CHAPTER IV

FIXED POINT THEOREMS

FOR

ASYMPTOTICALLY REGULAR

MAPPINGS

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4.1. The important concept of asymptotic regularity of a mapping at a point in Banach space was first introduced and studied by Browder and Petryshyn [1]. In a metric space its equivalent form is defined as

DEFINITION 1: A selfmapping $T$ of a metric space $(X,d)$ is said to be asymptotically regular at a point $x$ in $X$ if

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0.$$

The concept of asymptotic regularity is very useful not only in proving the existence of fixed points of selfmappings defined on the space but also in certain cases it is used to show that sequence of iterates at some point of the space converges to the fixed point of the selfmapping.

Mapping $T$ is said to be asymptotically regular on $X$ if it is asymptotically regular at every point of $X$. There exist mappings which are asymptotically regular at some point of the space $X$ but it is not on the space $X$ as given by the example of Guay and Singh [1].
Panja and Baisnab [1] proved the following theorem.

**THEOREM A:** Let $T$ be the selfmapping of complete metric space $(X,d)$ satisfying

$$(4.1.1) \quad d(Tx,Ty) \leq \alpha \left[ d(x,Tx) + d(y,Ty) \right] + \beta \, d(x,y)$$

$\forall \, x,y \in X$ where $0 \leq \alpha, \beta < 1$.

Then $T$ has a unique fixed point in $X$ if $T$ is asymptotically regular at some point in $X$.

In 1983, Quay and Singh [1] have proved the above theorem by replacing condition (4.1.1) by

$$(4.1.2) \quad d(Tx,Ty) \leq a \, d(x,y) + b \left[ d(x,Tx) + d(y,Ty) \right] + c \left[ d(x,Ty) + d(y,Tx) \right]$$

where $0 \leq a,c, a+2c < 1, b+c < 1$.

In 1987, Rao and Rao [1] have defined the concept of asymptotic regularity for a pair of selfmappings $S$ and $T$ defined on metric space $(X,d)$ at $x_0$ of $X$ if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ where sequence $\{ x_n \}$ is defined by $x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ for $n = 0,1,2, \ldots$. They have proved the following common fixed point theorem for a pair of mappings which is asymptotically regular at some point of the space.
THEOREM B. Let $S$ and $T$ be selfmaps on a complete metric space $(X, d)$ satisfying

$$(4.1.3) \quad d(Tx, Sy) \leq \alpha \left[ d(x, Tx) + d(y, Sy) \right]$$

$$+ \beta \left[ d(x, Sy) + d(x, Tx) \right] + \gamma d(x, y)$$

for all $x, y$ of $X$, $\alpha, \beta, \gamma \geq 0$ such that $\alpha + \beta < 1$ and $2\beta + \gamma < 1$ and pair $(S, T)$ is asymptotically regular at some point in $X$. Then $S$ and $T$ have a unique common fixed point in $X$.

4.2. The main aim of this section is to modify the fixed point theorem of Jaggi and Das [1] by using the concept of asymptotic regularity of mapping at a point.

Jaggi and Das [1] introduced the following fixed point theorem.

THEOREM C. Let $f$ be a selfmap defined on a metric space $(X, d)$ satisfying the following: for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$

$$(4.2.1) \quad d(fx, fy) \leq \alpha \frac{d(x, fx) d(y, fy)}{d(x, fy) + d(y, fx) + d(x, y)}$$

$$+ \beta d(x, y) \quad \forall \ x \neq y \in X$$
There exists $x_0 \in X$ such that \( \{ f^n x_0 \} \subseteq \{ f^k x_0 \} \)

with \( \lim_{k \to \infty} f^k x_0 \in X \).

Then $f$ has a unique fixed point $u = \lim_{k \to \infty} f^k x_0$.

We prove the above theorem by using the concept of asymptotic regularity of a mapping at some point of metric space and taking the constants $0 \leq \alpha, \beta < 1$.

**Theorem 1**: Let $T$ be a selfmapping of a complete metric space $(X,d)$ satisfying condition (4.2.1) for all $x \neq y$ of $X$ where $0 \leq \alpha, \beta < 1$. If $T$ is asymptotically regular at some point of $X$, then $T$ has a unique fixed point in $X$.

**Proof**: Let $T$ is asymptotically regular at a point $x_0$ of $X$.

By applying (4.2.1), we get

\[
\begin{align*}
d(T^n x_0, T^m x_0) &\leq \alpha \frac{d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^m x_0)}{d(T^{n-1} x_0, T^m x_0) + d(T^{m-1} x_0, T^n x_0) + d(T^{n-1} x_0, T^{m-1} x_0)} \\
&\quad + \beta \frac{d(T^{n-1} x_0, T^m x_0)}{d(T^{n-1} x_0, T^m x_0) + d(T^{m-1} x_0, T^n x_0) + d(T^{n-1} x_0, T^{m-1} x_0)} \\
&\quad \leq \alpha \frac{d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^m x_0) + d(T^{n-1} x_0, T^{m-1} x_0)}{d(T^{n-1} x_0, T^m x_0) + d(T^{m-1} x_0, T^n x_0) + d(T^{n-1} x_0, T^{m-1} x_0) + \beta[d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^n x_0) + d(T^m x_0, T^{m-1} x_0)]}
\end{align*}
\]
\[
\frac{\alpha}{1-\beta} \frac{d(T^{n-1}x_0, T^n x_0) d(T^{m-1}x_0, T^n x_0)}{d(T^{n-1}x_0, T^n x_0) + d(T^{m-1}x_0, T^n x_0) + d(T^{n-1}x_0, T^{m-1}x_0)} \\
+ \frac{\beta}{1-\beta} [d(T^{n-1}x_0, T^n x_0) + d(T^m x_0, T^{m-1}x_0)].
\]

By using the asymptotic regularity of $T$ at $x_0$, RHS tends to zero as $n, m$ tends to infinity, showing that $\{T^n x_0\}$ is a Cauchy sequence. Since $(X, d)$ is complete therefore there exists $z \in X$ such that \( \lim_{n \to \infty} T^n x_0 = z \).

