APPENDIX
ON COMMON FIXED POINT THEOREM OF THREE MAPPINGS

By

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In this paper we shall prove a common fixed point theorem for three mappings in complete metric space. The obtain theorem is a generalizations of Banch [1], Kannan [3] and Fisher [2].

Theorem. Let $E$, $F$, and $T$ be three continuous mappings of a complete metric space $X$ into itself satisfying

[1] $ET = TE$, $FT = TF$, $EX \subseteq TX$ and $FX \subseteq TX$

[II] $d(Ex, FY) \leq \alpha_1 \left[ \frac{d(Tx, Fy)}{d(Tx, Ty) + d(Ty, Fy)} \right]$

$+ \alpha_2 \{d(Tx, Ex) + d(Ty, Fy)\} + \alpha_3 \{d(Tx, Fy) + d(Ty, Ex)\}$

$+ \alpha_4 \{d(Tx, Ty)\}$

for all $x$, $y$ in $X$ with $Tx \neq Ty$, where $\alpha_1 \geq 0$, and $\alpha_1 + 2\alpha_3 + 2\alpha_3 + \alpha_4 < 1$

Then $E$, $F$, and $T$ have a unique common fixed point.

Proof. Let $x_0$ be an arbitrary point of $X$, and the sequence $( Tx_n )$ of elements of $X$ is define such that

$TX_{n+1} = EX_n$, $TX_{n+2} = FX_{n+1}$

for $n = 0, 1, 2, ...$

We can do this since $F(X) \subseteq T(X)$ and $E(X) \subseteq T(X)$. Then by [II] we have,

$d(Tx_{n+1}, TX_{n+2}) = d(EX_n, FX_{n+1})$

$\leq \alpha_1 \left[ \frac{d(Tx_n, FX_{n+1})}{d(Tx_n, TX_{n+1}) + d(TX_{n+1}, FX_{n+1})} \right]$

$+ \alpha_2 \{d(Tx_n, EX_n) + d(TX_{n+1}, FX_{n+1})\}$

$+ \alpha_3 \{d(Tx_n, FX_{n+1}) + d(TX_{n+1}, EX_n)\}$

$+ \alpha_4 \{d(Tx_n, TX_{n+1})\}$
\[ d(Tx_{n+1}, Tx_{n+2}) \leq a_1 \left[ \frac{d(Tx_{2n}, Tx_{2n+1}) d(Tx_{2n}, Sx_{2n+1})}{d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n}, Sx_{2n+1})} \right] \\
+ a_2 \left[ d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2}) \right] \\
+ a_3 \left[ d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2}) \right] \\
+ a_4 d(Tx_{2n}, Tx_{2n+1}) \]

\[ d(Tx_{n+1}, Tx_{n+2}) \leq a_1 \left( \frac{d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})}{d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})} \right) \\
+ a_2 \left[ d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2}) \right] \\
+ a_3 \left[ d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2}) \right] \\
+ a_4 d(Tx_{2n}, Tx_{2n+1}) \]

\[ (1 - a_5) d(Tx_{2n+1}, Tx_{2n+2}) \leq (a_1 + a_2 + a_3 + a_4) d(Tx_{2n}, Tx_{2n+1}) \]

Putting \[ h = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_5 - a_6} \] we find \( h < 1 \).

Since \( a_1 + 2a_2 + 2a_3 + a_4 < 1 \).

Hence

\[ d(Tx_{n+1}, Tx_{n+2}) \leq h d(Tx_{n}, Tx_{n+1}) \]

Similarly we can see,

\[ d(Tx_{2n}, Tx_{2n+1}) \leq h d(Tx_{2n-1}, Tx_{2n}) \]

proceeding in this way, we have,

\[ d(Tx_{2n+1}, Tx_{2n+2}) \leq h^{n+1} d(Tx_{0}, Tx_{1}) \]

By routine calculations the following inequalities hold for \( k \gg n \).

\[ d(Tx_{2n}, Tx_{2n+1}) \leq \sum_{i=1}^{n} d(Tx_{2i-1}, Tx_{2i}) \]

\[ \leq \sum_{i=1}^{n} \frac{h^{n+1}}{1-h} d(Tx_{0}, Tx_{1}) \]

\[ \leq \frac{k e}{1-h} d(Tx_{0}, Tx_{1}) \rightarrow 0 \text{ as } n \rightarrow \infty. \]
Since \( h < 1 \). Hence \( \{T_n\} \) is a cauchy sequence. By the completeness of \( X \), \( \{T_n\} \) converges to a point \( u \) in \( X \). It follows from [I] that \( \{E_n\} \) and \( \{F_n\} \) also converges to \( u \). Since \( E \), \( F \) and \( T \) are continuous, we have

\[ E(T_n) \rightarrow E(u), \quad F(T_n + 1) \rightarrow F(u). \]

From [I], \( T \) commutes with \( E \) and \( F \), therefore

\[ E(T_n) = T(E_n), \quad F(T_n + 1) = T(F_n + 1) \]

