CHAPTER III

COMMON FIXED POINT THEOREMS FOR
NEARLY DENSIFYING MAPPINGS

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3.1. C. Kuratowski [1] in 1966 has introduced and studied the important concept of "Measure of non-compactness of bounded set!"

Measure of non-compactness of bounded set $A$ of a metric space $X$, denoted by $\alpha(A)$, is the infimum of all $\varepsilon > 0$ such that $A$ admits a finite covering by sets with diameter less than $\varepsilon$.

Nussbaum [1] and Iseki [1] have studied and obtained many useful properties of the measure of non-compactness of bounded set $A$. Some of its important properties are:

(i) $0 \leq \alpha(A) \leq \delta(A)$ where $\delta(A)$ is the diameter of $A$.
(ii) If $X$ is complete and $\alpha(A) = 0$, then $A$ is compact.
(iii) $\alpha(\overline{A}) = 0$ iff $\alpha(A) = 0$, where $\overline{A}$ is the closure of $A$.
(iv) $\alpha(A \cup B) = \max \{\alpha(A), \alpha(B)\}$ for any two bounded subsets $A, B$ of $X$.

The concept of densifying mapping was introduced and studied by Furi and Vignoli [1].
**DEFINITION 1**: A mapping $T$ defined on a metric space $X$ to itself is called densifying if for every bounded subset $A$ of $X$ with $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

Sastry and Naidu [1] in 1982, have extended the idea of densifying mapping and introduced the concept of nearly densifying mapping.

**DEFINITION 2**: A self mapping $T$ on a metric space $X$ is said to be nearly densifying if $\alpha(T(A)) < \alpha(A)$, whenever $\alpha(A) > 0$, $A$ is bounded and $T$-invariant.

It follows that every densifying mapping is nearly densifying.

3.2. In this section, we obtain some common fixed point theorems of Jungck type [1] for nearly densifying mappings on complete metric space.

**THEOREM 1**: Let $S$ and $T$ be commuting, continuous and nearly densifying self mappings of a complete metric space $(X,d)$. Let $F$ be a lower-semi-continuous mapping of $X \times X$ into $\mathbb{R}^+$ satisfying the following conditions:
(3.2.1) \( F(x, x) = 0 \quad \forall x \in X \)

(3.2.2) \( F(x, y) \leq F(x, z) + F(z, y) \quad \forall x, y, z \in X \)

(3.2.3) \( F(Tx, Ty) \leq c_1 \frac{F(Sx, Tx) F(Sy, Ty)}{F(Sx, Sy)} + c_2 \left( F(Sx, Tx) + F(Sy, Ty) \right) + c_3 \left( F(Sx, Ty) + F(Tx, Sy) \right) \)

for \( Sx \neq Sy, Tx \neq Ty \). Constants \( c_1, c_2, c_3 \in \mathbb{R}^+ \) such that \( c_1 + 2c_2 + 2c_3 < 1 \) and for some \( x_0 \) in \( X \), the set \( A = \{ T^i J^j x_0 : i, j \geq 0 \} \) is bounded. Then \( S \) and \( T \) have a unique common fixed point.

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**Proof:** Clearly, \( A = \{ x_0 \} \cup S \mathring{A} \cup T \mathring{A} \). Since \( S \) and \( T \) are commuting and continuous, therefore \( S \mathring{A} \subseteq \mathring{A} \), \( T \mathring{A} \subseteq \mathring{A} \). Since \( S \) and \( T \) are nearly densifying and \((X, d)\) is complete, therefore \( \mathring{A} \) is compact.

Let \( H = \bigcap_{n=1}^{\infty} (ST)^n \mathring{A} \). Clearly, \( \{(ST)^n \mathring{A}\} \) is a decreasing sequence of non-empty compact subset of \( \mathring{A} \), therefore it follows that \( H \) is a non-empty compact set. Clearly \( SH \subseteq H \), \( TH \subseteq H \).

Let \( x \) be any element of \( H \), then \( x \in (ST)^{n+1} \mathring{A} \) for all \( n \). Therefore there exists \( \{x_n\} \subseteq (ST)^n \mathring{A} \) such
that $ST^n x = x$ for all $n$. Since $S, T$ are continuous and
$(ST)^n \bar{A}$ is compact and closed for all $n$, therefore
there exists a point $p \in (ST)^n \bar{A}$ for each $n$ and hence
$ST^n p = x$. Therefore $x \in SH$ and $x \in TH$. Thus $SH = H = TH$.