Let $z \neq Tz$. Then

\[
d(z, Tz) \leq d(z, T^n x_0) + d(T^n x_0, Tz) \\
\leq d(z, T^n x_0) + \alpha \frac{d(T^{n-1}x_0, T^n x_0) d(z, Tz)}{d(T^{n-1}x_0, Tz) + d(z, T^n x_0) + d(T^{n-1}x_0, z)} \\
+ \beta d(T^{n-1}x_0, z)
\]

Taking limit as $n \to \infty$, we get $d(z, Tz) \leq 0$ implies $z = Tz$.

To prove uniqueness, let $w$ be another fixed point of $T$. Then

\[
d(z, w) = d(Tz, Tw) \leq \alpha \frac{d(z, Tz) d(w, Tw)}{d(z, Tw) + d(w, Tz) + d(z, w)} + \beta d(z, w)
\]

implies $d(z, w) \leq \beta d(z, w) < d(z, w)$, a contradiction which proves $z = w$. 
COROLLARY 1: Let $T$ be a selfmapping of a metric space $(X,d)$ satisfying condition (4.2.1) where $0 \leq \alpha, \beta < 1$. If $T$ is asymptotically regular at a point $x$ in $X$ and sequence of iterates $\{T^n x\}$ has a subsequence converging to a point $z$ in $X$, then $z$ is the unique fixed point of $T$ and sequence $\{T^n x\}$ converges to $z$.

PROOF: Let $\lim_{k \to \infty} T^n_k x = z$ and $z \neq Tz$, Then

$$d(z,Tz) \leq d(z,T^n_k x) + d(T^n_k x, T^{n+1}_k x) + d(T^{n+1}_k x, Tz)$$

$$\leq d(z, T^n_k x) + d(T^n_k x, T^{n+1}_k x)$$

$$+ \alpha \frac{d(T^n_k x, T^{n+1}_k x) d(z,Tz)}{d(T^n_k x, Tz)+d(z,T^n_k x)+d(T^n_k x,z)}$$

$$+ \beta d(T^n_k x, z)$$

letting $k \to \infty$ and using the asymptotic regularity of $T$ at $x$, we have $d(z,Tz) \leq 0$ implies $z = Tz$. Uniqueness of $z$ follows as in Theorem 2. Now we show that $\{T^n x\}$ also converges to $z$.

$$d(z,T^n x) = d(Tz,T^n x) \leq d(Tz, T^{n+1} x) + d(T^{n+1} x, T^n x)$$

$$\leq \frac{\alpha d(z,Tz) d(T^n x, T^{n+1} x)}{d(z, T^{n+1} x) + d(T^n x, Tz) + d(z, T^n x)} + \beta d(T^n x, z)$$
\[ + \beta \ d(z, T^n x) + d(T^{n+1} x, T^n x) \]

letting \( n \to \infty \), it follows easily that \( \lim_{n \to \infty} T^n x = z \).

**COROLLARY 2:** Let \( T \) be a continuous selfmapping of a compact metric space \((X, d)\) satisfying condition (4.2.1). If \( T \) is asymptotically regular at a point \( x \) of \( X \), then \( T \) has a unique fixed point.

**PROOF:** Since space \((X, d)\) is compact therefore the subsequence \( \{T^{n_k} x\} \) of \( \{T^n x\} \) converges to a point say \( z \) in \( X \). Then \( \lim_{k \to \infty} T^{n_k} x = z \).

The rest of the proof is similar to that of Corollary 1.

4.3. In this section we use the contractive condition given by Jaggi [1] for a self-mapping to prove the common fixed point theorem, for three self-mappings using the concept of asymptotic regularity.

For three selfmappings \( P, Q \) and \( T \) of metric space \((X, d)\), the sequence \( \{x_n\} \) is given by \( T x_{2n+1} = P x_{2n} \), \( T x_{2n+2} = Q x_{2n+1} \) for \( n = 0, 1, 2, \ldots \). The set \( O(P, Q, T, x_0) = \{T x_n : n = 1, 2, 3 \ldots\} \) is called the orbit of \((P, Q, T)\).
at $x_0$. $T$ is said to be orbitally continuous at $x_0$ iff it is continuous on $O(P,Q,T,x_0)$. $X$ is called orbitally complete at $x_0$ iff every Cauchy sequence in $O(P,Q,T,x_0)$ converges in $X$. $(P,Q)$ is said to be asymptotically regular with respect to $T$ at $x_0$ if

$$\lim_{n \to \infty} d(Tx_n, Tx_{n+1}) = 0.$$  

Sastry et al [1].

Jaggi [1] has proved the following fixed point theorem for a continuous self map defined on complete metric space.

**Theorem D**: Let $f$ be a continuous selfmap defined on a complete metric space $(X,d)$ satisfying

$$(4.3.1) \quad d(fx, fy) \leq \frac{\alpha d(x, fx) \, d(y, fy)}{d(x, y)} + \beta \, d(x, y)$$

for all $x \neq y \in X$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then $f$ has a unique fixed point in $X$.

