CHAPTER V

COMMON FIXED POINT THEOREMS FOR TWO
AND THREE MAPPINGS ON METRIC AND
2-BIMETRIC SPACES

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5.1. Although Banach's contraction principle (1.1.1) has many fruitful applications in analysis but it has one serious drawback that it requires the continuity of the mapping throughout the space. In 1968, for the first time Kannan [1] has established an important result in fixed point theory where the continuity of the mapping is not required. He proved that a selfmapping T of a complete metric space \((X,d)\) satisfying

\[
(5.1.1) \quad d(Tx,Ty) \leq \alpha [d(x,Tx) + d(y,Ty)]
\]

for all \(x,y\) of \(X\), \(0 < \alpha < \frac{1}{2}\), the mapping \(T\) which is not necessarily continuous has a unique fixed point.

In 1975, Gupta and Ranganathan [1] proved a unique fixed point theorem for a selfmapping \(T\) of a complete metric space \((X,d)\) which is not necessarily continuous for \(p=0\) satisfying

\[
(5.1.2) \quad d(T^{p+1}x, T^{p+2}y) \leq \alpha d(T^pTx, T^{p+1}x) + \beta d(T^{p+1}y, T^{p+2}y) + \gamma d(Tx, T^{p+1}y)
\]

for all \(x,y\) in \(X\), \(p\) a non-negative integer and \(\alpha, \beta, \gamma > 0\) such that \(\alpha + \beta + \gamma < 1\).
Recently in 1990, Murthy and Pathak [1] have proved the following common fixed point theorem for two self-mappings defined on a metric space.

**THEOREM A:** Let \((X, d)\) be a metric space, \(T_1, T_2\) be selfmaps of \(X\), such that

\[
d(T_1^r x_1, T_2^s y) \leq \alpha \frac{d(x, T_1^r x) \cdot d(y, T_2^s y)}{d(x, T_2^s y) + d(y, T_1^r x) + d(x, y)} + \beta d(x, y)
\]

for all \(x, y \in X\), \(x \neq y\), where \(r, s > 0\) are integers and \(\alpha, \beta\) are non-negative real numbers such that \(\alpha + \beta < 1\).

If for some \(x \in X\) the sequence \(\{x_n\}\) consisting of points

\[
x_{2n+1} = T_1^{r_1} x_{2n}, \quad x_{2n+2} = T_2^{s_1} x_{2n+1}
\]

has a convergent subsequence \(\{x_{n_k}\}\) converging to a point \(p\), then \(T_1\) and \(T_2\) have a unique common fixed point \(p\) in \(X\).

5.2. The main object of this section is to generalize the results of fixed points of Fisher [6], Fisher and Khan [1] and Murthy and Pathak [1] for two and three self mappings of metric spaces.

**THEOREM 1:** Let \(S\) and \(T\) be selfmappings of a metric space \((X, d)\) satisfying:
(5.2.1) \[ d(S^p x, T^q y) \leq \frac{a_1 d(x, S^p x) d(x, T^q y) + a_2 d(y, T^q y) d(y, S^p x)}{d(x, T^q y) + d(y, S^p x)} + \frac{a_3 [d(x, T^q y)]^2 + a_4 [d(y, S^p x)]^2}{d(x, T^q y) + d(y, S^p x)} \]

for all \( x \neq y \in X \); \( p, q > 0 \) are integers; \( a_i (1 \leq i \leq 4) \) are non-negative real numbers such that \( a_1 + 2a_3 < 1, \)
\( a_2 + 2a_4 < 1 \). If for some \( x_0 \in X \), the sequence \( \{x_n\} \)
given by \( S^p x_{2n} = x_{2n+1}, T^q x_{2n+1} = x_{2n+2} \) has a convergent
subsequence \( \{x_{n_k}\} \) in \( X \) and \( d(x, T^q y) + d(y, S^p x) \neq 0 \),
otherwise \( d(S^p x, T^q y) = 0 \), then \( S \) and \( T \) have unique common fixed point.

**Proof:** By applying condition (5.2.1), we get

\[ d(x_{2n+1}, x_{2n+2}) = d(S^p x_{2n}, T^q x_{2n+1}) \]

\[ \leq \frac{a_1 d(x_{2n}, S^p x_{2n}) d(x_{2n}, T^q x_{2n+1})}{d(x_{2n}, T^q x_{2n+1}) + d(x_{2n+1}, S^p x_{2n})} + \frac{a_3 [d(x_{2n}, T^q x_{2n+1})]^2 + a_4 [d(x_{2n+1}, S^p x_{2n})]^2}{d(x_{2n}, T^q x_{2n+1}) + d(x_{2n+1}, S^p x_{2n})} \]
\[ a_1 d(x_{2n}, x_{2n+1}) d(x_{2n}, x_{2n+2}) + a_2 d(x_{2n+1}, x_{2n+2}) d(x_{2n+1}, x_{2n+1}) \leq \frac{a_3 [d(x_{2n}, x_{2n+2})]^2 + a_4 [d(x_{2n+1}, x_{2n+1})]^2}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} \]

implies, \[ d(x_{2n+1}, x_{2n+2}) \leq a_1 d(x_{2n}, x_{2n+1}) + a_3 d(x_{2n}, x_{2n+2}) \leq (a_1 + a_3) d(x_{2n}, x_{2n+1}) \]

\[ + a_3 d(x_{2n+1}, x_{2n+2}) \]

i.e. \[ d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n}, x_{2n+1}), \text{where } k_1 = \frac{a_1 + a_3}{1 - a_3} < 1 \]

Again by applying condition (5.2.1), we can easily get

\[ d(x_{2n+1}, x_{2n}) = d(S^px_{2n}, T^qx_{2n-1}) \leq k_2 d(x_{2n-1}, x_{2n}), \]

where \[ k_2 = \frac{a_2 + a_4}{1 - a_4} < 1 \]

For \( k = \max \{k_1, k_2\} \) we have,

\[ d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n}, x_{2n+1}), \]

\[ d(x_{2n+1}, x_{2n}) \leq k d(x_{2n-1}, x_{2n}). \]
On proceeding in the same manner, we have

\[ d(x_{2n+1}, x_{2n+2}) \leq k d(x_{2n}, x_{2n+1}) \leq k^2 d(x_{2n-1}, x_{2n}) \leq \ldots \leq k^{2n+1} d(x_0, x_1), \]

showing that \( \{ x_n \} \) is a monotonically decreasing sequence of positive real numbers which converges to some \( z \) of \( X \) and therefore its subsequence \( \{ x_{n_k} \} \) will also converge to the same limit \( z \).

