CHAPTER VII

FIXED POINT THEOREMS IN NONARCHIMEDEAN MENGERS PROBABILISTIC METRIC SPACE
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MENGER PROBABILISTIC METRIC SPACE

7.1. In the present chapter we extend some common fixed point theorems obtained by Chang [4] for single-valued and multi-valued mappings in non-archimedean Menger PM-space.

Before the statement of our results, we mention the following lemmas, which are required in the sequel.

**Lemma 1** (Schweizer and Sklar [2]). \( \Delta \) is a strictly increasing Archimedean t-norm, if and only if

\[
\Delta(s,t) = g^{-1}[g(s) + g(t)], \text{ for all } s,t \in [0,1],
\]

where \( g : [0,1] \to [0,\infty) \) is a continuous and strictly decreasing function with \( g(1) = 0, g(0) = +\infty \), and \( g^{-1} \) is the quasi-inverse of \( g \), i.e.

\[
gog^{-1}(t) = t, \text{ for all } t \in \text{Rang}(g) \text{ (the range of } g).\]

**Lemma 2** (Chang [4]). Let \( \Phi(t) \) satisfy condition \( \Phi_1 \) or \( \Phi_2 \), then
(i) $\tilde{\Phi}(t) < t$, for all $t > 0$;

(ii) if $t \leq \tilde{\Phi}(t)$, then $t = 0$.

**Lemma 3** (Chang [4]). Let $A \in \mathcal{A}$, and $x, y \in E$. Then

(i) $F_{x,A}(t) = 1$, for all $t > 0$, if and only if $x \in A$;

(ii) $F_{x,A}(\max(t_1, t_2)) \geq \Delta(F_{x,y}(t_1), F_{y,A}(t_2))$, for all $t_1, t_2 \geq 0$;

(iii) For any $B \in \mathcal{J}$, $F_{x,B}(t) \geq F_{A,B}(t), x \in A$.

**Theorem 1.** Let $(E, \tilde{\Phi}, \Delta)$ be a complete nonarchimedean Menger PM-space, $\Delta$ a strictly increasing Archimedean t-norm. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of mappings and $T$ be a continuous mapping of $E$ into itself. Suppose that each $S_i$ commutes with $T$ for all $i \in \mathbb{Z}^+$ (the set of all positive integers) and for all $x, y \in E$ and any $t > 0$, $F_{x,y}(t) \neq 0$, and that for any $i, j \in \mathbb{Z}^+$, $i \neq j$ and any $x, y \in E$ the condition holds:

$$(7.1.2) \quad g(F_{S_i x, S_j y}(t)) \leq \tilde{\Phi}(\max\{ g(F_{T x, T y}(t)), g(F_{T x, S_j x}(t)), g(F_{T y, S_j y}(t)), g(F_{T x, S_i x}(t)) \})$$

for all $t > 0$;

where $g$ is the function defined by (7.1.1) and $\tilde{\Phi}$ satisfies the...
condition \( (\phi_2) \). Suppose further that there exists an \( x_0 \in E \) such that the sequence \( \{ y_n \}_{n=1}^{\infty} \) defined by

\[
y_n = T x_n = S_n x_{n-1}, \quad n \geq 1
\]

satisfies the following condition.

\[
(7.1.3) \sup_{n \geq q} \{ g(F_{y_0, y_n} (t)) \} < + \infty, \quad \text{for all } t > 0
\]

Then \( \{ S_n \}_{n=1}^{\infty} \) and \( T \) have a unique common fixed point in \( E \).

**Proof.** Similar to the proof of Chang ([4], Theorem 3), it can be easily shown that for any \( n \in \mathbb{Z}^+ \).

\[
(7.1.4) \quad g(F_{y_n, y_{n+1}} (t)) \leq \Phi^n (\max \{ g(F_{y_0, y_1} (t)), \ldots, g(F_{y_0, y_n} (t)), \ldots, g(F_{y_0, y_{n+1}} (t)) \}), \quad \text{for all } t > 0.
\]

Now we prove that \( \{ y_n \} \) is a Cauchy sequence of \( E \).