**Theorem 2**: Let $P, Q$ and $T$ be three selfmappings of metric space $(X,d)$ satisfying

$$(4.3.2) \quad d(Px, Qy) \leq \frac{\alpha d(Tx, Px) \, d(Ty, Qy)}{d(Tx, Ty)} + \beta \, d(Tx, Ty)$$

for all $x \neq y$ of $X$, $0 \leq \alpha, \beta < 1$, $(P,Q)$ is asymptotically regular with respect to $T$ at some $x_0$ of $X$. $X$ is orbitally
complete at $x_0$ and $T$ is orbitally continuous at $x_0$.

If $T$ commutes with $P$ or $Q$ then $P, Q$ and $T$ have a unique common fixed point.

**Proof**: Since $(P, Q)$ is asymptotically regular with respect to $T$ at $x_0$ then
\[ \lim_{n \to \infty} d(Tx_n, Tx_{n+1}) = 0, \quad \ldots \ (1) \]

where sequence $\{Tx_n\}$ is given by

\[ Tx_{2n+1} = Px_{2n}, \quad Tx_{2n+2} = Qx_{2n+1}. \]

If $\{Tx_n\}$ is not a Cauchy sequence then there exists $\varepsilon > 0$ and strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k < n_k$ and

\[ d(Tx_{m_k}, Tx_{n_k}) \geq \varepsilon, d(Tx_{m_k}, Tx_{n_k-1}) < \varepsilon \]

for all positive integers $k$. Clearly, we have $d(Tx_{m_k}, Tx_{n_k}) \to \varepsilon$ as $k \to \infty$ by (1).

Let $B_1 = \{ k: m_k \text{ is even and } n_k \text{ is odd} \}$,

$B_2 = \{ k: m_k \text{ is even and } n_k \text{ is even} \}$,

$B_3 = \{ k: m_k \text{ is odd and } n_k \text{ is even} \}$ and

$B_4 = \{ k: m_k \text{ is odd and } n_k \text{ is odd} \}$.

Definitely, at least one of $B_i$ is infinite.
Let \( B_1 \) is infinite then for all \( k \in B_1 \) we have

\[
d(T_{m_k}^{n}, T_{n_k}^{n}) \leq d(T_{m_k}^{n}, T_{m_k+1}^{n}) + d(T_{m_k+1}^{n}, T_{n_k+1}^{n}) + d(T_{n_k+1}^{n}, T_{n_k}^{n}) \quad \cdots \quad (ii)
\]

By applying (4.3.2), and using (ii)

\[
d(T_{m_k}^{n}, T_{n_k}^{n}) \leq d(T_{m_k}^{n}, T_{m_k+1}^{n}) + \frac{d(T_{m_k}^{n}, T_{m_k+1}^{n})d(T_{n_k}^{n}, T_{n_k+1}^{n})}{d(T_{m_k}^{n}, T_{n_k}^{n})} + \alpha + \beta d(T_{m_k}^{n}, T_{n_k}^{n}) + d(T_{n_k+1}^{n}, T_{n_k}^{n}).
\]

Letting \( k \) tends to infinity and using (i),

\[
\varepsilon \leq \varepsilon + \frac{\alpha \varepsilon}{\varepsilon} + \beta \varepsilon + \sigma = \beta \varepsilon < \varepsilon, \quad \text{a contradiction.}
\]

Let \( B_2 \) is infinite, then for all \( k \in B_2 \)

\[
d(T_{m_k}^{n}, T_{n_k}^{n}) \leq d(T_{m_k}^{n}, T_{m_k+1}^{n}) + d(T_{m_k+1}^{n}, T_{n_k}^{n}) \quad \cdots \quad (iii)
\]

By applying (4.3.2) and using (iii)

\[
d(T_{m_k}^{n}, T_{n_k}^{n}) \leq d(T_{m_k}^{n}, T_{m_k+1}^{n}) + \frac{d(T_{m_k}^{n}, T_{m_k+1}^{n})d(T_{n_k-1}^{n}, T_{n_k}^{n})}{d(T_{m_k}^{n}, T_{n_k-1}^{n})} + \alpha + \beta d(T_{m_k}^{n}, T_{n_k-1}^{n})
\]
On taking $n \to \infty$ and using (i),

$$\varepsilon \leq o + \frac{\alpha (o) \varepsilon}{\varepsilon} + \beta \varepsilon = \beta \varepsilon \leq \varepsilon,$$

a contradiction.

Similarly, for $B_3$ and $B_4$, we can easily get the same result.

These contradictions show that $\{T_{x_n}\}$ is a Cauchy sequence, $X$ is $(P, Q, T)$-orbitally complete at $x_0$, therefore there exists $z$ of $X$ such that $\lim_{n \to \infty} T_{x_n} = z$.

Now applying (4.3.2), we have

$$d(px_{2n}, Qz) \leq \alpha \frac{d(Tx_{2n}, Tx_{2n+1}) d(Tz, Qz)}{d(Tx_{2n}, Tz)} + \beta d(Tx_{2n}, Tz)$$

letting $n \to \infty$, we get

$$d(z, Qz) \leq \alpha \frac{d(z, z)}{d(z, Tz)} \frac{d(Tz, Qz)}{d(z, Tz)} + \beta d(z, Tz)$$

or

$$d(z, Qz) \leq \beta d(z, Tz) \ldots \ldots \ldots (iv)$$

Similarly by considering $d(Pz, Qx_{2n+1})$, we get easily

$$d(Pz, z) \leq \beta d(Tz, z) \ldots \ldots \ldots (iv)$$

If $PT = TP$ then $PTx_{2n} = TPx_{2n} \to Tz$, as $n \to \infty$.