Since $F$ is lower semi-continuous, the real valued
mapping $\emptyset$ defined on $H$ given by $\emptyset(x) = F(Sx, Tx)$ is
lower semi-continuous and hence attains its infimum in $H$.
Let $\emptyset(u) = \inf \{ F(Sx, Tx) : x \in H \}$. Since $SH = H$ there
exists $v \in H$ such that $u = S v$. Suppose there is no
point $x$ in $X$ such that $Sx = Tx$. Now, by applying
(3.2.3), we get

$$F(STv, T^2v) = F(TSv, TTv)$$

$$\leq c_1 \frac{F(S^2v, TSv) F(STv, T^2v)}{F(S^2v, STv)} + c_2 \left\{ F(S^2v, TSv) + F(STv, T^2v) \right\}$$

$$+ c_3 \left\{ F(S^2v, T^2v) + F(STv, STv) \right\}$$

By using (3.2.1) and (3.2.2) it follows that

$$\leq c_1 F(STv, T^2v) + c_2 \left\{ F(S^2v, TSv) + F(STv, T^2v) \right\}$$

$$+ c_3 \left\{ F(S^2v, STv) + F(STv, T^2v) \right\}$$
or

$$F(STv, T^2v) \leq h F(S^2v, TSv), \text{ where } h = \frac{c_2 + c_3}{1 - c_1 c_2 - c_3} < 1$$

$$< F(S^2v, TSv)$$
implies $\emptyset(Tv) < \emptyset(Sv) = \emptyset(u)$, a contradiction. Since $\emptyset(u)$ is the infimum. Therefore there exists $z \in H$ such that $Sz = Tz$ and hence $S^2z = STz = TSz$.

Let $S^2z \neq Sz$, then by using (3.2.3) we get

$$F(S^2z, Sz) = F(TSz, Tz) \leq c_1 \frac{F(S^2z, TSz) F(Sz, Tz)}{F(S^2z, Sz)}$$

$$+ c_2 \left\{ F(S^2z, TSz) + F(Sz, Tz) \right\}$$

$$+ c_3 \left\{ F(S^2z, Tz) + F(TSz, Sz) \right\}$$

$$\leq 2c_3 F(S^2z, Sz) < F(S^2z, Sz),$$

a contradiction which shows $Sz = S^2z = TSz$ and this proves that $Sz$ is a common fixed point of mappings $S$ and $T$.

To prove uniqueness, let $w$ be another common fixed point of $S$ and $T$, then by (3.2.3), we have

$$F(Sz, w) = F(TSz, Tw) \leq c_1 \frac{F(S^2z, Tw) F(Sw, Tw)}{F(S^2z, Sw)}$$

$$+ c_2 \left\{ F(S^2z, TSz) + F(Sw, Tw) \right\}$$

$$+ c_3 \left\{ F(S^2z, Tw) + F(TSz, Sw) \right\}$$

$$\leq 2c_3 F(Sz, w) < F(Sz, w).$$
This contradiction proves that $S$ and $T$ have a unique common fixed point.

**Theorem 2**: If in theorem 1, the contractive condition (3.2.3) is replaced by

\[ F(Tx, Ty) < \max \left\{ F(Sx, Sy), F(Sx, Tx), F(Sy, Ty), \right. \]
\[
\left. \frac{1}{2} \left[ F(Sx, Tx) + F(Sy, Ty) \right], \frac{1}{2} \left[ F(Sx, Ty) + F(Tx, Sy) \right] \right\}. \]

Then $S$ and $T$ have a unique common fixed point.

**Proof**: We shall define the set $H$, mapping $\emptyset$ on $H$ and prove $SH = H = TH$ as in theorem 1, there exists $v \in H$ such that $u = Sv$ where $\emptyset(u) = \inf \left\{ F(Sx, Tx) : x \in H \right\}$. Suppose there is no point $x$ in $X$ such that $Sx = Tx$. Then by applying (3.2.4), we get

\[
F(STv, T^2v) = F(TSv, TVv)
\]
\[
< \max \left\{ F(S^2v, STv), F(S^2v, TSv), \right. \]
\[
F(STv, T^2v), \frac{1}{2} \left[ F(S^2v, TSv) + F(STv, T^2v) \right], \]
\[
\left. \frac{1}{2} \left[ F(S^2v, T^2v) + F(TSv, STv) \right] \right\}
\]