Applying (7.1.1) and (1.2.11), we have

\[
g(F_{y_n, y_{n+m}} (t)) \leq g(\Delta(F_{y_n, y_{n+1}} (t)), F_{y_{n+1}, y_{n+m}} (t))
\]

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\begin{align*}
&= g(F_{y_n, y_{n+1}}(t)) + g(F_{y_{n+1}, y_{n+m}}(t)) \\
&\leq g(F_{y_n, y_{n+1}}(t)) + \ldots \\
&\quad + g(F_{y_{n+m-1}, y_{n+m}}(t)), \text{ for all } t > 0.
\end{align*}

Taking

\[ u_o(t) = \sup_{n \geq 1}\{g(F_{y_{o}, y_{n}}(t))\}, \]

by (7.1.3) we have

\[ u_o(t) < + \infty, \text{ for all } t > 0. \]

By (7.1.4) and (7.1.5) it follows that

\[ g(F_{y_n, y_{n+m}}(t)) \leq \sum_{i=1}^{n+m-1} \Phi^i(u_o(t)) \to 0 \text{ as } n \to \infty, \]

for all \( m \in \mathbb{Z}^+, t > 0, \) and therefore

\[ F_{y_n, y_{n+m}}(t) \to 1 \text{ as } n, m \to \infty, \text{ for all } t > 0. \]
Thus \( \{y_n\} \) is a Cauchy sequence. Since \( E \) is complete, we assume that

\[
y_n = T x_n \rightarrow z \in E.
\]

Consequently,

\[
S_n x_{n-1} \rightarrow z.
\]

Since \( T \) is continuous and \( S_n \) commutes with \( T \) for all \( n \in \mathbb{Z}^+ \), we have

\[
TT x_n \rightarrow Tz \text{ and } S_n T x_{n-1} = T S_n x_{n-1} \rightarrow Tz.
\]

for all \( n \in \mathbb{Z}^+ \).

Next, we prove that \( z \) is a common fixed point of \( \{S_n\}_{n=1}^{\infty} \) and \( T \).

In fact, for arbitrary given \( i \in \mathbb{Z}^+ \), any \( n \in \mathbb{Z}^+ \), \( n > i \), and any \( t > 0 \),

\[
g(F_{S t z, T z}(t)) \leq g(F_{S t z, S_T x_{n-1}}(t)) + g(F_{S_T x_{n-1}, T z}(t))
\]

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\[
\leq \bar{\Phi} \left( \max\{g(F_{Tz,TTx_{n-1}}(t)), g(F_{Tz,S_{i,n-1},Tz}(t)), \\
g(F_{TTx_{n-1},S_{i,n-1},Tz}(t)), g(F_{Tz,S_{i,n-1},Tz}(t)), \\
g(F_{TTx_{n-1},S_{i,n-1},Tz}(t)), g(F_{S_{i,n-1},Tz}(t)) \right) + g(F_{S_{i,n-1},Tz}(t)),
\]
for all \( t > 0 \).

Letting \( n \to \infty \) and using the continuity of \( g \) and the right continuity of \( \bar{\Phi} \), we get

\[
g(F_{S_{i,z},Tz}(t)) \leq \bar{\Phi}(g(F_{S_{i,z},Tz}(t))
\]
for all \( t > 0 \).

Now Lemma 2, gives

\[
g(F_{S_{i,z},Tz}(t)) = 0,
\]
for all \( t > 0 \) and therefore
\[ F_{S_i^z, Tz}(t) = 1 \text{ for all } t > 0. \]

Hence \( S_i^z = Tz. \)

Further, we have

\[ g(F_{z, S_i^z}(t)) \leq g(F_{z, y_n'}(t)) + g(F_{y_n', S_i^z}(t)) \]

\[ = g(F_{z, y_n}(t)) + g(F_{S_n x_{n-1}, S_i^z}(t)) \]

\[ \leq g(F_{z, y_n}(t)) + \bar{\Phi}(\max\{g(F_{y_{n-1}, Tz}(t)), \]

\[ g(F_{y_{n-1}, y_n}(t)), g(F_{Tz, S_i^z}(t)), \]

\[ g(F_{y_{n-1}, S_i^z}(t)), g(F_{Tz, y_n}(t))\}), \]

\[ g(F_{y_{n-1}, S_i^z}(t)) \leq g(F_{y_{n-1}^z, Tz}(t)) + g(F_{z, S_i^z}(t)). \]

Substituting (7.1.7) into (7.1.6), letting \( n \to \infty \) and using the continuity of \( g \) and the right continuity of \( \bar{\Phi} \), we have
\[ g(\text{F}_{z,S_i^z}(t)) \leq \Phi (g(\text{F}_{z,S_i^z}(t))). \text{ for all } t > 0. \]

Now by Lemma 2 yields
\[ g(\text{F}_{z,S_i^z}(t)) = 0 \]
for all \( t > 0 \) and therefore
\[ \text{F}_{z,S_i^z}(t) = 1, \text{ for all } t > 0. \]
Hence
\[ z = S_i^z. \]
Since \( i \in Z^+ \) is arbitrary, it follows that \( z \) is a common fixed point of \( (S_n^i)_{n=1}^\infty \) and \( T \).
It can be easily proved that \( z \) is unique.