Since $T$ is orbitally continuous at $x_0$,

By applying (4.3.2)
\[ d(PTx_{2n}, Qx_{2n+1}) \leq \alpha \left( \frac{d(TTx_{2n}, PTx_{2n})}{d(TTx_{2n}, Tx_{2n+1})} \cdot d(Tx_{2n+1}, Qx_{2n+1}) \right) + \beta d(TTx_{2n}, Tx_{2n+1}) \]

Letting \( n \) tends to infinity.

\[ d(Tz, z) \leq \alpha \frac{d(Tz, Tz) \cdot d(z, z)}{d(Tz, Tz)} + \beta d(Tz, z) \]

or \( d(Tz, z) \leq \beta d(Tz, z) \), implies \( Tz = z \).

Similarly if \( QT = TQ \) then by considering \( d(Px_{2n}, QTx_{2n+1}) \) and taking limit, as \( n \to \infty \) we get \( Tz = z \). Therefore from (iv) and (v) we get,

\[ Pz = Qz = Tz = z. \]

To prove uniqueness of \( z \), suppose \( w \) is another fixed point such that \( z \neq w \).

By applying (4.3.2)

\[ d(z, w) = d(Pz, Qw) \leq \alpha \frac{d(Tz, Pz) \cdot d(Tw, Qw)}{d(Tz, Qw)} + \beta d(Tz, Tw) \]

\[ \leq \alpha \frac{d(z, z) \cdot d(w, w)}{d(z, w)} + \beta d(z, w) \]

or \( d(z, w) \leq \beta d(z, w) < d(z, w) \), this contradiction implies \( z = w \).

Thus \( z \) is a unique common fixed point of \( P, Q \) and \( T \).
**COROLLARY 3:** Let $P, Q$ and $T$ be three selfmappings of a metric space $(X,d)$ satisfying condition (4.3.2) of Theorem 2, such that $\alpha + \beta < 1$ and for some $x_0 \in X$ there exists an orbit $O(P,Q,T,x_0)$ such that $X$ is orbitally complete at $x_0$ and $T$ is orbitally continuous at $x_0$. If $T$ commutes with $P$ or $Q$ then $P,Q$ and $T$ have a unique common fixed point.

**PROOF:** If $n$ is odd then

$$d(Tx_n, Tx_{n+1}) = d(Px_{n-1}, Qx_n)$$

$$\leq \alpha \frac{d(Tx_{n-1}, Tx_n) \cdot d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n)}$$

$$+ \beta \cdot d(Tx_{n-1}, Tx_n)$$

implies $d(Tx_n, Tx_{n+1}) \leq k \cdot d(Tx_{n-1}, Tx_n)$, where $k = \frac{\beta}{1-\alpha} < 1$

If $n$ is even then

$$d(Tx_n, Tx_{n+1}) = d(Qx_{n-1}, Px_n) = d(Px_n, Qx_{n-1})$$

$$\leq \alpha \frac{d(Tx_n, Tx_{n+1}) \cdot d(Tx_{n-1}, Tx_n)}{d(Tx_n, Tx_{n-1})}$$

$$+ \beta \cdot d(Tx_n, Tx_{n-1})$$

implies $d(Tx_n, Tx_{n+1}) \leq k \cdot d(Tx_{n-1}, Tx_n)$

Thus we have $d(Tx_n, Tx_{n+1}) \leq k \cdot d(Tx_{n-1}, Tx_n) \forall n \ldots (vi)$
Thus \( \{d(Tx_n, Tx_{n+1})\} \) is a decreasing sequence of non-negative real numbers and therefore will converge to a real number, say \( \omega \).

Letting \( n \to \infty \) in (vi) we get \( \omega \leq \kappa \omega \) which gives \( \omega = 0 \).

Thus \( \lim_{n \to \infty} d(Tx_n, Tx_{n+1}) = 0 \) and therefore by Theorem 2 it follows that \( P, Q \) and \( T \) have a unique common fixed point.

**Theorem 3:** If in Theorem 2, contractive condition (4.3.2) is replaced by

\[
(4.3.3) \quad d(Px, Qy) \leq q \max \left\{ d(Tx, Ty), \frac{d(Tx, Px)}{d(Tx, Ty)}, \frac{d(Tx, Qy)}{d(Ty, Px)} \right\},
\]

where \( q \in (0, 1) \) then \( P, Q \) and \( T \) have a unique common fixed point when \( Tx \neq Ty \).

**Proof:** As in Theorem 2, we can prove \( d(Tx_m, Tx_n) \to \mathcal{E} \) as \( k \to \infty \). The sets \( B_1, B_2, B_3 \) and \( B_4 \) are defined in the same way as in Theorem 2.

Let \( B_1 \) is infinite then for all \( k \in B_1 \) we have

on applying (4.3.3)

\[
d(Tx_{m_k+1}, Tx_{n_k+1}) = d(px_{m_k}, Qx_{n_k}) \leq q \max \left\{ d(Tx_{m_k}, Tx_{n_k}) \right\},\]


\[
d(T_{x_{m_k}}, T_{x_{n_k}+1}) \quad d(T_{x_{n_k}}, T_{x_{n_k}+1})
\]
\[
\frac{d(T_{x_{m_k}}, T_{x_{n_k}})}{d(T_{x_{m_k}}, T_{x_{n_k}})}
\]
\[
\frac{d(T_{x_{m_k}}, T_{x_{n_k}+1})d(T_{x_{n_k}}, T_{x_{m_k}+1})}{d(T_{x_{m_k}}, T_{x_{n_k}})}
\]

By using condition (ii) of Theorem 2

\[
d(T_{x_{m_k}}, T_{x_{n_k}}) \leq d(T_{x_{m_k}}, T_{x_{m_k}+1}) + q \max\{d(T_{x_{m_k}}, T_{x_{n_k}}),
\]
\[
\frac{d(T_{x_{m_k}}, T_{x_{m_k}+1}) d(T_{x_{n_k}}, T_{x_{n_k}+1})}{d(T_{x_{m_k}}, T_{x_{n_k}})}
\]
\[
[ d(T_{x_{m_k}}, T_{x_{n_k}}) + d(T_{x_{n_k}}, T_{x_{n_k}+1}) ] [d(T_{x_{n_k}}, T_{x_{m_k}}) + d(T_{x_{m_k}}, T_{x_{m_k}+1})]
\]
\[
\frac{d(T_{x_{m_k}}, T_{x_{n_k}})}{d(T_{x_{m_k}}, T_{x_{n_k}})}
\]
\[
+ d(T_{x_{n_k}+1}, T_{x_{n_k}}).
\]

Letting \( n \) tends to infinity and using (i)

\[
\epsilon \leq o + q \max\{\epsilon, o, \epsilon\} + o = q \epsilon < \epsilon,
\]

a contradiction.