For all \( n = 0, 1, 2, \ldots \). Taking \( n \rightarrow \infty \) we have

\[ E(u) = T(u) = F(u) \]

Now

\[ \text{[V]} \quad E(u) = T(u) = F(u) \]

And

\[ \text{[VI]} \quad T(T(u)) = T(E(u)) = E(T(u)) = E(F(u)) = T(F(u)) = F(T(u)) \]

Equating to \( F(u) \), we have

\[ d(E(u), F(E(u))) \leq \alpha_1 \left[ d(T(u), E(E(u))) + d(T(E(u)), F(F(u))) \right] + \alpha_2 \left[ d(T(u), F(F(u))) + d(T(F(u)), E(E(u))) \right] + \alpha_3 \left[ d(T(u), E(E(u))) + d(T(E(u)), F(F(u))) \right] + \alpha_4 \left[ d(T(u), F(F(u))) + d(T(F(u)), E(E(u))) \right] \]

\[ d(E(u), E(E(u))) \leq (\alpha_1 + 2\alpha_2 + \alpha_4) d(E(u), F(E(u))) \]

leading to a contradiction, since

\[ \alpha_1 + 2\alpha_2 + \alpha_4 \leq \alpha_1 + 2\alpha_2 + 2\alpha_4 + \alpha_4 < 1. \]

Hence

\[ \text{[VII]} \quad E(u) = F(E(u)) \]

Using [VII] and [VII] we have

\[ E(u) = F(E(u)) = T(E(u)) = E(u) \]

which shows that \( E(u) \) is the common fixed point of \( E, F \) and \( T \).

Let \( z \) and \( w \) be two points in \( X \) such that

\[ E(z) = T(z) = w \quad \text{and} \quad E(w) = T(w) = z \]

Then by [II] we have

\[ d(z, w) = d(z, w) \]

\[ \leq \alpha_1 \left[ d(z, w) + d(T(z), T(w)) \right] + \alpha_2 \left[ d(T(z), T(w)) + d(T(w), T(z)) \right] + \alpha_3 \left[ d(T(z), T(w)) + d(T(w), T(z)) \right] \]

\[ d(E(z), E(w)) \leq (\alpha_1 + 2\alpha_2 + \alpha_4) d(E(z), E(w)) \]

leading to a contradiction, since

\[ \alpha_1 + 2\alpha_2 + \alpha_4 \leq \alpha_1 + 2\alpha_2 + 2\alpha_4 + \alpha_4 < 1. \]
Hence $x = y$. This implies the uniqueness of common fixed point for $E$, $F$ and $T$. This completes the proof of the theorem.

REMARKS:

1. Taking $E = F$ and $T = I$ identity mapping $I$ and $x_1 = x_2 = x_3 = 0$ we get Banach Contraction Principle Theorem [1].
2. Taking $E = F$ and $T = I$ and $x_1 = x_2 = x_3 = 0$ we get Kannan Theorem [3].
3. Taking $E = F$ and $T = I$ and $x_1 = x_2 = x_3 = 0$ we get Fisher Theorem [2].

REFERENCES

Common fixed point theorems in T-orbitally complete metric space

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Summary: A number of authors have proved fixed point theorems for the mappings in metric space. Dhage\(^3\) proved some non-unique fixed point theorems for orbitally continuous self-mappings on a T-orbitally complete metric space.

Afterward Qureshi\(^4\) and Pande\(^5\) proved some results on non-unique fixed point of the mappings on a T-orbitally complete metric space.

In this paper we shall prove common fixed point theorems in T-orbitally complete metric space. The theorems obtained are generalizations of the results of Qureshi and Pande\(^5\).

**Theorem:** A - Let \( M \) be a metric space with two metrics \( d_1 \) and \( d \) with

(i) \( d_1(x, y) \leq d(x, y) \) for all \( x, y \in M \),

(ii) \( M \) is orbitally complete with respect to \( d_1 \), and

(iii) The mappings \( S: M \to M \) and \( T: M \to M \) is orbitally continuous with respect to \( d_1 \) and

\[
[d(Sx, Ty)]^2 \leq [a_1 d(x, y) d(y, Ty) + a_2 d(x, Sx) d(Sx, Ty) + a_3 d(x, Ty) d(y, Sx)]
\]

for all \( x, y \in M \) and \( a_1, a_2, a_3 \) are real numbers such that \( a_1 + a_2 = h, h \in (0, 1) \).

Then and \( T \) have common fixed point and if \( a_3 < 1 \) then \( S \) and \( T \) have unique fixed point.