Now, we show that \( z \) is a common fixed point of \( S^p \) and \( T^q \). Let \( S^p z \neq z \). Then we have

\[
d(S^p z, x_{2n_k+1}) = d(S^p z, T^q x_{2n_k})
\]

\[
a_1 d(z, S^p z) d(z, x_{2n_k+1}) + a_2 d(x_{2n_k}, x_{2n_k+1}) d(x_{2n_k}, S^p z) \leq \frac{a_3 [d(z, x_{2n_k+1})]^2 + a_4 [d(x_{2n_k}, S^p z)]^2}{d(z, x_{2n_k+1}) + d(x_{2n_k}, S^p z)}
\]

Taking limit as \( n \to \infty \), we get
\[ d(S^p z, z) \leq \frac{a_1 d(z, S^p z) d(z, z) + a_2 d(z, z) d(z, S^p z)}{d(z, z) + d(z, S^p z)} \]
\[ + \frac{a_3 [d(z, z)]^2 + a_4 [d(z, S^p z)]^2}{d(z, z) + d(z, S^p z)} \]
\[ \leq a_4 d(z, S^p z) < d(z, S^p z), \]

a contradiction which shows that \( S^p z = z \). Similarly we can prove \( T^q z = z \).

Let \( z' \) be another common fixed point of \( S^p \) and \( T^q, s.t. \ z' \neq z' \).

Then by applying (5.2.1), we get
\[ d(z, z') = d(S^p z, T^q z) \]
\[ \leq \frac{a_1 d(z, z) d(z, z') + a_2 d(z', z') d(z', z)}{d(z, z') + d(z', z)} \]
\[ + \frac{a_3 [d(z, z')]^2 + a_4 [d(z', z)]^2}{d(z, z') + d(z', z)} \]
\[ \leq \frac{(a_3 + a_4)}{2} d(z, z') < d(z, z'). \]

This contradiction proves that \( z = z' \).

If \( d(x, T^q y) + d(y, S^p x) = 0 \) implies \( d(S^p x, T^q y) = 0 \), then \( d(S^p z, T^q z') = 0 \) i.e.
\[ d(z, z') = 0 \] showing that \( z = z' \).
Hence in either case, we see that \( z \) is a unique common fixed point of \( S^p \) and \( T^q \).

Now, \( S^p(Sz) = S^{p+1}z = S(S^pz) = Sz \), implies that \( Sz \) is a fixed point of \( S^p \). But \( S^p \) has a unique fixed point \( z \), therefore \( Sz = z \). Similarly \( Tz = z \).

Let \( w \) be another common fixed point of \( S \) and \( T \).

Then by using (5.2.1), we get

\[
d(z,w) = d(Sz,Tw) = d(S^p z, T^q w) \\
\leq \frac{(a_3 + a_4)}{2} d(z,w) < d(z,w),
\]

a contradiction which proves that \( S \) and \( T \) have a unique common fixed point.

**Remark 1:** For \( p = q = 1 \), we have the following particular cases:

(a) For \( a_1 = a_2 = k \) and \( a_3 = a_4 = 0 \), we get the result of Fisher and Khan [1].

(b) For \( a_1 = a_2 = 0 \) we get the result of Fisher [6].

**Theorem 2:** Let \( T_1, T_2 \) and \( P \) be three self mappings of a metric space \((X,d)\) satisfying
(5.2.2) \( d((T_1P)^r x, (T_2P)^s y) \)

\[
\leq \alpha \frac{d(x, (T_1P)^r x) d(y, (T_2P)^s y)}{d(x, (T_2P)^s y) + d(y, (T_1P)^r x) + d(x, y)} + \beta d(x, y)
\]

for all \( x \neq y \in X \); \( r, s > 0 \) are integers; \( \alpha, \beta \) are non-negative real numbers such that \( \alpha + \beta < 1 \). If for any \( x_0 \in X \), the sequence \( \{x_n\} \) consisting of points

\[
x_{2n+1} = (T_1P)^r x_{2n}, \quad x_{2n+2} = (T_2P)^s x_{2n+1}
\]

has a convergent subsequence \( \{x_{n_k}\} \) in \( X \), then \((T_1P)^r, (T_2P)^s \)

have a common fixed point. Further if \( T_1 \) or \( T_2 \) commutes

with \( P \), then \( T_1, T_2 \) and \( P \) have a unique common fixed point.

**PROOF**: By applying condition (5.2.2), we have

\[
d(x_{2n+1}, x_{2n+2}) = d((T_1P)^r x_{2n}, (T_2P)^s x_{2n+1})
\]

\[
\leq \alpha \frac{d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1})
\]

\[
\leq k d(x_{2n}, x_{2n+1}) \text{ where } k = \alpha + \beta < 1.
\]

Again,

\[
d(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n})
\]

\[
= d((T_1P)^r x_{2n}, (T_2P)^s x_{2n-1})
\]
\[ \leq k \, d(x_{2n-1}, x_{2n}) \]

On continuing the process, we get

\[ d(x_{2n+1}, x_{2n+2}) \leq k \, d(x_{2n}, x_{2n+1}) \leq k^2 d(x_{2n-1}, x_{2n}) \leq \cdots \leq k^{2n+1} d(x_0, x_1) \]

Thus \( \{x_n\} \) is a monotonic decreasing sequence of positive real numbers converges to some \( z \in X \). Since \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \), therefore it will also converge to some \( z \).

Now, we show that \( z \) is a common fixed point of \( (T_1P)^r \) and \( (T_2P)^s \).

Let \( (T_1P)^r z \neq z \). By applying (5.2.2), we get

\[ d((T_1P)^r z, x_{2n_k}) = d((T_1P)^r z, (T_2P)^s x_{2n_k-1}) \]

\[ \leq k \, d(z, (T_1P)^r z) \, d(x_{2n_k-1}, x_{2n_k}) \]

\[ \leq \frac{k \, d(z, (T_1P)^r z) \, d(x_{2n_k-1}, x_{2n_k})}{d(z, x_{2n_k}) + d(x_{2n_k-1}, (T_1P)^r z) + d(z, x_{2n_k-1})} \]

on taking limit \( n \to \infty \), we get

\[ d((T_1P)^r z, z) \leq \frac{\alpha d(z, (T_1P)^r z) \, d(z, z)}{d(z, z) + d(z, (T_1P)^r z) + d(z, z)} + \beta \, d(z, z) \]
implies \( d((T_1P)^r z, z) \leq 0 \) which proves \((T_1P)^r z = z\).