Let \( B_2 \) is infinite then for all \( k \in B_2 \), we get
by applying (4.3.3) and using condition (iii) of Theorem 2.

\[
d(T_{m_k}^{n_k}, T_{n_k}) \leq d(T_{m_k}^{m_k}, T_{x_{m_k+1}}^{n_k}) + q \max \left\{ d(T_{m_k}^{m_k}, T_{n_{k-1}}^{n_k}), \right.
\frac{d(T_{m_k}^{m_k+1}, T_{n_{k-1}}^{n_k})}{d(T_{m_k}^{m_k}, T_{n_k}^{n_k})}, \right.
\frac{d(T_{m_k}^{m_k}, T_{n_{k-1}}^{n_k})}{d(T_{m_k}^{m_k}, T_{n_k}^{n_k})} \left. d(T_{m_k}^{m_k+1}, T_{n_{k-1}}^{n_k}) \right\}
\]

on taking \( n \to \infty \)

\[\mathcal{E} \leq o + q \max \left\{ \mathcal{E}, o, \mathcal{E}\right\} = q \mathcal{E} < \mathcal{E}, \text{ a contradiction.}\]

For sets \( B_3 \) and \( B_4 \) we can arrive at the same situation.

These contradictions show that \( \{ T_{x_n} \} \) is a Cauchy sequence.

\( X \) is \((P,Q,T)\) orbitally complete at \( x_0 \), therefore there exists \( z \) of \( X \) such that

\[ \lim_{n \to \infty} T_{x_n} = z \]

Now, \( d(Px_{2n}, Qz) \leq q \max \left\{ d(T_{2n}^{2n}, Tz), \right. \frac{d(T_{2n}^{2n+1}, Tz, Qz)}{d(T_{2n}^{2n}, Tz)}, \left. \frac{d(T_{2n}^{2n}, Qz)}{d(T_{2n}^{2n}, Tz)} \right\} \)}

\( d(T_{2n}^{2n+1}, Tz, T_{2n}^{2n+1}) \)}
Letting \( n \to \infty \), we get

\[
d(z,Qz) \leq q \max \left\{ d(z,Tz), \frac{d(z,Qz)}{d(z,Tz)}, \frac{d(Tz,z)}{d(Tz,Tz)} \right\}
\]

implies \( d(z,Qz) \leq q d(z,Tz) \). Similarly, by considering \( d(Pz, Qx_{2n+1}) \) we can get easily \( d(Pz,z) \leq q d(Tz,z) \).

Rest of the proof is similar to that of Theorem 2.

**COROLLARY 4**: Let \( P, Q \) and \( T \) be three selfmappings of a metric space \((X,d)\) satisfying condition (4.3.3) of Theorem 3 and rest of the conditions of Corollary 3 then \( P, Q \) and \( T \) have a unique common fixed point.

**PROOF**: If \( n \) is odd, then

\[
d(Tx_n, Tx_{n+1}) = d(Px_{n-1}, Qx_n)
\]

\[
\leq q \max \left\{ d(Tx_{n-1}, Tx_n), \frac{d(Tx_{n-1}, Tx_n) \cdot d(Tx_n, Tx_{n+1})}{d(Tx_{n-1}, Tx_n)}, \frac{d(Tx_{n-1}, Tx_{n+1}) \cdot d(Tx_n, Tx_n)}{d(Tx_{n-1}, Tx_n)} \right\}
\]

implies \( d(Tx_n, Tx_{n+1}) \leq q d(Tx_{n-1}, Tx_n) \).

If \( n \) is even, then
\[ d(Tx_n, Tx_{n+1}) = d(Qx_{n-1}, Px_n) = d(Px_n, Qx_{n-1}) \]

\[ \leq q \max \left\{ \frac{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})d(Tx_{n-1}, Tx_n)}{d(Tx_n, Tx_{n-1})}, \frac{d(Tx_n, Tx_{n+1})}{d(Tx_n, Tx_{n-1})} \right\} \]

implies \[ d(Tx_n, Tx_{n+1}) \leq q d(Tx_{n-1}, Tx_n) \]

Rest of the proof is similar to that of Corollary 3.

4.4. This section is completely devoted to generalize the result of Sastry, Naidu, Rao and Rao [1] of common fixed point of three mappings under asymptotic regularity to the common fixed point of four mappings.

Sastry et al [1], in 1984 proved the following theorem for three mappings under asymptotic regularity.