**Proof:** Let \( x_0 \in M \) be arbitrary, then we define a sequence

\[
Sx_{2n} = x_{2n+1}, \quad Tx_{2n+1} = x_{2n+2}.
\]

Now for \( x = x_n \) and \( y = x_{n+1} \), from (iii) we obtain,

\[
\begin{align*}
[d(Sx_{2n}, Tx_{2n+1})]^2 & \leq [a_1 d(x_{2n}, x_{2n+1})] \\
& \quad + a_2 d(x_{2n}, Sx_{2n}) d(Sx_{2n}, Tx_{2n+1}) \\
& \quad + a_3 d(x_{2n}, Tx_{2n+1}) d(x_{2n+1}, Sx_{2n}) \\
& \quad + [a_2 d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}) \\
& \quad \quad + a_2 d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})] \\
& \quad \quad + a_3 d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}) \\
& \quad \quad \quad + [a_2 d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2})]
\end{align*}
\]

hence, we have,

\[
[d(x_{2n+1}, x_{2n+2})]^2 \leq [a_1 d(x_{2n}, x_{2n+1})] \\
[d(x_{2n+1}, x_{2n+2})] \\
[d(x_{2n+1}, x_{2n+2})]^2 \leq h \left[ d(x_{2n}, x_{2n+1}) d(x_{2n+1}, x_{2n+2}) \right]
\]

Where \( a_1 + a_2 = h, h \in (0, 1) \),

\[
d(x_{2n+1}, x_{2n+2}) \leq h d(x_{2n}, x_{2n+1}).
\]
continuing in this manner, we get,

\[ d(x_{n+1}, x_n) \leq \frac{1}{h} d(x_0, x_1), \]

and hence,

\[ d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, x_1), \]

Where \( p \) is any positive integer.

Therefore by (i), we have,

\[ d_1(x_n, x_{n+p}) \leq d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, x_1). \]

It follows that \( \{x_n\} \) is a cauchy sequence with respect to \( d_1 \). Let \( M \) be T-orbitally complete, there is some \( u \in M \) such that

\[ u = \lim_{n \to \infty} T^n x_n. \]

By orbital continuity of \( T, T^n u = \lim_{n \to \infty} T^n x_n = u. \)

Thus \( u \) is the fixed point of \( T \). Similarly we can show that \( u \) is fixed point of \( S \).

Now claim to its uniqueness. If possible, let \( v \) be another fixed point for \( S \) and \( T \).

i.e. \( u = Su = Tv \) and \( v = Sv = Tv, u \neq v \).

From (iii) we obtain,

\[ [d(u,v)]^2 = [d(Su, Tv)]^2 \]

\[ \leq \left[ d_1 (u,v) d(v, Tv) + a_2 d(u,Su)d(Su,Tv) + a_1 d(u, Tv)d(v,Su) \right]^2 \]

\[ \leq a_1 [d(u,v)]^2. \]

Leading to a contradiction, since \( a_1 < 1 \).

Thus \( S \) and \( T \) have a unique fixed point.

This completes the proof.

Theorem B: Let \( S: M \to M \) and \( T: M \to M \) be an orbitally continuous

Self-mappings of a metric space \( M \), and let \( M \) be a \( T \)-\( T \)-arbitrarily complete.

If \( S \) and \( T \) satisfy the condition,

\[ [d(Sx, Ty)]^2 \leq [a_1 d(x, y) d(y, Ty) + a_2 d(x, Sx) d(Sx, Ty) + a_3 d(x, Ty) d(y, Sx)] \]

... (B.1)

for all \( x, y \in M \) and \( a_1, a_2, a_3 \) are real numbers such that \( 0 < a_1 + a_2 < 1 \), then for each \( x \in M \), the sequence \( \{T^n x\} \) converges to a fixed point of \( S \) and \( T \), and if \( a_3 < 1 \) then \( S \) and \( T \) have unique fixed point.

Proof: Let \( x \in M \) be arbitrary, then we define a sequence

\[ S \times 2^n = x_{2^n+1}, T x_{2^n+1} = x_{2^n+2} \]

... (B.2)

If for some \( n \in \mathbb{N}, x_n = x_{n+1}. \)

Then \( \{x_n\} \) is a cauchy sequence and the limit of \( x_n \) is a fixed point of \( T \). Similarly we can show that \( x_n \) is a fixed point of \( S \).

Suppose that \( x_n \neq x_{n+1} \) for each \( n = 0, 1, 2, \ldots \), then for \( x = x_{2^n+1} \) and \( y = x_{2^n+1}. \)

Also from (B.1), we have,

\[ [d(Sx_{2^n}, Tx_{2^n+1})]^2 \leq [a_1 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, Tx_{2^n+1}) \]

\[ + a_2 d(x_{2^n}, Sx_{2^n}) d(Sx_{2^n}, Tx_{2^n+1}) + a_3 d(x_{2^n}, Tx_{2^n+1}) d(y, Sx_{2^n})], \]

\[ [d(x_{2^n+1}, x_{2^n+2})]^2 \leq [a_1 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, x_{2^n+2}) \]

\[ + a_2 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, x_{2^n+2}) + a_3 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, x_{2^n+2})], \]

\[ [d(x_{2^n+1}, x_{2^n+2})]^2 \leq [a_1 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, x_{2^n+2}) \]

\[ + a_2 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, x_{2^n+2}) + a_3 d(x_{2^n}, x_{2^n+1}) d(x_{2^n+1}, x_{2^n+2})], \]
\[
[d(x_{2n+1,h}^{2n+2})] \leq (a_1 + a_2) [d(x_{2h}, x_{2h+1})]
\]
\[
d(x_{2h+1,h}^{2h+2}) \leq h [d(x_{2h}, x_{2h+1})] = (a_1 + a_2) [d(x_{2h}, x_{2h+1})]
\]
\[
\text{Where } a_1 + a_2 = h, h \in (0,1).
\]
d(x_{2n+1,h}^{2n+2}) \leq h d(x_{2h}, x_{2h+1})
d(x_{2h+1,h}^{2n+2}) \leq h d(x_0, x_1)
\]