Similarly we can prove \((T_2P)^s z = z\).

To prove uniqueness, let \( z' \) be another common fixed point of \((T_1P)^r \) and \((T_2P)^s \). Then

\[
d(z, z') = d((T_1P)^r z, (T_2P)^s z') \\
\leq \alpha \frac{d(z, z) d(z', z')}{d(z, z') + d(z', z) + d(z, z')} + \beta d(z, z') \\
\leq \beta d(z, z') < d(z, z')
\]

a contradiction, which shows that \( z = z' \).

Now, \((T_1P)^r (T_1P)z = (T_1P)^r+1 z = (T_1P)(T_1P)^r z = (T_1P)z\),

shows that \((T_1P)z\) is a fixed point of \((T_1P)^r\). But \((T_1P)^r\)

has unique fixed point \( z \). Therefore \( T_1Pz = z \).

Similarly \( T_2Pz = z \). Let \( T_1 \) commutes with \( P \).

Now, we show that \( z \) is a common fixed point of \( T_1, T_2 \)

and \( P \). Let \( Pz \neq z \).

Then by applying (5.2.2) we have,

\[
d(Pz, z) = d(PT_1Pz, T_2Pz) = d(T_1PPz, T_2Pz) = d((T_1P)^r Pz, (T_2P)^s z) \\
\leq \alpha \frac{d(Pz, (T_1P)^r Pz) d(z, (T_2P)^s z)}{d(Pz, (T_2P)^s z) + d(z, (T_1P)^r Pz) + d(Pz, z)} + \beta d(Pz, z)
\]
\[ \leq \alpha \frac{d(Pz, Pz) \cdot d(z, z)}{d(Pz, z) + d(z, Pz) + d(Pz, z)} + \beta d(Pz, z) \]

\[ \leq \beta d(Pz, z) \]

or \((1-\beta) d(Pz, z) \leq 0\) implies \(d(Pz, z) \leq 0\) shows \(Pz = z\).

Thus \(T_1 Pz = T_2 Pz = z\) implies \(T_1 z = T_2 z = z = Pz\).

Similarly, if \(T_2\) commutes with \(P\), we can prove

\[ T_1 z = T_2 z = Pz = z. \]

To prove uniqueness, let \(w\) be another common fixed point of \(T_1, T_2\) and \(P\).

Then,

\[ d(z, w) = d(T_1 Pz, T_2 Pw) = d((T_1 P)^r z, (T_2 P)^s w) \]

\[ \leq \alpha \frac{d(z, T_1 Pz) \cdot d(w, T_2 Pw)}{d(z, T_2 Pw) + d(w, T_1 Pz) + d(z, w)} + \beta d(z, w) \]

\[ \leq \beta d(z, w) < d(z, w), \]

a contradiction, which proves that \(T_1, T_2\) and \(P\) have a unique common fixed point.
REMARK 2: (a) For $P = I_x$, we get Theorem 1 [1] of Murthy and Pathak [1].

(b) For $P = I_x; r = S = 1; T_1 = T_2 = f$

we get the result of Jaggi and Das [1].

5.3. In 1963, to generalize the concept of distance function (i.e. metric) Gahler [1] has introduced the very important concept of area function (i.e. 2-metric). The concept of 2-metric gives the fundamental properties of the area function for a triangle determined by a triplet in Euclidean spaces. Gahler has studied the various properties of 2-metric spaces in his papers [2], [3], [4].

Let $X$ be a non-empty set. A non-negative real valued function $d$ defined on $X \times X \times X$ is called 2-metric, if following conditions are satisfied.

(I) to each pair of distinct points $x, y$ in $X$, there exists a point $z$ in $X$ such that $d(x, y, z) \neq 0$

(II) $d(x, y, z) = 0$ when at least of two of $x, y, z$ are equal.

(III) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z$ in $X$ and

(IV) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w$ in $X$.

When $d$ is a 2-metric on $X$, the ordered pair $(X, d)$ is called a 2-metric space.
A sequence \( \{x_n\} \) in a 2-metric space \((X,d)\) is said to be convergent with limit \(x\) in \(X\) if

\[
\lim_{n \to \infty} d(x_n, x, a) = 0 \quad \text{for all } a \in X
\]

A sequence \( \{x_n\} \) in a 2-metric space \((X,d)\) is said to be a Cauchy sequence if

\[
\lim_{n, m \to \infty} d(x_n, x_m, a) = 0 \quad \text{for all } a \in X
\]

A 2-metric space is said to be complete if every Cauchy sequence in it is convergent.

Let \( T : X \to X \) if for all \( a \in X \),

\[
d(T^n x, u, a) \to 0 \quad \text{as } i \to \infty,
\]

implies \( d(T T^n x, Tu, a) \to 0 \) as \( i \to \infty \), then \( T \) is called orbitally continuous at \( x \in X \). Iseki, K. [6], Mishra, S.N. [2]

A 2-metric space \((X,d)\) is said to be \((T, x_o)\)-orbitally complete for some \( x_o \in X \), if every Cauchy sequence of the form \( \{T^n x_o\} \) is convergent to some \( x \in X \).

\[
\lim_{i \to \infty} d(T^n x, x, a) = 0 \quad \text{for all } a \in X.
\]

To generalize Banach's contraction principle Maia [1] in 1968 has introduced and studied the concept of Bimetric spaces. Two metrics \( d_1, d_2 \) are defined on
nonempty set \( X \), then triplet \((X,d_1,d_2)\) is called Bimetric space. He proved the following important theorem.

**Theorem B**: Let \((X,d_1,d_2)\) be a Bimetric space and \( T \) be a selfmap of \( X \) such that:

(a) \( d_1(x,y) \leq d_2(x,y) \)
(b) \( X \) is complete with respect to \( d_1 \)
(c) \( T \) is continuous with respect to \( d_1 \)
(d) \( d_2(Tx,Ty) \leq \alpha d_2(x,y) \)

for every \( x,y \in X \) and \( \alpha \in [0,1) \). Then \( T \) has a unique fixed point.