**COROLLARY 1.** On taking \( T = I \) (Identity mapping) we get Theorem 3 of Chang [4].

**THEOREM 2.** Let \( (S_n^i) \) and \( (T_n^i) : E \rightarrow \mathcal{L} \) be sequences of multivalued mappings. Suppose that for any \( x,y \in E \), \( t > 0 \), \( F_{x,y}(t) \neq 0 \). Suppose further that for any \( x,y \in E \), and \( i,j \in Z^+ \), \( i \neq j \) and any...
t > 0 the following condition holds:

\[
(7.1.8) \quad g(F_{S_i,x}, T_{j,y} (t)) \leq \Phi \left( g(F_{x,y} (t)), g(F_{x,S_i x} (t)), g(F_{y,T_j y} (t)), \\
g(F_{x,T_j y} (t)), g(F_{y,S_i x} (t)) \right),
\]

where \( \Phi : [0, \infty)^5 \rightarrow [0, \infty) \) is nondecreasing for each variable, right-continuous and for any \( t \geq 0 \),

\[
\Psi(t) = \max\{ \Phi(t,t,t,2t,0), \Phi(t,t,t,0,2t), \Phi(0,0,t,t,0), \\
\Phi(0,t,0,0,0) \} < t,
\]

where the function \( \Psi(t) : [0, \infty) \rightarrow [0, \infty) \) is nondecreasing, right-continuous and

\[
\sum_{n=1}^{\infty} \Psi^n(t) < +\infty, \text{ for all } t > 0.
\]

Suppose further that there exists an \( x_0 \in E \) and the sequence \( \{x_n\} \),

where

\[
x_{2n-1} \in S_n x_{2n-2} \text{ and } x_{2n} \in T_n x_{2n-1}, \quad n \geq 1
\]

\[
(7.1.9) \quad g(F_{x_{2n-1}, x_{2n}} (t)) \leq g(F_{S_n x_{2n-2}, T_n x_{2n-1}} (t))
\]

and
\[(7.1.10)\quad g(F_{x_{2n}x_{2n+1}}(t)) \leq g(F_{S_nx_{2n-2}T_nx_{2n-1}}(t)),\]
\[n = 1, 2, \ldots, \quad t > 0.\]

Then \(\{S_n\}_{n=1}^{\infty}\) and \(\{T_n\}_{n=1}^{\infty}\) have a common fixed point in \(E\).

**Proof.** First we prove that \(\{x_n\}\) is a Cauchy sequence of \(E\).

Applying (7.1.8), (7.1.9), (1.2.13) and (1.2.14), for any \(n \in \mathbb{Z}^+\) and \(t > 0\), we have

\[(7.1.11)\quad g(F_{x_{2n-1}x_{2n}}(t)) \leq g(F_{S_nx_{2n-2}T_nx_{2n-1}}(t))\]
\[\leq \Phi(g(F_{x_{2n-2}x_{2n-1}}(t)), g(F_{x_{2n-2}x_{2n-1}}(t)),\]
\[g(F_{x_{2n-1}x_{2n}}(t)), g(F_{x_{2n-1}x_{2n}}(t)),\]
\[g(F_{x_{2n-1}x_{2n-1}}(t))).\]

If there exists some \(t_0 > 0\) such that

\[g(F_{x_{2n-2}x_{2n-1}}(t_0)) < g(F_{x_{2n-1}x_{2n}}(t_0)),\]

(7.1.11) yields
g(F_{x_{2n-1},x_{2n}}(t_0)) \leq \Phi (g(F_{x_{2n-1},x_{2n}}(t_0)),
\begin{align*}
g(F_{x_{2n-1},x_{2n}}(t_0)),
g(F_{x_{2n-1},x_{2n}}(t_0)),
2g(F_{x_{2n-1},x_{2n}}(t_0),0)
\end{align*}
\leq \Psi (g(F_{x_{2n-1},x_{2n}}(t_0))
\leq g(F_{x_{2n-1}',x_{2n}}(t_0))
\text{a contradiction.}