By using (3.2.1) and (3.2.2) it follows easily that

\[ F(STv, T^2v) < F(S^2v, TSv) \]

i.e.
\[ \varnothing(Tv) < \varnothing(Sv) = \varnothing(u), \text{ a contradiction. Since } \varnothing(u) \text{ is the infimum. Therefore there exists } z \in H \text{ such that } Sz = Tz \text{ and hence } S^2z = STz = TSz. \]

Let \( S^2z \neq Sz \), then by applying (3.2.4), we get
\[
F(S^2z, Sz) = F(TSz, Tz) < \max \left\{ F(S^2z, Sz), F(S^2z, TSz), F(Sz, Tz), \frac{1}{2} \left[ F(S^2z, TSz) + F(Sz, Tz) \right] \right\} \]
\[
= \max \left\{ F(S^2z, Sz), F(S^2z, S^2z), F(Sz, Sz), \frac{1}{2} \left[ F(S^2z, S^2z) + F(Sz, Sz) \right], \frac{1}{2} \left[ F(S^2z, Sz) + F(S^2z, Sz) \right] \right\} \]
implies \( F(S^2z, Sz) < F(S^2z, Sz) \), a contradiction. Therefore \( Sz = S^2z = TSz \), showing that Sz is a common fixed point of S and T.

For uniqueness of Sz. Let w is another fixed point, \( F(Sz, w) = F(TSz, Tw) \)
\[
< \max \left\{ F(S^2z, Sw), F(S^2z, TSz), F(Sw, Tw), \frac{1}{2} \left[ F(S^2z, TSz) + F(Sw, Tw) \right] \right\}, \]
\[ \frac{1}{2} \left[ F(S^2 z, Tw) + F(TSz, Sw) \right] \]

\[ = \max \left\{ F(Sz, w), F(Sz, Sz), F(w, w), \frac{1}{2} \left[ F(Sz, Sz) + F(w, w) \right] \right\} \]

\[ \frac{1}{2} \left[ F(Sz, w) + F(Sz, w) \right] \]

or

\[ F(Sz, w) < F(Sz, w), \] a contradiction implies

\[ Sz = w. \] Hence \( S \) and \( T \) have a unique common fixed point.

**THEOREM 3**: Let \( S \) and \( T \) be a commuting, continuous and nearly densifying selfmappings of a complete metric space \((X,d)\) satisfying

\[ F(Tx, Ty) < \frac{F(Sx, Tx) F(Sx, Ty) + F(Sy, Ty) F(Sy, Tx)}{F(Sx, Ty) + F(Sy, Tx)} \ (3.2.5) \]

For \( Sx \neq Sy, Tx \neq Ty \) where \( F \) be lower semi-continuous mapping of \( X \times X \) into \( R^+ \) such that \( F(x, x) = 0 \) for all \( x \) in \( X \) and for some \( x_0 \) in \( X \), the set \( A = \left\{ S^j T^i x_0 : i, j > 0 \right\} \) is bounded. Then \( S \) and \( T \) have a unique common fixed point.

**PROOF**: The set \( H \) is defined in the same manner and we can prove \( SH = H = TH \) as in the proof of theorem 1.

The real valued mapping \( \emptyset \) on \( H \) given by \( \emptyset(x) = F(Sx, Tx) \) is lower semi-continuous and hence attains its infimum at
u of H. There exists v ∈ H such that u = Sv. Suppose there is no point x in X such that Sx = Tx. Then by applying (3.2.5), we have

\[ F(STv, TTv) = F(TSv, TTv) < \frac{F(S^2v, TSv) F(S^2v, T^2v) + F(STv, T^2v) F(STv, TSv)}{F(S^2v, T^2v) + F(STv, TSv)} \]

or \[ F(STv, T^2v) < F(S^2v, TSv) \]

implies \( \emptyset(Tv) < \emptyset(Sv) = \emptyset(u) \), a contradiction, since \( \emptyset(u) \) is the infimum. Therefore there exists z ∈ H such that \( Sz = Tz \) and hence \( S^2z = STz = TSz \).

Let \( S^2z \neq Sz \) then by using (3.2.5) we have

\[ F(S^2z, Sz) = F(TSz, Tz) < \frac{F(S^2z, TSz) F(S^2z, Tz) + F(Sz, Tz) F(Sz, TSz)}{F(S^2z, Tz) + F(Sz, TSz)} \]

\[ = \frac{F(S^2z, S^2z) F(S^2z, Sz) + F(Sz, Sz) F(Sz, S^2z)}{F(S^2z, Sz) + F(Sz, S^2z)} \]

implies \( F(S^2z, Sz) < o \), a contradiction therefore \( Sz = S^2z = TSz \) showing that Sz is a common fixed point of S and T. Uniqueness of the common fixed point can be easily seen as in proof of Theorem 1.
3.3. In this section we assume the lower semi-continuous mapping $F$ of $X \times X$ into $\mathbb{R}^+$ is symmetric too and we have obtained some common fixed point theorems of Jungck [1] type of complete metric spaces.