Since 1968 a number of generalization of Maia's theorem have appeared. Many mathematicians such as Iseki, K. [2],[3], Rus, I.A.[2], Singh, S.P.[3], Rhoades, B.E.[2], Mishra, S.N. [1] and others have worked in this field.

We combine both these concepts and introduce the concept of 2-Bimetric space. This concept gives very fruitful results.

**Definition**: Let \( X \) be a non-empty set containing at least three points and \( d_1,d_2 \) be two non-negative real valued functions defined on \( X \times X \times X \). The triplet \((X,d_1,d_2)\) is called 2-Bimetric space if following conditions are satisfied for \( i = 1,2 \).
(1) to each pair of distinct points \(x, y\) in \(X\), there exists a point \(z\) in \(X\) such that \(d(x, y, z) \neq 0\).

(2) \(d(x, y, z) = 0\) when at least two of \(x, y, z\) are equal.

(3) \(d(x, y, z) = d(x, z, y) = d(y, z, x)\) for all \(x, y, z\) in \(X\) and

(4) \(d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)\) for all \(x, y, z, w\) in \(X\).

In 1974, Ciric [1] has proved a non-unique fixed point theorem for orbitally continuous self mapping \(T\) defined on a orbitally complete metric space \((X, d)\) which satisfies the condition

\[
\min \left\{d(Tx, Ty), d(x, Tx), d(y, Ty)\right\} - \min \left\{d(x, Ty), d(y, Tx)\right\} \\
\leq q \ d(x, y) , \forall \ x, y \in X \text{ and for some } q \in (0, 1)
\]

Achari [2], Pachpatte [1],[2], Dhage [1], Bajaj[1] and others have also proved some non-unique fixed point theorems on Ciric type maps.

Dhage and Dhothale [1] have obtained the Ciric type non-unique fixed point theorem for a selfmapping \(T\) defined on a Bimetric space \((X, d_1, d_2)\). In 1991, Pathak and Dubey [1] have generalized it in the following way:
**THEOREM B**: Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \). Let \( T \) be a selfmap of \( X \), and \( X \) satisfying the following conditions:

(i) \( d_1(x,y) \leq a \left[ d(x,Tx)+d(y,Ty) \right] \); \( a \geq 1 \) and for every \( x,y \in X \),

(ii) \( X \) is 'T' orbitally complete with respect to \( d_1 \),

(iii) \( T \) is orbitally continuous with respect to \( d_1 \),

(iv) \[
\min \left\{ \frac{1}{2} d(x,Tx) d(x,Ty) d(Tx,Ty), \frac{1}{2} [d(x,y)]^2 d(x,Ty), \frac{1}{2} d(x,Tx)d(y,Ty) \left[ d(x,Ty)+d(Tx,y) \right]^{-1} \right\} - \min \left\{ d(x,Tx) d(x,y) \right\}
\]

\[
< \frac{1}{2} qd(x,Tx) d(y,x) \left[ d(x,Ty) + d(Tx,y) \right],
\]

for all \( x,y \in X \) and \( q \in [0,1) \), then \( T \) has a fixed point.

We have generalized the above theorem B for a pair of self-mappings and prove non unique fixed point of Ciric type.

To prove our main result, we have used a property(S) due to Sharma A.K. [3].

A 2-metric space \((X,d)\) is said to have property(S), if \( d(x,y,z) = 0 \) implies that at least two of the points \( x,y,z \) are equal and conversely.

**THEOREM 3**: Let \( T_1 \) and \( T_2 \) be two selfmappings of 2-Bimetric space \((X,d_1,d_2)\) satisfying the following conditions:
(5.4.1) \( d_1(x, y, a) \leq \alpha \left[ d_2(x, T_1x, a) + d_2(y, T_1x, a) + d_2(y, T_2y, a) + d_2(T_1x, T_2y, a) \right]; \)
\[
\alpha \geq 1/2 \text{ and for every } x, y, a \in X.
\]

(5.4.2) \( X \) is \((T_1, T_2)\)-orbitally complete 2-metric space with respect to \( d_1 \).

(5.4.3) \( T_1, T_2 \) are orbitally continuous with respect to \( d_1 \).

(5.4.4) \[
\min \left\{ d_2(x, T_1x, a) d_2(x, T_2y, a) d_2(T_1x, T_2y, a), \right.
\]
\[
1/2 \left[ d_2(x, y, a) \right]^2 d_2(x, T_2y, a),
\]
\[
1/2 d_2(x, T_1x, a) d_2(y, T_2y, a) \left[ d_2(x, T_2y, a) + d_2(T_1x, y, a) \right]
\]
\[
+ d_2(T_1x, y, a) \right\}.
\]
\[
- \min \left\{ d_2(x, T_1x, a) d_2(x, y, a) d_2(T_1x, T_2y, a), \right.
\]
\[
1/2 d_2(x, T_2y, a) d_2(y, T_1x, a) [d_2(x, T_2y, a) + d_2(T_1x, y, a)] \right\}
\]
\[
\leq 1/2 q d_2(x, T_1x, a) d_2(x, y, a) \left[ d_2(x, T_2y, a) + d_2(T_1x, y, a) \right]
\]
\[
\forall x, y, a \in X \text{ and } q \in [0, 1). \text{ Then } T_1 \text{ and } T_2 \text{ have a common fixed point.}
\]

**Proof:** Let \( x_0 \) an arbitrary point in \( X \). We define a sequence \( \{x_n\} \) as follows :

\[
x_{2n+1} = T_1x_{2n}, \quad x_{2n+2} = T_2x_{2n+1} \quad \text{for } n = 0, 1, 2, \ldots
\]

If \( x_{2n} = x_{2n+1} \), then \( \{x_n\} \) is a Cauchy sequence and limit of \( \{x_n\} \) is a common fixed point of \( T_1 \) and \( T_2 \).
Therefore we assume that \( x_{2n} \neq x_{2n+1} \) for each \( n \in \mathbb{N}_0 \)
where \( \mathbb{N}_0 \) is a set of non-negative integers.