Hence for any \( P \in 1^* \), we have,
\[
d(x_{2n+1,h}^{2n+2}) \leq h^{2n+1} d(x_0, x_1).
\]

Since \( \lim h^{2n+1} = 0 \), it follows that \( x_0 \) is a Cauchy sequence.

Also \( M \) is \( T \)-orbitally complete, there exists some \( u \in M \) such that \( u = \lim T^n x_0, x \in M \).

By orbital continuity of \( T \), we get \( Tu = \lim T^n x_0 = u \), and hence \( u \) is a fixed point of \( T \). Similarly, we can show that \( u \) is the fixed point of \( S \).

Uniqueness- follows from theorem (A). This completes the proof.

Theorem - C- Let \( B = (x_0, r) = \{ x \in M, d(x, x_0) < r \} \)

Where \( (M, d) \) is an orbitally complete metric space. Let \( S \) and \( T \) be an orbitally continuous mappings of \( B \) into \( M \) and satisfies
\[
[d(Sx, Ty)]^2 \leq [a_1 d(x, y) + a_2 d(x, Sx)]^2
\]
\[
d(x, Ty) + a_2 d(x, Ty) d(y, Sx))
\]

where \( a_1 + a_2 = h, h \in (0,1) \), for all \( x, y \in B \) and \( \text{dist}(x_0, Tx_0) = h (1-h) r \).

Then \( S \) and \( T \) have a fixed point, and if \( a_1 < 1 \) then \( S \) and \( T \) have unique fixed point.

\textbf{Proof}: Let \( x_0 \in M \) be arbitrary, and we define a sequence
\[
Sx_{2n} = x_{2n+1}, T_{2n+1} = x_{2n+2}.
\]

Now for \( x = x_0 \) and \( y = x_1 \), we have,
\[
[d(Sx_0, Tx_1)]^2 \leq a_1 d(x_0, x_1) d(x_1, Tx_1)
\]
\[
+ a_2 d(x_0, Sx_0) d(Sx_0, Tx_1) + a_2 d(x_0, Tx_1)
\]
\[
d(x_0, Sx_0)
\]

\[
[d(x_0, x_1)]^2 \leq [a_1 d(x_0, x_1) d(x_1, x_2) + a_2 d(x_0, x_1)]
\]
\[
[d(x_1, x_2)]^2 \leq [a_1 d(x_0, x_1) d(x_1, x_2)
\]
\[
+ a_2 d(x_0, x_1)] d(x_1, x_2)
\]
\[
d(x_1, x_2)]^2 \leq (a_1 + a_2) d(x_0, x_1) d(x_1, x_2)
\]
\[
i.e.
\]
\[
[d(x_1, x_2)]^2 \leq h [d(x_0, x_1) d(x_1, x_2)]
\]
\[
\text{where } a_1 + a_2 = h, h \in (0,1).
\]
d(x_1, x_2) \leq h d(x_0, x_1)
d(x_1, x_2) \leq h (1-h) r.

Hence \( d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)
\]
d(x_0, x_2) \leq (1-h) r + h (1-h) r.

Suppose that
\[
d(x_0, x_2) \leq (1 + h ... + h^{n-1}) (1-h) r.
\]

Similarly
\[
d(x_0, x_2) \leq h^{2n-1} (1-h) r.
\]

Taking \( x = x_{n-1} \) and \( y = x_n \), we have,
\[ d(x_n, x_{n+1})^2 \leq [a_1 d(x_{n-1}, x_n) d(x_n, x_{n+1})
+ a_3 d(x_{n-1}, x_n) d(x_n, x_n)] 
\]

\[ [d(x_n, x_{n+1})]^2 \leq [a_1 d(x_{n-1}, x_n) d(x_n, x_{n+1})
+ a_3 d(x_{n-1}, x_n) d(x_n, x_n)] \]

\[ [d(x_n, x_{n+1})]^2 \leq (a_1 + a_3) [d(x_{n-1}, x_n) d(x_n, x_{n+1})] \]

\[ d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \]

where \( a_1 + a_3 = h, h \in (0, 1) \).

Thus the sequence \( \{x_0, x_1 = T^n x_0\} \), \( n \geq 0 \) is contained \( B \).

Also

\[ d(x_n, x_m) \leq d(x_n, x_{n+1}) d(x_{n+1}, x_{n+1}) \]

\[ + \cdots + d(x_{m-1}, x_m) \]

\[ d(x_n, x_m) \leq (h^n + \cdots + h^{n-1}) (1-h) r. \]

\[ d(x_n, x_m) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Since \( B \) is also orbitally complete, so \( u = \lim_{n \rightarrow \infty} T x^n \).