**Theorem 4:** Let $S$ and $T$ be commuting, continuous and nearly densifying self mappings of a complete metric space $(X,d)$. Let $F$ be symmetric and lower semi-continuous mapping of $X \times X$ into $\mathbb{R}^+$ satisfying the following conditions:

\[(3.3.1) \quad F(x,x) = 0 \quad \forall x \in X \]
\[(3.3.2) \quad F(x,y) \leq F(x,z) + F(z,y) \quad \forall x,y,z \in X \]
\[(3.3.3) \quad [F(Tx,Ty)]^2 \leq c_1 \left\{ F(Sx,Tx) F(Sy,Ty) + F(Sx,Ty)F(Sy,Tx) \right\} + c_2 \left\{ F(Sx,Tx) F(Sy,Tx) + F(Sx,Ty)F(Sy,Ty) \right\} + c_3 \quad F(Sx,Sy) F(Tx,Ty) \]

where constants $c_i \in \mathbb{R}^+$, $i = 1,2,3$, such that $c_1 + 2c_2 + c_3 < 1$ and for some $x_0$ in $X$, the set

$A = \left\{ s^i T^j x_0 : i,j \geq 0 \right\}$

is bounded.

Then $S$ and $T$ have a unique common fixed point.

**Proof:** Clearly $A = \left\{ x_0 \right\} \cup SA \cup TA$, since $S$ and $T$ are commuting and continuous therefore $SA \subseteq A$, $TA \subseteq A$. 
Since $S$ and $T$ are nearly densifying and $(X,d)$ is complete, therefore $\bar{A}$ is compact.

Let $H = \bigcap_{n=1}^{\infty} (ST)^n \bar{A}$. Clearly, $\{ (ST)^n \bar{A} \}$ is a decreasing sequence of non-empty compact subsets of $\bar{A}$ and hence $H$ is a non-empty compact set. Clearly $SH \subseteq H$, $TH \subseteq H$.

Let $x$ be any element of $H$ then $x \in (ST)^{n+1} \bar{A}$ for all $n$. Therefore there exists $\{ x_n \} \subseteq (ST)^n \bar{A}$ such that $STx_n = x$ for all $n$. Since $S, T$ are continuous and $(ST)^n \bar{A}$ is compact and closed for all $n$ and therefore there exists a point $p \in (ST)^n \bar{A}$, for each $n$ and thus $STp = x$. Therefore $x \in SH$, $x \in TH$. Thus $SH = H = TH$.

Since $F$ is lower semi-continuous, the real valued mapping $\Phi$ on $H$ given by $\Phi(x) = F(Sx, Tx)$ is lower semi-continuous and hence attains its infimum in $H$. Let

$\Phi(u) = \inf \{ F(Sx, Tx) : x \in H \}$. Since $SH = H$, there exists $v \in H$ such that $u = Sv$. Suppose there is no point $x$ in $X$ such that $Sx = Tx$. By applying (3.3.3) and using (3.3.1), (3.3.2), we get.

$$[F(STv, TTv)]^2 = [F(TSv, TTv)]^2$$
\[
\leq \left\{ c_1 F(S^2v, TSv) + c_2 F(S^2v, T^2v) + c_3 F(S^2v, STv) \right\} F(STv, T^2v)
\]

implies

\[
F(STv, T^2v) \leq h F(S^2v, TSv) < F(S^2v, TSv),
\]

where \( h = \frac{c_1 + c_2 + c_3}{1 - c_2} < 1 \).