Put \( x = x_{2n} \), \( y = x_{2n+1} \), by applying (5.4.4), we have

\[
\begin{align*}
\min \{ & d_2(x_{2n}, x_{2n+1}, a) d_2(x_{2n}, x_{2n+2}, a) d_2(x_{2n+1}, x_{2n+2}, a), \\
& \quad 1/2 [d_2(x_{2n}, x_{2n+1}, a)]^2 d_2(x_{2n}, x_{2n+2}, a), \\
& \quad 1/2 d_2(x_{2n}, x_{2n+1}, a) d_2(x_{2n+1}, x_{2n+2}, a) [d_2(x_{2n}, x_{2n+2}, a) \\
& \quad + d_2(x_{2n+1}, x_{2n+1}, a)] \} \\
\leq & \min \{ d_2(x_{2n}, x_{2n+1}, a) d_2(x_{2n}, x_{2n+1}, a) d_2(x_{2n+1}, x_{2n+2}, a), \\
& \quad 1/2 d_2(x_{2n}, x_{2n+2}, a) d_2(x_{2n+1}, x_{2n+1}, a) [d_2(x_{2n}, x_{2n+2}, a) \\
& \quad + d_2(x_{2n+1}, x_{2n+1}, a)] \} \\
\leq & 1/2 q d_2(x_{2n}, x_{2n+2}, a) d_2(x_{2n}, x_{2n+1}, a) [d_2(x_{2n}, x_{2n+2}, a) \\
& \quad + d_2(x_{2n+1}, x_{2n+1}, a)] \\
\text{i.e. } \min \{ & d_2(x_{2n+1}, x_{2n+2}, a), 1/2 d_2(x_{2n}, x_{2n+1}, a), \\
& \quad 1/2 d_2(x_{2n+1}, x_{2n+2}, a) \} \leq 1/2 q d_2(x_{2n}, x_{2n+1}, a) \\
\end{align*}
\]

Either, \( d_2(x_{2n}, x_{2n+1}, a) \leq q \ d_2(x_{2n}, x_{2n+1}, a) \)

\[ \leq d_2(x_{2n}, x_{2n+1}, a), \text{ a contradiction.} \]

or, \( d_2(x_{2n+1}, x_{2n+2}, a) \leq q \ d_2(x_{2n}, x_{2n+1}, a) \) \ldots (1)
Similarly, \( d_2(x_{2n}, x_{2n+1}, a) \leq q \, d_2(x_{2n-1}, x_{2n}, a) \).

By repeating the process in a similar manner, we have:

\[
d_2(x_{2n+1}, x_{2n+2}, a) \leq q \, d_2(x_{2n}, x_{2n+1}, a) \leq q^2 \, d_2(x_{2n-1}, x_{2n}, a) \leq \cdots \leq q^{2n+1} \, d_2(x_0, x_1, a),
\]

and

\[
d_2(x_{2n}, x_{2n+1}, a) \leq q^{2n} \, d_2(x_0, x_1, a) \cdots (3)
\]

Now to show that \( \{ x_n \} \) is a Cauchy sequence with respect to \( d_1 \) we proceed as follows:

If \( a = x_{2n} \) and \( a = x_{2n-1} \) in equalities (1) and (2) respectively,

\[
d_2(x_{2n+1}, x_{2n+2}, x_{2n}) \leq q \, d_2(x_{2n}, x_{2n+1}, x_{2n}) = 0 \quad \cdots (4)
\]

\[
d_2(x_{2n}, x_{2n+1}, x_{2n-1}) \leq q \, d_2(x_{2n-1}, x_{2n}, x_{2n-1}) = 0 \quad \cdots (5)
\]

Put \( 2n = m \) in (4) and (5)

\[
d_{2m}(x_{m+1}, x_{m+2}, x_m) = 0 \quad \text{and} \quad d_2(x_m, x_{m+1}, x_{m-1}) = 0
\]

Now,

\[
d_2(x_0, x_1, x_m) \leq d_2(x_0, x_1, x_{m-1}) + d_2(x_0, x_{m-1}, x_m) + d_2(x_{m-1}, x_1, x_m)
\]
\[ d_2(x_0, x_1, x_{m-1}) + q^{m-1} d_2(x_0, x_0, x_1) + q^{m-2} d_2(x_0, x_1, x_1) \leq d_2(x_0, x_1, x_m) \leq d_2(x_0, x_1, x_{m-1}) \leq \cdots \leq \cdots \leq \]
\[ d_2(x_0, x_1, x_1) = 0. \]

In general case, it can be shown easily that
\[ d_2(x_n, x_{n+1}, x_m) = 0 \quad \cdots \quad (6) \]

For any \( n, m \in \mathbb{N}, n < m \) and \( a \in X \), we have
\[ d_2(x_n, x_m, a) \leq d_2(x_n, x_{n+1}, a) + d_2(x_{n+1}, x_m, a) + d_2(x_n, x_m, x_{n+1}) \]
\[ \leq q^n d_2(x_0, x_1, a) + q^{n+1} d_2(x_0, x_1, a) + d_2(x_{n+2}, x_m, a) \leq \]
\[ \leq \sum_{i=0}^{m-1} q^{n+i} d_2(x_0, x_1, a) < \frac{q^n}{1-q} d_2(x_0, x_1, a). \]

Thus \( \{x_n\} \) is a Cauchy sequence with respect to \( d_2 \).

By applying (5.4.1)
\[ d_1(x_n, x_m, a) \leq \alpha [ d_2(x_n, x_{n+1}, a) + d_2(x_m, x_{m+1}, a) + d_2(x_{n+1}, x_{m+1}, a) + d_2(x_m, x_{n+1}, a) ] \]
\[ < \alpha [ q^n + q^m + \frac{2q^{n+1}}{1-a} ] d_2(x_0, x_1, a). \]
Since \( n < m \) and \( q < 1 \implies q^n < q^m \).

\[
\leq \frac{2\alpha q^n}{1-q} d_2(x_o, x_1, a)
\]

Letting \( n \) and \( m \) tend to infinity \( \text{RHS} \) tends to zero i.e.

\[
\lim_{n,m \to \infty} d_1(x_n, x_m, a) = 0
\]

Thus \( \{x_n\} \) is a Cauchy sequence \( \text{w.r.t.} \ d_1 \).