Thus \( u \) is a fixed point of \( T \). Similarly we can prove that \( u \) is the fixed point of \( S \).

The uniqueness follows from theorem (A).

This completes the proof.

Remark: Taking \( a_3 = 0 \) and \( a_1 = \frac{a}{2} \) and \( B \),

\[ \alpha = - \frac{1}{2}, \text{ then we have Qureshi and Pande}. \]

References:

A fixed point theorem

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Many authors like Furi and Vignoli [1], Iski [2] and Pande [6] proved fixed point
theorems for densifying mapping in a complete metric space. The purpose of the present
paper is to establish the generalizations of their results.

We need the concept of the measure of non-compactness by Kuratowski [4] to define this.

Let ‘A’ be a bounded subset of a complete metric space X. By the real number \( a(A) \),
we denote the infimum of all positive numbers \( \varepsilon \) such that ‘A’ admits a finite covering consisting of subsets with diameters less than \( \varepsilon \).

Then we have the following results:
(i) \( 0 \leq a(A) \leq \delta(A) \), where \( \delta(A) \) is the diameter of \( A \).
(ii) \( a(A) = 0 \) iff \( A \) is precompact.
(iii) \( a(A \cup B) = \max \{ a(A), a(B) \} \).

Many useful properties about measure of non-compactness \( a \) of bounded sets in a
metric space, see Nussbaum [5].

The following notion of densifying mapping was introduced by Furi and Vignoli [1].

Definition: Let \((X,d)\) be a metric space, \(T\) be a continuous mapping of \(X\) into itself. The
mapping \(T\) is called densifying, if for every bounded subset \(A\) of \(X\) with \(a(A) > 0\), we have
\(a(TA) < a(A)\).

Here, we prove the following theorem:

Theorem: Let \(T\) be a continuous densifying mapping of a bounded complete metric space
\((X,d)\) into itself. If for every \(x,y\) in \(X: x \neq y, x \neq Tx, y \neq Ty\),
\[ d(Tx,Ty) \leq c_1 \cdot d(x,y) + c_2 \left( d(x,Tx) + d(y,Ty) \right) + c_3 \left( d(x,Ty) + d(y,Tx) \right) \]
\[ + c_4 \left( \frac{d(x,Tx,d(y,Ty))}{d(x,y)} + \frac{d(x,Ty,d(y,Tx))}{d(x,y)} \right) + c_5 \left( \frac{d(x,Tx,d(x,Ty))}{d(x,Tx)} + d(Tx,Ty) \right) \]

where \(c_1, c_2, c_3, c_4, c_5\) are non-negative and \(c_1 + 2c_2 + 2c_3 + 2c_4 + c_5 = 1\), then \(T\) has a
unique fixed point.
P.K. Dubey & K.C. Srivastava

Proof: Let \( x_0 \) be a point of \( X \), and we define a sequence,

\[ x_0, x_1 = T(x_0), \ldots, x_{n+1} = T(x_n), \ldots \]

Put \( A = \{x_0, x_1, \ldots, x_n, \ldots\} \). Then \( T(A) \subset A \) and by the continuity of \( T \), we have,

\[ T(A) \subset \overline{T(A)} \subset A \]

Hence \( A \) is invariant under \( T \), and is bounded.

Suppose \( \alpha(A) > 0 \). Since \( A = T(A) \cup \{x_0\} \), we have

\[ \alpha(A) = \max \{\alpha(T(A)), \alpha\{x_0\}\} \]

\[ = \alpha(T(A)) \]

The mapping \( T \) is densifying, so \( \alpha(A) = 0 \), which implies that \( A \) is precompact.

Since \( X \) is complete metric space \( A \) is compact. By the hypothesis, \( d(x, Tx) \) is continuous on the compact subset \( A \). Hence \( d(x,Tx) \) has a minimum point \( u \in A \). To prove that \( u \) or \( Tu \) is fixed point of \( T \), suppose \( u \neq Tu \) and \( Tu \neq T^2u \), then we have

\[ d(Tu, T^2u) < c_1 d(u,v) + c_2 \{d(u, Tu) + d(Tu, T^2u)\} \]

\[ + c_3 \{d(u, Tu) + d(Tu, Tu)\} \]

\[ + c_4 \left\{ \frac{d(u, Tu)}{d(u, Tu)} + \frac{d(u, T^2u)}{d(Tu, T^2u)} \right\} \]

\[ + c_5 \left\{ \frac{d(Tu, Tu)}{d(Tu, Tu)} + \frac{d(T^2u, Tu)}{d(Tu, T^2u)} \right\} \]

i.e.

\[ d(Tu, T^2u) < \frac{c_1 + c_2 + c_3 + c_4}{1 - c_3 - c_4 - c_5} d(u, Tu) \]

leading to a contradiction, since \( c_1 + 2c_2 + 2c_3 + 2c_4 + c_5 = 1 \).

Thus \( u \) or \( Tu \) is a fixed point of \( T \).