Thus \( \emptyset (Tv) < \emptyset (Sv) = \emptyset (u) \), a contradiction. Since \( \emptyset (u) \)
is the infimum therefore there exists \( z \in H \) such that \( Sz = Tz \) and hence

\[
S^2z = STz = TSz \ldots \ldots \ldots \ldots \ (1)
\]

Let \( S^2z \neq Sz \), then by applying (3.3.3) and using (3.3.1), (1) we get

\[
[F(S^2z, Sz)]^2 = [F(TSz, Tz)]^2
\]

\[
\leq c_1 \left\{ F(S^2z, TSz) F(Sz, Tz) + F(S^2z, Tz) F(Sz, TSz) \right\}
\]

\[
+ c_2 \left\{ F(S^2z, TSz) F(Sz, TSz) + F(S^2z, Tz) F(Sz, Tz) \right\}
\]

\[
+ c_3 F(S^2z, Sz) F(TSz, Tz)
\]

\[
\leq (c_1 + c_3) [F(S^2z, Sz)]^2 < [F(S^2z, Sz)]^2
\]

\[
[\text{since } c_1 + c_3 \leq c_1 + 2c_2 + c_3 < 1]
\]
a contradiction, which proves $S^2z = TSz = STz$
and thus $Sz$ is a common fixed point of $S$ and $T$.

To prove uniqueness let $w$ be another common fixed
point of $S$ and $T$, then by applying (3.3.3),

$$[F(Sz,w)]^2 = [F(TSz, Tw)]^2$$

$$\leq c_1 \left\{ F(S^2z, TSz) F(Sw, Tw) + F(S^2z, Tw) F(Sw, TSz) \right\}$$

$$+ c_2 \left\{ F(S^2z, TSz) F(Sw, TSz) + F(S^2z, Tw) F(Sw, Tw) \right\}$$

$$+ c_3 F(S^2z, Sw) F(TSz, Tw)$$

$$\leq (c_1 + c_3) \left[ F(Sz,w) \right]^2 < \left[ F(Sz, w) \right]^2 .$$

This contradiction proves that $S$ and $T$ have unique common
fixed point.

**Theorem 5:** Let $S$ and $T$ be commuting, continuous and
nearly densifying self mappings of a complete metric space
$(X,d)$ satisfying

(3.3.4) $F(Tx, Ty) < \max \left\{ \frac{F(Sx, Tx) F(Sy, Ty)}{F(Sx, Sy)}, \frac{F(Sy, Tx) F(Sx, Ty)}{F(Tx, Ty)}, F(Sx, Sy) \right\}$
for \( Sx \neq Sy, Tx \neq Ty \) where \( F \) be symmetric and lower semi-continuous mapping of \( X \times X \) into \( \mathbb{R}^+ \) such that \( F(x,x) = 0 \) \( \forall x \in X \) and for some \( x_0 \) in \( X \), the set \( A = \{ S^i T^j x_0 : i, j \geq 0 \} \) is bounded. Then \( S \) and \( T \) have a unique common fixed point.

**PROOF:** The set \( H \) is defined in the same manner and we can prove \( SH = H = TH \) as in the proof of theorem 4. The real valued mapping \( \emptyset \) on \( H \) given by \( \emptyset (x) = F(Sx, Tx) \) is lower semi-continuous and hence attains its infimum at \( u \) of \( H \). There exists \( v \in H \) such that \( u = Sv \). Suppose there is no point \( x \) in \( X \) such that \( Sx = Tx \). Then by applying (3.3.4) we have,

\[
F(STv, TTv) = F(TSv, TTv)
\]

\[
< \max \left\{ \frac{F(S^2v, TSv) F(STv, T^2v)}{F(S^2v, STv)}, \frac{F(STv, TSv) F(S^2v, T^2v)}{F(TSv, T^2v)}, \frac{F(S^2v, STv)}{F(S^2v, STv)} \right\}
\]

\[
= \max \left\{ F(STv, T^2v), o, F(S^2v, STv) \right\}
\]

implies \( F(STv, T^2v) < F(S^2v, STv) \),

Thus \( \emptyset (Tv) < \emptyset (Sv) = \emptyset (u) \), a contradiction.
Hence there exists \( z \in H \) such that \( S^2z = Tz \) and therefore \( S^2z = STz = TSz \).

Let \( S^2z \neq Sz \) then by applying (3.3.4) we get,

\[
F(S^2z, Sz) = F(TSz, Tz)
\]

\[
< \max \left\{ \frac{F(S^2z, TSz)}{F(S^2z, Sz)}, F(S^2z, Sz) \right\}
\]

\[
= \max \left\{ \frac{F(S^2z, S^2z)}{F(S^2z, Sz)}, F(S^2z, Sz) \right\}
\]

implies \( F(S^2z, Sz) < F(S^2z, Sz) \). This contradiction proves that \( Sz = S^2z = TSz \).

Thus \( Sz \) is a common fixed point of \( S \) and \( T \).