Therefore for any } t > 0,
\begin{align*}
g(F_{x_{2n-1},x_{2n}}(t)) \leq g(F_{x_{2n-2},x_{2n-1}}(t))
\end{align*}

Similarly, by (7.1.8), (7.1.10), (1.2.13) and (1.2.14) we have
\begin{align*}
g(F_{x_{2n},x_{2n+1}}(t)) \leq g(F_{x_{2n-1},x_{2n}}(t))
\end{align*}

Therefore in view of (7.1.11), we get
\begin{align*}
g(F_{x_{2n-1}',x_{2n}}(t)) \leq \Phi (g(F_{x_{2n-2},x_{2n-1}}(t)),
g(F_{x_{2n-2},x_{2n-1}}(t)),
g(F_{x_{2n-2},x_{2n-1}}(t)),
2g(F_{x_{2n-2},x_{2n-1}}(t),0)
\end{align*}
\[ \leq \psi \left( g(F_{x_{2n+1}x_{2n}})(t) \right) \]

Similarly

\[ g(F_{x_{2n+1}x_{2n}})(t) \leq \psi \left( g(F_{x_{2n-1}x_{2n}})(t) \right). \]

In general, for any \( t > 0 \),

\[ g(F_{x_nx_{n+1}})(t) \leq \psi \left( g(F_{x_{2n-1}x_{2n}})(t) \right) \leq \psi^m(g(F_{x_0x_1})(t)), \text{ for all } t > 0, \]

and so for any \( n, m \in \mathbb{Z}^+ \)

\[ g(F_{x_nx_{n+m}})(t) \leq g(F_{x_nx_{n+1}})(t) + \psi^m(g(F_{x_{2n-1}x_{2n}})(t)) \leq \sum_{i=n}^{n+m-1} \psi^i(g(F_{x_0x_1})(t))) \rightarrow 0 \text{ as } n \rightarrow \infty \]

for all \( m \in \mathbb{Z}^+ \) and for all \( t > 0 \).

This implies that

\[ F_{x_nx_{n+m}}(t) \rightarrow 1 \text{ for all } t > 0. \]
Therefore \( \{x_n\} \) is a Cauchy sequence in \( E \) and since \( E \) is complete it converges to a point \( z \in E \).

Consequently, the subsequences \( \{x_{2n-1}\} \) and \( \{x_{2n}\} \) also converge to \( z \).

Now we prove that \( z \) is a common fixed point of \( \{S_n\}_{n=1}^{\infty} \) and \( \{T_n\}_{n=1}^{\infty} \). In fact, for any \( i,n \in \mathbb{Z}^+ \), \( n > i \), by Lemma 3, (7.1.1) and (7.1.8) we have

\[
g(F_{z,T_i,z}^i(t)) \leq g(F_{z,x_{2n-1}}(t)) + g(F_{x_{2n-1},T_i,z}^i(t))
\]

\[
\leq g(F_{z,x_{2n-1}}(t)) + g(F_{s_n^{x_{2n-2}},T_i,z}^i(t))
\]

\[
\leq g(F_{z,x_{2n-1}}(t)) + \Phi(g(F_{x_{2n-2},z}^{x_{2n-2}}(t)), g(F_{z,T_i,z}(t)), g(F_{x_{2n-2},T_i,z}(t)), g(F_{z,x_{2n-1}}(t)))
\]

for any \( t > 0 \).

Letting \( n \to \infty \) and considering the continuity of \( g \) and the right continuity of \( \Phi \), we obtain

\[
g(F_{z,T_i,z}^i(t)) \leq \Phi(0,0,g(F_{z,T_i,z}(t)), g(F_{z,T_i,z}(t)), 0)
\]
\[ \psi(g(F_{z,T_i z})(t)) \leq \psi(F_{z,T_i z}(t)) \quad \text{for all } t > 0, \]

and so

\[ F_{z,T_i z}(t) = 1 \quad \text{for all } t > 0. \]

This shows that \( z \in T_i z \) for all \( i \in Z^+ \).

Similarly \( z \in S_i z \) for all \( i \in Z^+ \).

Therefore, \( z \) is a common fixed point of \( \{S_n\}_{n=1}^\infty \) and \( \{T_n\}_{n=1}^\infty \). This completes the proof.

**Corollary 2.** If we take \( S_n = T_n \) for all \( n \in Z^+ \) in Theorem 2, we obtain Theorem 5 of Chang [4].