**THEOREM E.** Let P,Q and T be self maps on a metric space \((X,d)\) satisfying:

1. \((4.4.1)\) \( PT = TP \) or \( QT = TQ \)

2. \((4.4.2)\) \[ d(Px,Qy) \leq h \max \left\{ d(Tx,Ty), d(Tx, Px), d(Ty, Qy), \right. \]

\[ \left. d(Tx, Qy), d(Ty, Px) \right\} \]
For all \( x, y \in X \), where \( h \in (0, 1) \), and

\[(4.4.3) \quad (P, Q) \text{ is asymptotically regular with respect to } T \text{ at } x_0 \in X.
\]

\( X \) is orbitally complete at \( x_0 \) and \( T \) is orbitally continuous at \( x_0 \). Then \( P, Q \) and \( T \) have a unique common fixed point in \( X \). Before proving our main result we have the following definitions:

Let \( S, T \) be two selfmappings of a metric space \((X, d)\) and \( x_0 \) be an arbitrary point of \( X \). Then the set

\[O(S, T, x_0) = \{x_n : n \in \mathbb{N}_0\}\]  

\((\mathbb{N}_0 \text{ denotes the set of all non-negative integers})\)

where \( x_n = Sx_{n-1} \) when \( n \) is odd, \( x_n = Tx_{n-1} \) when \( n \) is even, is called the orbit of \( S \) and \( T \) with respect to \( x_0 \).

If \( O(S, T, x_0) \) is complete, then \( X \) is said to be \((S, T, x_0)\)-orbitally complete. If \( X \) is \((S, T, x_0)\)-orbitally complete for every \( x_0 \) of \( X \), then \( X \) is said to be \((S, T)\)-orbitally complete. A complete metric space is \((S, T)\)-orbitally complete but the converse need not to be true.Ciric [2].
Let \( P, Q, S, T \) be selfmappings on a metric space \((X, d)\) for a point \( x_0 \in X\), if there exists a sequence \( \{x_n\} \) in \( X \) such that,

\[ P_{x_{2n}} = S_{x_{2n+1}} = y_{2n+1} \quad \text{(say)}, \]

and \( Q_{x_{2n+1}} = T_{x_{2n+2}} = y_{2n+2} \quad \text{(say)}, \)

for \( n = 0, 1, 2, \ldots \).

Then \( O(P, Q, S, T, x_0) = \{ y_n : n = 1, 2, \ldots \} \), is called the orbit of \((P, Q, S, T)\) at \( x_0 \). The pair \((S, T)\) is said to be orbitally continuous at \( x_0 \), if \((S, T)\) is continuous on \( O(P, Q, S, T, x_0) \).

\( X \) is called orbitally complete at \( x_0 \) if and only if every Cauchy sequence in \( O(P, Q, S, T, x_0) \) converges in \( X \).

The pair \((P, Q)\) is said to be asymptotically regular \((a.r.)\) with respect to \((S, T)\) at \( x_0 \), if there exists a sequence \( \{x_n\} \) in \( X \) such that

\[ P_{x_{2n}} = S_{x_{2n+1}} = y_{2n+1} \quad \text{(say), for } n = 0, 1, \ldots \]

\[ Q_{x_{2n+1}} = T_{x_{2n+2}} = y_{2n+2} \quad \text{(say),} \]

and \( d(y_n, y_{n+1}) \to 0 \) as \( n \to \infty \).

i.e. \( d(S_{x_{2n+1}}, T_{x_{2n+2}}) \to 0 \) as \( n \to \infty \).

**Theorem 4:** Let \( P, Q, S, T \) be selfmappings on a metric space \((X, d)\) satisfying

\[ (4.4.4) \quad PS = SP \quad \text{and} \quad QT = TQ. \]
(4.4.5) \[ d(P_x, Q_y) \leq h \max \left\{ d(S_x, T_y), d(S_x, P_x), d(T_y, Q_y), \\ d(S_x, Q_y), d(T_y, P_x) \right\} \]
for all \( x, y \) in \( X \), where \( 0 < h < 1 \).

(4.4.6) Pair \((P, Q)\) is asymptotically regular w.r.t. \((S, T)\) at \( x_0 \).

(4.4.7) \( X \) is orbitally complete at \( x_0 \) and \((S, T)\) is orbitally continuous at \( x_0 \in X \).

Then \( P, Q, S \) and \( T \) have a unique common fixed point in \( X \).

**PROOF:** By (4.4.6), \((P, Q)\) is asymptotically regular with respect to \((S, T)\) at \( x_0 \in X \), there exists a sequence \( \{x_n\} \)
in \( X \), such that

\[ P_{x_{2n}} = S_{x_{2n+1}} = y_{2n+1} \quad \text{(say)} \]

\[ Q_{x_{2n+1}} = S_{x_{2n+2}} = y_{2n+2} \quad \text{(say)} \]

for \( n = 0, 1, 2, \ldots \),

Suppose \( \{y_n\} \) is not a Cauchy sequence. Then there exists an \( \varepsilon > 0 \) and we define strictly monotonic increasing sequences of positive integers \( \{m_k\} \) and \( \{n_k\} \) with \( m_k < n_k \)
and \( d(y_{m_k}, y_{n_k}) \geq \varepsilon \) and \( d(y_{m_k}, y_{n_k-1}) < \varepsilon \) for all \( k = 1, 2, \ldots (1) \)

Taking \( n \rightarrow \infty \), it is clear that by (1)

\[ d(y_{m_k}, y_{n_k}) \rightarrow \varepsilon . \quad . \quad . \quad . \quad (2) \]
Following four conditions are possible:

(i) \( B_1 = k : m_k \) is even and \( n_k \) is odd

(ii) \( B_2 = k : m_k \) is even and \( n_k \) is even

(iii) \( B_3 = k : m_k \) is odd and \( n_k \) is even

(iv) \( B_4 = k : m_k \) is odd and \( n_k \) is odd.