Now to prove uniqueness, let \( v \) be another fixed point of \( T \). Then,

\[ d(u, v) < c_1 d(u,v) + c_2 \{d(u, Tu) + d(v, Tv)\} \]

\[ + c_3 \{d(u, Tu) + d(v, Tu)\} \]

\[ + c_4 \left\{ \frac{d(u, Tu)}{d(u, v)} + \frac{d(u, T^2v)}{d(Tu, Tv)} \right\} \]

\[ + c_5 \left\{ \frac{d(Tu, Tu)}{d(Tu, v)} + \frac{d(T^2u, T^2v)}{d(Tu, Tv)} \right\} \]

\[ d(u, v) < c_1 + c_2 + c_3 + c_4 + c_5 d(u,v) \]

again leading to a contradiction, since \( c_1 + 2c_2 + 2c_3 + 2c_4 + c_5 = 1 \). Thus \( u \) is the unique fixed point of \( T \).

This completes the proof of the theorem.
The Mathematics Education

Remarks

(i) If $c_1 = c_2 = c_3 = c_4 = 0$, then we have Furi and Vignoli theorem [1].
(ii) If $c_1 = c_2 = c_3 = c_4 = 0$ or $c_1 = c_2 = c_3 = c_4 = c_5 = 0$, then we have Iseki theorems [2].
(iii) If $c_4 = c_3 = 0$ then we have Iseki theorem [3].
(iv) If $c_5 = 0$ then we have Pande theorems [5].

References

A note on fixed point theorem for densifying mapping

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(Received 25 January 1990)

In this paper, we prove some fixed point theorems for densifying mappings. These theorems are generalized the results of Furi and Vignoli (1), Iseki (2), Qureshi and Pandey (3) and Das (4).

We need the concept of the measure of compactness by C. Kuratowski to define this. Let A be a bounded subset of a complete metric space X. By the real number \( a(A) \), we denote the infimum of all positive number \( \varepsilon \) such that \( A \) admits a finite covering consisting of subsets with diameters less than \( \varepsilon \). Then we have the following results:

1. \( 0 < a(A) \leq \delta(A) \) where \( \delta(A) \) is the diameter of \( A \).
2. \( a(A) = 0 \) iff \( A \) is pre-compact.
3. \( a(A \cap B) = \min \{a(A), a(B)\} \).

For some fundamental properties about measure of non-compactness "\( a \)" of bounded sets in a metric space, see Iseki (3) and Nussbaum (4).

The notion of densifying mapping was introduced in (1). Let \( (X, d) \) be a metric space, \( T \) be a mapping of \( X \) into itself. The mapping \( T \) is called densifying, if every bounded subset \( F \) of \( X \) with \( a(A) > \delta \) we have \( a(T(A)) < a(A) \).

Here we prove the following theorem:

Theorem 1 — Let \( T \) be a continuous densifying mapping of a bounded complete metric space \( X, d \), into itself. Then for every \( x, y \) in \( X \),

\[
x \neq y, x \neq Tx, y \neq Ty,
\]

\[
\min \{ d(Tx, Ty), \frac{d(x, y) + d(y, Ty)}{2}, d(y, Ty) \} + k \min \{ d(x, Ty), d(y, Tx) \} \leq c_1 d(x, y) + c_2 \left[ d(x, Tx) + d(y, Ty) \right]
\]

\[
+ c_3 \left[ d(x, Ty) + d(y, Tx) \right] + c_4 \left[ \frac{d(x, Tx)}{d(x, y)} + c_5 \left[ \frac{d(x, Ty)}{d(x, y)} + \frac{d(y, Tx)}{d(Tx, Ty)} \right] \right]
\]

\[
[185]
\]
where $k$, $c_1$, $c_2$, $c_3$, $c_4$, and $c_5$ are real numbers such that $c_1 + 2c_2 + 2c_3 + c_4 + c_5 = 1$, then $T$ has a fixed point, if \[
frac{c_1 + 2c_2 + c_3}{k} < 1,
\] then $T$ has a unique fixed point.

Proof: Let $x$ be a point of $A$ and consider the sequence $x_n, x_1 = T x_n, x_{n+1} = T x_{n+1}$. Then $T(A) \subseteq A$, and by the continuity of $T$, we have $T(A) \subseteq T(A) \subseteq A$. Hence $A$ is invariant under $T$, and is bounded. Suppose $\alpha(A) > 0$, since $A = T(A) \cup \{x\}$, we have $\alpha(A) = \max\{\alpha(T(A)), \alpha(x)\} = \alpha(A)$.

The mapping $T$ is densifying, so $\alpha(A) = 0$, which implies that $A$ is precompact. Since $X$ is complete metric space, $A$ is compact. By hypothesis, $d(x, Tx)$ is continuous on the compact subset $A$. Hence $d(x, Tx)$ is minimum point $\xi$ in $A$.

