, a|| = ||EFx_0 - E^2x_0, a|| \]

\[ \leq \alpha ||Gx_0 - x_0, a|| + \beta \left( ||Gx_0 - Gx_0, a|| + ||x_0 - x_0, a|| \right) \]

\[ + \gamma (||Gx_0 - Gx_0, a|| + ||Gx_0 - x_0, a||) \]

\[ + \delta \frac{||Gx_0 - Gx_0, a|| \cdot ||x_0 - x_0, a||}{||Gx_0 - x_0, a||} \]

\[ \leq (\alpha + 2\gamma) ||Gx_0 - x_0, a|| < ||Gx_0 - x_0, a|| \text{, a contradiction} \]

which implies \( Gx_0 = x_0 \) and by using (8) we have, \( EFx_0 = Gx_0 = x_0 \) and \( Ex_0 = FGx_0 = Fx_0 \). ... ... (9)
By applying (6.4.6) and using (6.4.4), (6.4.5), (7), (8) and (9), we have

\[ ||E_0 - x_0, a|| = ||E_0 - E^2 x_0, a|| = ||E_0 - E E_0, a|| \]

\[ \leq \alpha ||E_0 - x_0, a|| + \beta [ ||E_0 - E_0, a|| + ||x_0 - x_0, a|| ] \]

\[ + \gamma [ ||E_0 - x_0, a|| + ||x_0 - E_0, a|| ] \]

\[ + \delta \frac{||E_0 - E_0, a|| + ||x_0 - x_0, a||}{||E_0 - x_0, a||} \]

\[ \leq (\alpha + 2\gamma) ||E_0 - x_0, a|| < ||E_0 - x_0, a|| \cdot \]

This contradiction implies that \( E_0 = x_0 \), by using (9), we get

\[ Fx_0 = Gx_0 = E_0 = x_0 \quad \cdots \quad \cdots \quad (10) \]

Thus \( x_0 \) is a common fixed point of \( E, F \) and \( G \).

Finally, we have to show that \( x_0 \) is unique fixed point of \( E, F \) and \( G \). Let \( y_0 \) be another fixed point of \( E, F \) and \( G \), s.t. \( x_0 \neq y_0 \).

Again applying (6.4.6) and using (6.4.4), (6.4.5), (7), (8), (9) and (10), we have

\[ ||x_0 - y_0, a|| = ||E^2 x_0 - E^2 y_0, a|| = ||E E_0 - E E y_0, a|| \]

\[ \leq \alpha ||x_0, y_0, a|| + \beta [ ||x_0 - x_0, a|| + ||y_0 - y_0, a|| ] \]
+ \gamma \left[ ||x_o - y_o, a|| + ||y_o - x_o, a|| \right]

+ \delta \frac{||x_o - x_o, a|| \cdot ||y_o - y_o, a||}{||x_o - y_o, a||}

\leq (\alpha + 2\gamma) ||x_o - y_o, a||

< ||x_o - y_o, a||, a contradiction which implies x_o = y_o.

Hence x_o is a unique common fixed point of E, F and G.

**REMARK:** If in theorem 5,

(i) G = I and \delta = 0, we get Theorem 4.

(ii) If X is Banach space and \delta = 0 we have corollary of Khan and Imdad [1].

(iii) If F = G = I and \gamma = \delta = 0 in Banach space X we have Theorem of Iseki [1].

(iv) If G = I and \beta = \gamma = \gamma = \delta = 0 and taking Banach space in place of 2-Banach space, we get Theorem of Khan [4].

(v) If \delta = 0 it is corollary of Murthy and Sharma[1].

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