Suppose at least one of \( B_1 \), \( B_2 \), \( B_3 \) and \( B_4 \) is infinite.

we consider condition (i). Let \( B_1 \) is infinite for all \( k \in B_1 \).

\[
d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k+1}) + d(y_{m_k+1}, y_{n_k+1}) + d(y_{n_k+1}, y_{n_k}) \quad \text{...(3)}
\]

Applying (4.4.5)

\[
d(y_{m_k+1}, y_{n_k+1}) = d(Sx_{m_k+1}, Tx_{n_k+1}) = d(Px_{m_k}, Qx_{n_k})
\]

\[
\leq h \max \{ d(Sx_{m_k}, Tx_{n_k}), d(Sx_{m_k}, Px_{m_k}),
\]
\[
d(Tx_{n_k}, Qx_{n_k}), d(Sx_{m_k}, Qx_{n_k}),
\]
\[
d(Tx_{n_k}, Px_{m_k}) \}
\]

\[
\leq h \max \{ d(y_{m_k}, y_{n_k}), d(y_{m_k}, y_{m_k+1}),
\]
\[
d(y_{n_k}, y_{n_k+1}), d(y_{m_k}, y_{n_k+1})d(y_{n_k}, y_{m_k+1}) \}
\]
By using (3), we have

\[ d(y_m, y_n) \leq d(y_m, y_{m+1}) + h \max \left\{ d(y_m, y_n), d(y_m, y_{m+1}) \right\} \]

\[ d(y_n, y_{n+1}) + [d(y_m, y_n) + d(y_n, y_{n+1})] \]

\[ \left\{ d(y_m, y_n) + d(y_m, y_{m+1}) \right\} + d(y_n, y_{n+1}) \cdot \]

On taking \( n \to \infty \) and using (1) and (2),

\[ \varepsilon \leq o + h \max \left\{ \varepsilon, o, o, [\varepsilon + o], [\varepsilon + o] \right\} + o \]

or \( \varepsilon \leq h \varepsilon \cdot \varepsilon \), since \( o < h < 1 \). This contradiction implies that \( B_1 \) is not infinite.

Suppose \( B_2 \) is infinite, for all \( k \in B_2 \)

\[ d(y_m, y_n) \leq d(y_m, y_{m+1}) + d(y_{m+1}, y_n) \quad \cdots (4) \]

By applying (4.4.5)

\[ d(y_{m+1}, y_n) = d(Sx_{m+1}, Tx_n) = d(Px_m, Qx_{n-1}) \]

\[ \leq h \max \left\{ d(Sx_m, Tx_{n-1}), d(Sx_m, Px_m), \right\} \]

\[ d(Tx_{n-1}, Qx_{n-1}), d(Sx_m, Qx_{n-1}), d(Tx_{n-1}, Px_m) \right\} ; \]
\[ \leq h \max \left\{ d(y_{m_k}, y_{n_{k-1}}), d(y_{m_k}, y_{m_k+1}), d(y_{n_{k-1}}, y_{n_k}), \\
\quad d(y_{m_k}, y_{n_k}), d(y_{n_k}, y_{m_k+1}) \right\} \]

Now, using (4), we have

\[ d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k+1}) + h \max \left\{ d(y_{m_k}, y_{n_{k-1}}), \\
\quad d(y_{m_k}, y_{m_k+1}), d(y_{n_{k-1}}, y_{n_k}), d(y_{m_k}, y_{n_k}) \right\} \]

\[ \left[ d(y_{n_k}, y_{m_k}) + d(y_{m_k}, y_{m_k+1}) \right]. \]

Letting \( n \) tends to infinity and using (1) and (2), we get \( \varepsilon \leq o + h \max \left\{ \varepsilon, o, o, [\varepsilon + o] \right\} \)
or \( \varepsilon \leq h \varepsilon < \varepsilon \), a contradiction, this shows that \( B_2 \) is not infinite.

Similar is the situation in the remaining cases i.e. for \( B_3 \) and \( B_4 \), we can easily get the same result.

Hence \( \{y_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is \( (P,Q,S,T) \)-orbitally complete at \( x_o \), it follows that there exists a point \( z \in X \) such that \( y_n \to z \) as \( n \to \infty \).
By applying (4.4.5)
\[ d(S_{2n+1}, Qz) = d(P_{2n}, Qz) \]
\[ \leq h \max \{ d(S_{2n}, Tz), d(S_{2n}, S_{2n+1}), d(Tz, Qz), \]
\[ d(S_{2n}, Qz), d(Tz, S_{2n+1}) \} \]

On taking \( n \to \infty \).

\[ d(z, Qz) \leq h \max \{ d(z, Tz), d(z, z), d(Tz, Qz), d(z, Qz), d(Tz, z) \} \]

Since \( d(z, Qz) \leq h \cdot d(z, Qz) < d(z, Qz) \), a contradiction.

So that
\[ d(z, Qz) \leq h \max \{ d(z, Tz), d(Tz, Qz) \} \ldots (5) \]

Similarly, again applying (4.4.5),
\[ d(P_{2n+1}, T_{2n+1}) = d(P_z, Q_{2n+1}) \]
\[ \leq h \max \{ d(S_z, T_{2n+1}), d(S_z, P_z), d(T_{2n+1}, T_{2n+2}), \]
\[ d(S_z, T_{2n+1}), d(T_{2n+2}, P_z) \} \]

Letting \( n \to \infty \).

\[ d(P_z, z) \leq h \max \{ d(S_z, z), d(S_z, P_z), d(z, z), d(S_z, z), d(z, P_z) \} \]

Since \( d(P_z, z) \leq h \cdot d(z, P_z) < d(P_z, z) \), a contradiction.

\[ \Rightarrow d(P_z, z) \leq h \max \{ d(S_z, z), d(S_z, P_z) \} \ldots \ldots (6) \]
Now, if $PS = SP$ then $PSx_{2n} = SPx_{2n} \to Sz$, as $n \to \infty$.

since $S$ is orbitally continuous at $x_0$.