To prove that $\xi$ or $T \xi$ is a fixed point of $T$, suppose $\xi \neq T \xi$ and $T \xi \neq T^2 \xi$, then we have

\[
\min \left\{ d(T \xi, T^2 \xi), \{ d(\xi, T \xi) + d(T \xi, T^2 \xi) \}, d(T \xi, T^2 \xi) \right\} + k \min \left\{ d(\xi, T \xi), d(T \xi, T \xi) \right\} < c_1 \min \left\{ d(\xi, T \xi), d(T \xi, T \xi) \right\}
\]

\[
+ c_2 \left\{ d(\xi, T \xi) + d(T \xi, T^2 \xi) \right\} + c_3 \left\{ d(\xi, T^2 \xi) \right\}
\]

\[
+ c_4 \left\{ d(T \xi, T^2 \xi) \right\} + d(\xi, T \xi) + d(T \xi, T \xi) + d(T^2 \xi, T \xi)
\]

\[
= \min \left\{ d(T \xi, T^2 \xi), \{ d(\xi, T \xi) + d(T \xi, T^2 \xi) \}, k \min \left\{ d(\xi, T \xi), d(T \xi, T \xi) \right\} \right\} + c_2 \left\{ d(\xi, T \xi) + d(T \xi, T^2 \xi) \right\} + c_3 d(\xi, T^2 \xi) + c_4 d(T \xi, T^2 \xi) + d(\xi, T \xi) + d(T \xi, T \xi) + d(T^2 \xi, T \xi)
\]

i.e.

\[
\min \left\{ d(T \xi, T^2 \xi), \{ d(\xi, T \xi) + d(T \xi, T^2 \xi) \}, k \min \left\{ d(\xi, T \xi), d(T \xi, T \xi) \right\} \right\} + c_2 \left\{ d(\xi, T \xi) + d(T \xi, T^2 \xi) \right\} + c_3 d(\xi, T^2 \xi) + c_4 d(T \xi, T^2 \xi) + d(\xi, T \xi) + d(T \xi, T \xi) + d(T^2 \xi, T \xi)
\]

Since $c_1 + 2c_2 + 2c_3 + c_4 + c_5 = 1$, $c_1 + c_2 + c_3 < c_1 + 2c_2 + 2c_3 + c_4 + c_5 = 1$, and $c_2 + c_3 + c_5 < c_1 + 2c_2 + 2c_3 + c_4 + c_5 = 1$ does not hold. Therefore, we have

\[
T \xi \neq T^2 \xi \Rightarrow \frac{c_1 + c_2 + c_3 + c_5}{1 - c_2 - c_3 - c_4} d(\xi, T \xi) < d(\xi, T)\]

which contradicts, therefore "" or "" is a fixed point of $T$.

To prove uniqueness, let $\eta$ be another fixed point of $T$.

Then

\[
\min \left\{ d(T \xi, T \eta), \{ d(\xi, T \eta) + d(\xi, T \eta) \}, k \min \left\{ d(\xi, T \eta), d(\xi, T \eta) \right\} \right\}
\]

\[
+ c_2 \left\{ d(\xi, T \xi) + d(\eta, T \eta) \right\} + c_3 \left\{ d(T \xi, T \eta) \right\} + c_4 \left\{ d(T \xi, T \eta) \right\} + d(\xi, T \xi) + d(\xi, T \eta)
\]
K.C. Srivastava & P.K. Dubey

\[ + c_4 \left[ \frac{d(\xi, T\eta)}{d(\xi, \eta)} \right] + c_2 \left[ \frac{d(\xi, \eta)}{d(\xi, \eta)} \right] \]

i.e.

\[ d(\xi, \eta) < \frac{(c_1 + 2c_2 + c_4)}{k} d(\xi, \eta). \]

Leading to contradiction. Since \( \frac{(c_1 + 2c_2 + c_4)}{k} < 1 \). Thus \( \xi \) is the unique fixed point of \( T \).

Therefore complete the proof of the theorem.

**Remark**

Since \( \min \{ d(x, T_1), d(y, T_2) \} = 0 \)

(i) Let \( d(x, Ty) = 0 = d(y, Tx) \), then condition (1.1) becomes,

\[ \min \{ d(Tx, Ty), d(T_1) + d(T_2) \} = d(Tx, Ty) \]

\[ < c_1 d(x, y) + c_2 \left[ d(x, T_1) + d(y, T_2) \right] \]

\[ + c_4 \left[ \frac{d(x, T_1) d(y, T_2)}{d(x, y)} \right] \]

i.e.

\[ d(Tx, Ty) < c_1 d(x, y) + c_2 \left[ d(x, T_1) + d(y, T_2) \right] \]

\[ + c_4 \left[ \frac{d(x, T_1) d(y, T_2)}{d(x, y)} \right] \]

Now taking \( c_2 = c_4 = c_5 = 0 \), then we get result due to M. Furi and A. Vignoli (1).

(ii) Taking \( x = Ty, y = x \) in L.H.S of condition (i) and taking \( c_4 = c_5 = 0 \), then we get result due to K. Iseki (2).

(iii) Taking \( x = Ty, y = x \) in L.H.S of condition (i) and taking \( c_5 = 0 \) we get results due to K. Qureshi and R.K. Pande (5).