By (5.4.2), \( X \) is \((T_1, T_2)\)-orbitally complete \( \text{w.r.t.} \ d_1 \), it converges to some point \( u \in X \), \( \forall \ a \in X \), i.e.

\[
\lim_{n \to \infty} d_1(T^n x_o, u, a) = 0
\]

By (5.4.3) \( T_1 \) and \( T_2 \) are orbitally continuous \( \text{w.r.t.} \ d_1 \) we get

\[
\lim_{n \to \infty} d_1(T_1^{2n+1} x_o, T_1 u, a) = 0
\]

and

\[
\lim_{n \to \infty} d_1(T_2^{2n+2} x_o, T_2 u, a) = 0
\]

From the definition of 2-metric space (for \( d_1 \))

\[
d_1(u, T_1 u, a) \leq d_1(u, T_1 u, T_1 x_o) + d_1(u, T_1^{2n+1} x_o, a) + d_1(T_1^{2n+1} x_o, T_1 u, a)
\]
Letting $n$ tends to infinity, we have

$$d_1(u, T_1u, a) = 0 \implies T_1u = u.$$ 

Similarly, we can show that $d_1(u, T_2u, a) = 0$,

which implies $T_2u = u$.

Hence $u$ is a common fixed point of $T_1$ and $T_2$.

**REMARK:** If $T_1 = T_2$ then Theorem 3 is another generalization of Theorem B in 2-bimetric space $(X, d_1, d_2)$.

**THEOREM 4:** Let $(X, d_1, d_2)$ be a 2-bimetric space. Let $T_1$ and $T_2$ be self maps of $X$ satisfying the conditions (5.4.1) and (5.4.4). If conditions (5.4.2) and (5.4.3) replaced by,

(5.4.5) $(X, d_1)$ is a complete 2-metric space.

(5.4.6) $T_1$ and $T_2$ are continuous with respect to $d_1$.

$$\forall x, y, a \in X \text{ and } q \in [0, 1). \text{ Then } T_1 \text{ and } T_2 \text{ have a common fixed point.}$$
**Proof:** Let \( x_0 \) be an arbitrary point in \( X \). We define a sequence \( \{x_n\} \) by \( x_{2n+1} = T_1 x_{2n} \) and \( x_{2n+2} = T_2 x_{2n+1} \) as in Theorem 1. Then we get \( \{x_n\} \) is a Cauchy sequence with respect to \( d_1 \) and since \((X,d_1)\) is complete 2-metric space, it converges to some \( u \in X \), i.e.

\[
\lim_{n \to \infty} d_1(x_n, u, a) = 0, \forall a \in X.
\]

Since \( T_1 \) and \( T_2 \) are continuous w.r.t. \( d_1 \) so that

\[
\lim_{n \to \infty} d_1(x_{2n+1}, u, a) = \lim_{n \to \infty} d_1(T_1 x_{2n}, u, a) = 0 \quad \text{and}
\]

\[
\lim_{n \to \infty} d_1(x_{2n+2}, u, a) = \lim_{n \to \infty} d_1(T_2 x_{2n+1}, u, a) = 0
\]

Now,

\[
d_1(T_1 u, u, a) \leq d_1(T_1 u, u, T_1 x_{2n}) + d_1(T_1 u, T_1 x_{2n}, a)
\]

\[+d_1(T_1 x_{2n}, u, a)
\]

on letting \( n \) tends to infinity

\[
d_1(T_1 u, u, a) = 0 \quad \text{which gives } T_1 u = u.
\]

Similarly,

\[
d_1(T_2 u, u, a) = 0 \quad \text{implies } T_2 u = u.
\]

Thus \( u \) is a common fixed point of \( T_1 \) and \( T_2 \).
COROLLARY 1: Let \((X, d_1, d_2)\) be a 2-Bimetric space,

Let \(T_1\) and \(T_2\) be a selfmaps of \(X\) satisfying the conditions (5.4.1), (5.4.3) and (5.4.4) of theorem 3,

replacing condition (5.4.2) by,

(5.4.7)...There exists a point \(x_0 \in X\) such that the sequence
\[
\{ x_n \} \text{ defined by },
\]

\[
x_{2n+1} = T_1 x_{2n} \text{ and } x_{2n+2} = T_1 x_{2n+1}, \text{ for } n = 0, 1, 2, \ldots
\]

has a subsequence converging to some \(u \in X\) with respect to \(d_1\),

then \(T_1\) and \(T_2\) have a common fixed point.

PROOF: Let \(x_0\) be an arbitrary point in \(X\). We define

a sequence \(\{ x_n \} \) by,

\[
x_{2n+1} = T_1 x_{2n} \text{ and } x_{2n+2} = T_2 x_{2n+1}
\]

Theorem 4. Then we get \(\{ x_n \}\) is a Cauchy sequence with

respect to \(d_1\). By (5.4.7) it has a convergent subsequence

\(\{ x_{n_k} \}\) converges to \(u\) in 2-metric space \((X, d_1)\), by S.Iyer[1]

\(\{ x_n \}\) also converges to \(u\) i.e.

\[
\lim_{n \to \infty} d_1(x_n, u, a) = 0
\]

Since \(T_1\) and \(T_2\) are continuous with respect to \(d_1\),

Rest of the proof is similar to Theorem 4.
Recently, Namdeo and Ansari [1] have proved the following theorem for orbitally continuous mapping in complete metric space.

**Theorem C**: Let $T$ be an orbitally continuous self map of a complete metric space $(X, d)$. If $T$ satisfies the following condition:

$$
\min \left\{ d(x, Tx) d(x, y), d(y, Ty) d(x, y), d(Tx, Ty) d(x, y),
\left[ d(x, Tx) d(y, Ty), d(x, Tx) d(Tx, Ty) \right]^2 \right\} \\
- \min \left\{ d(x, Tx) d(x, Ty), d(y, Tx) d(y, Ty) \right\} \leq q \left[ d(x, y) \right]^2.
$$

For all distinct $x, y \in X$, then the sequence $\{ T^n x \}$ converges to a fixed point of $T$.

Now we prove the following result for pair of mappings in 2-Bimetric space.

**Theorem 5**: Let $(X, d_1, d_2)$ be 2-Bimetric space. Let $T_1$ and $T_2$ be selfmaps of $X$ satisfying the following conditions:

(5.4.8) $d_1(x, y, a) \leq \beta d_2(x, y, a): \beta \geq 1$ and $x, y, a \in X$.

(5.4.9) $X$ is $(T_1, T_2)$–orbitally complete 2–metric space with respect to $d_1$.