By applying (4.4.5)

$$d(PSx_{2n}, Qx_{2n+1}) \leq h \max \left\{ d(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, PSx_{2n}), d(Tx_{2n+1}, Qx_{2n+1}), d(SSx_{2n}, Qx_{2n+1}), d(Tx_{2n+1}, PSx_{2n}) \right\}$$

Letting $n$ tends to infinity

$$d(Sz, z) \leq h \max \left\{ d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z)d(z, Sz) \right\}$$

$$= h \ d(Sz, z) < d(Sz, z), \text{since } 0 < h < 1.$$

This contradiction implies $Sz = z$ \hspace{1cm} (7)

and similarly $QT = TQ$, then $QTx_{2n+1} = TX_{2n+1} \to Tz$

as $n \to \infty$, and since $T$ is also orbitally continuous at $x_0$.

Again, applying (4.4.5)

$$d(Px_{2n}, QTx_{2n+1})$$

$$\leq h \max \left\{ d(Sx_{2n}, TTx_{2n+1}), d(SSx_{2n}, Px_{2n}), d(TTx_{2n+1}, QTx_{2n+1}), d(SSx_{2n}, QTx_{2n+1}), d(TTx_{2n+1}, Px_{2n}) \right\}$$

Letting $n \to \infty$

$$d(z, Tz) \leq h \max \left\{ d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(Tz, z) \right\}$$
\[ d(z, Qz) \leq h \max \left\{ d(z, z), d(z, Qz) \right\} = h \ d(z, Qz) \text{ and} \]

\[ d(Pz, z) \leq h \max \left\{ d(z, z), d(z, Pz) \right\} = h \ d(Pz, z) \]

These contradictions imply \( Qz = z \) and \( Pz = z \).

Thus \( z \) is a common fixed point of \( P, Q, S \) and \( T \).

For uniqueness of \( z \), let \( w \) is another fixed point of \( P, Q, S \) and \( T \) such that \( z \neq w \).

By (4.4.5), we have

\[ d(z, w) = d(Pz, Qw) \leq h \max \left\{ d(Sz, Tw), d(Tz, Pz), d(Tw, Qw), d(Sz, Qw), d(Tw, Pz) \right\} \]

\[ \leq h \max \left\{ d(z, w), d(z, z), d(w, w), d(z, w), d(w, z) \right\} \]

or \( d(z, w) \leq h \ d(z, w) < d(z, w) \), a contradiction.

This shows that \( z = w \), i.e. \( z \) is a unique common fixed point of \( P, Q, S \) and \( T \).

**COROLLARY 5:** Let \( P, Q, S \) and \( T \) be selfmappings on a metric space \( (X, d) \) satisfying
(4.4.8) If $PS = SP$ and $QT = T$

(4.4.9) $d(Px, Qy) \leq h \max \left\{ d(Sx, Ty), d(Sx, Px), d(Ty, Qy), \right\}$

$$\frac{1}{2} \left[ d(Sx, Qy) + d(Ty, Px) \right]$$, for all $x, y \in X$

for all $x, y \in X$, where $0 < h < 1$, for $x_0 \in X$, there exists an orbit $O(P, Q, S, T, x_0)$ such that $X$ is orbitally complete at $x_0$ and $(S, T)$ is orbitally continuous at $x_0$ then $P, Q, S$ and $T$ have a unique common fixed point.

**Proof**: For a positive integer $n$, by (4.4.9), we get

$$d(y_{2n+1}, y_{2n+2}) = d(Sx_{2n+1}, Tx_{2n+2}) = d(Px_{2n}, Qx_{2n+1})$$

$$\leq h \max \left\{ d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Sx_{2n+1}), \right\}$$

$$d(Tx_{2n+1}, Tx_{2n+2}),$$

$$\frac{1}{2} \left[ d(Sx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Sx_{2n+1}) \right]$$

$$= h \max \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \right\}$$

$$d(y_{2n+1}, y_{2n+2}),$$

$$\frac{1}{2} \left[ d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1}) \right]$$

$$\Rightarrow d(y_{2n+1}, y_{2n+2}) \leq h \max \left\{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \right\}$$

$$\frac{1}{2} [d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]$$
Following three cases are possible.

(i) \( d(y_{2n+1}, y_{2n+2}) \leq h \ d(y_{2n+1}, y_{2n+2}) \) is impossible

(ii) \( d(y_{2n+1}, y_{2n+2}) \leq h \ d(y_{2n}, y_{2n+1}) \)

(iii) \( d(y_{2n+1}, y_{2n+2}) \leq h \ \frac{1}{2} \lbrack d(y_{2n}, y_{2n+1})+d(y_{2n+1}, y_{2n+2}) \rbrack \)

or \( d(y_{2n+1}, y_{2n+2}) \leq h_1 \ d(y_{2n}, y_{2n+1}) \), where \( h_1 = \frac{h}{2-h} < 1 \)

If \( q = \max\{h, h_1\} \), then

\[ d(y_{2n+1}, y_{2n+2}) \leq q \ d(y_{2n}, y_{2n+1}) \] \( \cdots (i) \)

By continuing above process, we get \( \{d(y_n, y_{n+1})\} \) is a decreasing sequence of non-negative real numbers and it converges to a non-negative real number, say \( \alpha \).

Letting \( n \to \infty \) in (i) we get \( \alpha \leq q \ \alpha < \alpha \), a contradiction.

\( \Rightarrow \alpha = 0 \). Thus \( d(y_n, y_{n+1}) \to 0 \) as \( n \to \infty \).

Hence by Theorem 4, we get a unique common fixed point of \( P, Q, S \) and \( T \).

REMARK: If we take \( S = T \) in Theorem 4, we have Theorem E.