For uniqueness, let \( w \) is another fixed point of \( S \) and \( T \). By applying (3.3.4)

\[
F(Sz, w) = F(TSz, Tw)
\]

\[
< \max \left\{ \frac{F(S^2z, Tw)}{F(S^2z, Sw)}, F(S^2z, Sw) \right\}
\]
\[ F(\text{Sw}, \text{TSz}) \frac{F(S^2z, \text{Tw})}{F(\text{TSz}, \text{Tw})}, \frac{F(S^2z, \text{Sw})}{F(\text{TSz}, \text{Tw})} \]

\[ = \max \left\{ \frac{F(\text{Sz}, \text{w})}{F(\text{Sz}, \text{w})}, \frac{F(\text{w}, \text{Sz})}{F(\text{Sz}, \text{w})}, F(\text{Sz}, \text{w}) \right\} \]

or

\[ F(\text{Sz}, \text{w}) < F(\text{Sz}, \text{w}) \text{ a contradiction, implies S and T} \]

has a unique common fixed point.

**Theorem 6:** If in theorem 5, contractive condition (3.3.4) is replaced by

\[ (3.3.5) F(\text{Tx}, \text{Ty}) < \frac{\max \left\{ F(\text{Sx}, \text{Tx}) F(\text{Sy}, \text{Ty}), F(\text{Sx}, \text{Ty}) F(\text{Sy}, \text{Tx}) \right\}}{\max \left\{ F(\text{Sx}, \text{Sy}), F(\text{Tx}, \text{Ty}) \right\}} \]

for \( \text{Sx} \neq \text{Sy} \) and \( \text{Tx} \neq \text{Ty} \). Then S and T have a unique common fixed point.

**Proof:** We shall define the set \( H \) in similar way and can prove \( \text{SH} = H = \text{TH} \), as in the proof of theorem 4. The real valued mapping \( \phi \) on \( H \) given by \( \phi(x) = F(\text{Sx}, \text{Tx}) \) is lower semi-continuous and hence attains its infimum at \( u \) of \( H \). There exists \( v \in H \) such that \( u = \text{Sv} \). Suppose there is no point \( x \) in \( X \) such that \( \text{Sx} = \text{Tx} \). By applying (3.3.5),

\[ F(\text{STv}, \text{T}^2\text{v}) = F(\text{TSv}, \text{ITv}) \]

\[ < \frac{\max \left\{ F(S^2\text{v}, \text{TSv}) F(\text{STv}, \text{T}^2\text{v}), F(S^2\text{v}, \text{T}^2\text{v}) F(\text{STv}, \text{TSv}) \right\}}{\max \left\{ F(S^2\text{v}, \text{STv}), F(\text{TSv}, \text{T}^2\text{v}) \right\}} \]
implies

Either \( F(STv, T^2v) < F(S^2v, STv) \) or \( F(STv, T^2v) < F(S^2v, TSv) \) implies \( \emptyset(Tv) < \emptyset(Sv) = \emptyset(u) \), a contradiction since \( \emptyset(u) \) is the infimum. Therefore there exists \( z \in H \) such that \( Sz = Tz \) and hence \( S^2z = STz = TSz \).

Let \( S^2z \neq Sz \) by applying (3.3.5)

\[
F(S^2z, Sz) = F(TSz, Tz)
\]

\[
\frac{\max \{F(S^2z, Tz), F(Sz, Tz), F(S^2z, Tz) \} \cdot F(Sz, TSz)}{\max \{F(S^2z, Sz), F(TSz, Tz) \}} < \frac{\max \{F(S^2z, Sz) \} \cdot F(Sz, Sz)}{\max \{F(S^2z, Sz), F(S^2z, Sz) \}}
\]

In each case, we get

\( F(S^2z, Sz) < F(S^2z, Sz) \) a contradiction which implies that \( Sz = S^2z = TSz \).

Thus \( Sz \) is a common fixed point of \( S \) and \( T \).

Now for uniqueness, let \( w \) be another common fixed point of \( S \) and \( T \).
\[ F(Sz, w) = F(TSz, Tw) \]

\[
\max \left\{ \frac{F(S^2z, TSz) F(Sw, Tw), F(S^2z, Tw) F(Sw, TSz)}{\max \{ F(S^2z, Sw), F(TSz, Tw) \}} \right\}
\]

\[
= \frac{\max \{ F(Sz, Sz) F(w, w), F(Sz, w) F(w, Sz) \}}{\max \{ F(Sz, w), F(Sz, w) \}}
\]

In each case, we have

\[ F(Sz, w) < F(Sz, w) \], a contradiction, which implies \( S \) and \( T \) have a unique common fixed point.

This completes the proof.