(5.4.10) $T_1, T_2$ are orbitally continuous w.r.t. $d_1$.

(5.4.11) $\min \left\{ d_2(x, T_1 x, a) d_2(x, y, a), d_2(y, T_2 y, a) d_2(x, y, a),
\left[ d_2(x, T_1 x, T_2 y, a) d_2(x, y, a), d_2(x, T_1 x, a) d_2(y, T_2 y, a),
\left[ d_2(x, T_1 x, a) d_2(T_1 x, T_2 y, a) \right]^2 \right\}$


\[
- \min \left\{ d_2(x, T_1 x, a), d_2(y, T_2 y, a), d_2(y, T_1 x, a) \right\} X
\]
\[
d_2(y, T_2 y, a) \right\} \leq q \left[ d_2(x, y, a) \right]^2
\]

\( \forall x, y, a \in X \) and \( q \in (0, 1) \). Then \( T_1 \) and \( T_2 \) have a common fixed point.

**Proof:** Let \( x_0 \) be an arbitrary point in \( X \). We define a sequence \( \{x_n\} \) by,

\[
x_{2n+1} = T_1 x_{2n}, \quad x_{2n+2} = T_2 x_{2n+1} \quad \text{for } n = 0, 1, 2, \ldots
\]

If for some \( n \), \( x_{2n} = x_{2n+1} \). Then \( \{x_n\} \) is a Cauchy sequence and the limit of \( \{x_n\} \) is a common fixed point of \( T_1 \) and \( T_2 \).

Therefore without the loss of generality we may suppose that \( x_{2n} \neq x_{2n+1} \) for each \( n = 0, 1, 2, \ldots \).

By applying (5.4.11), for \( x = x_{2n}, y = x_{2n+1} \)

\[
\min \left\{ d_2(x_{2n}, x_{2n+1}, a), d_2(x_{2n}, x_{2n+1}, a), d_2(x_{2n+1}, x_{2n+2}, a) \right\}
\]
\[
d_2(x_{2n}, x_{2n+1}, a), \quad d_2(x_{2n+1}, x_{2n+2}, a) \leq d_2(x_{2n}, x_{2n+1}, a),
\]
\[
d_2(x_{2n}, x_{2n+1}, a), \quad d_2(x_{2n+1}, x_{2n+2}, a),
\]
\[
d_2(x_{2n}, x_{2n+1}, a) \leq d_2(x_{2n+1}, x_{2n+2}, a)[d_2(x_{2n}, x_{2n+1}, a)]^2
\]
\[
- \min \left\{ d_2(x_{2n}, x_{2n+1}, a), d_2(x_{2n+1}, x_{2n+2}, a),
\]
\[
d_2(x_{2n+1}, x_{2n+1}, a) \leq d_2(x_{2n+1}, x_{2n+2}, a),
\]
\[
d_2(x_{2n+1}, x_{2n+1}, a) \leq d_2(x_{2n+1}, x_{2n+2}, a)
\]
\[
\leq q \left[ d_2(x_{2n}, x_{2n+1}, a) \right]^2
\]
or 
\[
\min \left\{ [d_2(x_{2n}, x_{2n+1}, a)]^2, d_2(x_{2n}, x_{2n+1}, a) \cdot \right. \\
\left. d_2(x_{2n+1}, x_{2n+2}, a) \right\}
\]

\[- \min \left\{ d_2(x_{2n}, x_{2n+1}, a), d_2(x_{2n}, x_{2n+2}, a) \right\} \cdot 0 \]

\[\leq q \left[ d_2(x_{2n}, x_{2n+1}, a) \right]^2 \]

Either 
\[
[d_2(x_{2n}, x_{2n+1}, a)]^2 \leq q \left[ d_2(x_{2n}, x_{2n+1}, a) \right]^2 < [d_2(x_{2n}, x_{2n+1}, a)]^2,
\]

a contradiction,

or 
\[
d_2(x_{2n}, x_{2n+1}, a) \cdot d_2(x_{2n+1}, x_{2n+2}, a) \]

\[\leq q \left[ d_2(x_{2n}, x_{2n+1}, a) \right]^2 \]

which implies 
\[
d_2(x_{2n+1}, x_{2n+2}, a) \leq q \cdot d_2(x_{2n}, x_{2n+1}, a) .
\]

By continuing the above process, we get 
\[
d_2(x_{2n+1}, x_{2n+2}, a) \leq q \cdot d_2(x_{2n}, x_{2n+1}, a) \leq \ldots \leq q^{n-1}d_2(x_n, x_1, a)
\]

In view of theorem 3, we get \{x_n\} a Cauchy sequence with respect to \(d_2\).

Now, by applying (5.4.8), for \(x = x_n\), \(y = x_m\), \(m > n\) \(\forall m,n \in \mathbb{N}^+\),

\[
d_1(x_n, x_m, a) \leq \beta \quad d_2(x_n, x_m, a) \leq \beta \frac{q^n}{1-q} d_2(x_0, x_1, a)
\]
Letting $n$ tends to infinity, RHS tends to zero. It follows that $\{x_n\}$ is a Cauchy sequence w.r.t. $d_1$. Since $X$ is $(T_1, T_2)$ orbitally complete 2-metric space with respect to $d_1$. There exists a point $u \in X$ such that $\forall a \in X, d_1(T^n x_0, u, a) = 0$.

Since $T_1$ and $T_2$ are orbitally continuous w.r.t. $d_1$ so that $\lim_{n \to \infty} d_1(T_1^{2n+1} x_0, T_1 u, a) = 0$ and $\lim_{n \to \infty} d_1(T_2^{2n+2} x_0, T_2 u, a) = 0$.

Now, $d_1(u, T_2 u, a) \leq d_1(u, T_2 u, T_2^{2n+2} x_0) + d_1(u, T_2 x_0, a) + d_1(T_2^{2n+2} x_0, T_2 u, a)$

Assuming $n$ tends to infinity, we have $d_1(u, T_2 u, a) = 0$ which implies $T_2 u = u$.

Similarly, it can be seen easily that $d_1(u, T_1 u, a) = 0 \Rightarrow T_1 u = u$. Hence $u$ is common fixed point of $T_1$ and $T_2$.

Iseki [2] A proved the following result:

Let $X$ be a metric space with two metrics $d_1$ and $d_2$ and satisfying the following conditions:

(i) $d_1(x, y) < d_2(x, y), \forall x, y \in X$ (ii) $X$ is complete w.r.t. $d_1$

(iii) Two mappings $f, g : X \to X$ are continuous w.r.t. metric $d_1$ and

(iv) $d_2(f(x), g(y)) \leq \alpha d_2(x, y) + \beta [d_2(x, f(x)) + d_2(y, g(y))]$ $+ \gamma [d_2(x, g(y)) + d_2(y, f(x))]$

for every $x, y$ in $X$, where $\alpha, \beta, \gamma \in R^+$ and $\alpha + 2\beta + 2\gamma < 1$, then $f$ and $g$ have a unique common fixed point.

We generalize above result of Iseki [2] A for

2-Bimetric space.

**THEOREM 6**: Let $(X, d_1, d_2)$ be a 2-Bimetric space. Let $T_1$ and $T_2$ be self maps of $X$ satisfying the conditions:

(5.4.12) $d_1(x, y, a) \leq \alpha \max d_2(x, y, a), d_2(x, T_1 x, a), d_2(y, T_2 y, a)$

(5.4.13) $(X, d_1)$ is a complete 2-metric space.

(5.4.14) $T_1$ and $T_2$ are continuous with respect to $d_1$. 
\( (5.4.15) \quad d_2(T_1x, T_2y, a) \leq b d_2(x, y, a) + c[d_2(x_1, T_1x, a) + d_2(y, T_2y, a)] \\
+ d[d_2(x, T_2y, a) + d_2(y, T_1x, a)] \\
\) 

for every \( x, y, a \in X \); where \( a \geq 1, b, c, d \in \mathbb{R}^+ \) and \( b + 2c + 2d < 1 \). 

Then \( T_1 \) and \( T_2 \) have a unique common fixed point.

**Proof:** We define the sequence \( \{x_n\} \) as in Theorem 3. On applying (5.4.15), we have

\[
d_2(x_{2n+1}, x_{2n+2}, a) = d_2(T_1x_{2n}, T_2x_{2n+1}, a) \\
\leq b d_2(x_{2n}, x_{2n+1}, a) \\
+ c[d_2(x_{2n}, x_{2n+1}, a) + d_2(x_{2n+1}, x_{2n+2}, a)] \\
+ d[d_2(x_{2n}, x_{2n+2}, a) + d_2(x_{2n+1}, x_{2n+1}, a)] \\
\leq b d_2(x_{2n}, x_{2n+1}, a) \\
+ c[d_2(x_{2n}, x_{2n+1}, a) + d_2(x_{2n+1}, x_{2n+2}, a)] \\
+ d(d_2(x_{2n}, x_{2n+1}, a) + d_2(x_{2n+1}, x_{2n+2}, a) \\
+ d_2(x_{2n}, x_{2n+2}, x_{2n+1})] \\
\text{. . . (1)}
\]

On taking \( a = x_{2n} \) in (1), we get

\[
d_2(x_{2n+1}, x_{2n+2}, x_{2n}) \leq b d_2(x_{2n}, x_{2n+1}, x_{2n}) \\
+ c[d_2(x_{2n}, x_{2n+1}, x_{2n}) + d_2(x_{2n+1}, x_{2n+2}, x_{2n})] \\
+ d[d_2(x_{2n}, x_{2n+1}, x_{2n}) + d_2(x_{2n+1}, x_{2n+2}, x_{2n})] \\
\leq (c+2d) d_2(x_{2n+1}, x_{2n+2}, x_{2n}) < d_2(x_{2n+1}, x_{2n+2}, x_{2n}) \\
\]

which implies \( d_2(x_{2n+1}, x_{2n+2}, x_{2n}) = 0 \).

(By (1), we get,

\[
d_2(x_{2n+1}, x_{2n+2}, a) \leq q d_2(x_{2n}, x_{2n+1}, a), \text{ where } q = \frac{b+c+d}{1-(c+d)} < 1
\]

Now, it can be shown easily as in Theorem 3, \( \{x_n\} \) is a Cauchy sequence with respect to \( d_2 \).

By applying (5.4.12), we get

\[
d_1(x_n, x_m, a) \leq a \max d_2(x_n, x_m, a), d_2(x_n, x_{n+1}, a), d_2(x_m, x_{m+1}, a)
\]

from which we can get easily as in Theorem 3 and by using (5.4.15).
the following three cases:

(1) \( d_1(x_n, x_m, a) \leq \alpha \sum_{i=0}^{m-1} q^{n+i} d_2(x_0, x_1, a) < \alpha \frac{q^n}{1-q} d_2(x_0, x_1, a) \)

(2) \( d_1(x_n, x_{n+1}, a) \leq \alpha \sum_{i=0}^{m-1} q^{n+i} d_2(x_0, x_1, a) < \alpha q^n d_2(x_0, x_1, a) \)

Similarly,

(3) \( d_1(x_n, x_m, a) \leq \alpha q^m d_2(x_0, x_1, a) < \alpha q^n d_2(x_0, x_1, a) \) [since \( m > n \)]

Letting \( n \) tends to infinity RHS tends to zero in each case. \( \lim_{n \to \infty} d_1(x_n, x_m, a) = 0. \)

Thus \( \{x_n\} \) is a Cauchy sequence w.r.t. \( d_1 \), since \( (X, d_1) \) is complete, it converges to some \( u \in X \).

Then as in Theorem 3, we get \( T_1u = u \) and \( T_2u = u \).

Thus \( u \) is common fixed point of \( T_1 \) and \( T_2 \).

To prove uniqueness of \( u \), suppose \( w \) be another fixed point of \( T_1 \) and \( T_2 \) then there exists a point \( v \in X \) such that

\[ d_2(u, w, v) \neq 0. \]

Now applying (5.4.15), we get

\[ d_2(u, w, v) = d_2(T_1u, T_2w, v) \leq b d_2(u, w, v) + c[d_2(u, u, v) + d_2(w, w, v)] \]

\[ + d[d_2(u, w, v) + d_2(u, w, v)] \]

\[ = (b + 2d) d_2(u, w, v) < d_2(u, w, v), \]

a contradiction which implies \( u = w \). Hence \( u \) is a unique common fixed point of \( T_1 \) and \( T_2 \). This com

This completes the proof of the theorem.

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