Further we prove the following theorem which generalizes the theorem 2 of Das (6).

**Theorem 2** — Let \( T \) be a continuous densifying mapping of a bounded complete metric space \( (X, d) \) into itself such that for some positive integer \( m \), \( T^m \) is continuous and for every \( x, y \in X \) with \( x \neq y \), \( x \neq Ty, y \neq Tx \),

\[ d(T^m x, T^m y) < c_1 d(x, y) + c_2 d(x, T^m x) + c_3 d(y, T^m y) \]

\[ + c_4 \left[ \frac{d(x, T^m x) d(y, T^m y)}{d(x, y)} \right] + c_5 \left[ \frac{d(x, T^m x) d(y, T^m y)}{d(x, y)} \right] \]

\[ + c_6 \left[ \frac{d(x, T^m y) d(y, T^m x)}{d(x, y)} \right] \]
The Mathematics Education

\[ + c_4 \frac{d(x, T^m y) d(y, T^m y)}{d(T^n x, T^m y)} + c_5 \frac{d(x, T^m y) d(T^n x, T^m y)}{d(x, T^n x) + d(T^n x, T^m y)} \]

where \( c_4 + c_5 + c_6 + c_6 < 1 \), \( c_4 > 0 \), \( c_5 < 0 \), \( c_6 + c_8 + 2c_6 + c_8 + 2c_6 + c_8 \) and \( c_1 + c_4 + c_5 + c_6 < 1 \).

Then \( T \) has a unique fixed point.

**Proof:** Let \( x_0 \) be a point of \( X \), and consider the sequence \( x_0, x_1 = T x_0, \ldots, x_{n+1} = T x_n, \ldots \). Then \( T(A) \subset A \) and by the continuity of \( T \), we have \( T(A) \subset T(A) \subset A \).

Hence \( A \) is invariant under \( T \), and is bounded. Suppose \( \alpha(A) > 0 \), since \( A = T(A) \cup \{ x_0 \} \), we have

\[ \alpha(A) = \max \{ \alpha(T(A)), \alpha(x_0) \} \]

\[ = \alpha(T(A)). \]

The mapping \( T \) is densifying. So \( \alpha(A) = 0 \), which implies that \( A \) is precompact. Since \( X \) is complete metric space, \( A \) is compact.

Define \( f : X \to [0, \infty] \) by \( f(x) = d(x, T^m x) \) for every \( x \in X \). Continuity of \( d \) and \( T^m \) ensure the continuity of \( f \). Then compactness of \( A \) yields a point \( \xi \in A \) such that

\[ f(\xi) = \inf \{ f(x) : x \in A \} \]

Then \( f(\xi) \neq 0 \) gives \( \xi \neq T^m \xi \). If \( T^m \xi = T^{2m} \xi \), then \( T^{m/2} \xi \) is a fixed point of \( T^m \). So we assure that \( T^m \xi \neq T^{2m} \xi \).

Then

\[ f(T^m \xi) = d(T^m \xi, T^{2m} \xi) \]

\[ < c_1 d(\xi, T^m \xi) + c_2 d(\xi, T^{2m} \xi) + c_3 d(T^m \xi, T^{3m} \xi) \]

\[ + c_4 d(T^m \xi, T^{2m} \xi) + c_5 d(T^m \xi, T^m \xi) \]

\[ + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} + c_6 \frac{d(\xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} \]

\[ < (c_1 + c_2) d(\xi, T^m \xi) + (c_3 + c_4) d(T^m \xi, T^{2m} \xi) \]

\[ + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} + d(T^m \xi, T^{2m} \xi) \]

\[ < (c_1 + c_2) \left[ d(\xi, T^m \xi) + d(T^m \xi, T^{2m} \xi) \right] + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} \]

\[ + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} \]

\[ < (c_1 + c_2) \left[ d(\xi, T^m \xi) + d(T^m \xi, T^{2m} \xi) \right] + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} \]

\[ + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} + d(T^m \xi, T^{2m} \xi) \]

\[ < (c_1 + c_2) \left[ d(\xi, T^m \xi) + d(T^m \xi, T^{2m} \xi) \right] + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} \]

\[ + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} + d(T^m \xi, T^{2m} \xi) \]

\[ < (c_1 + c_2) \left[ d(\xi, T^m \xi) + d(T^m \xi, T^{2m} \xi) \right] + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} \]

\[ + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} + d(T^m \xi, T^{2m} \xi) \]

\[ < (c_1 + c_2) \left[ d(\xi, T^m \xi) + d(T^m \xi, T^{2m} \xi) \right] + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} \]

\[ + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} + d(T^m \xi, T^{2m} \xi) \]

\[ < (c_1 + c_2) \left[ d(\xi, T^m \xi) + d(T^m \xi, T^{2m} \xi) \right] + c_6 \frac{d(\xi, T^m \xi) d(T^m \xi, T^{2m} \xi)}{d(T^m \xi, T^{2m} \xi)} \]

\[ + c_6 \frac{d(T^m \xi, T^{2m} \xi) d(T^{2m} \xi, T^m \xi)}{d(T^{2m} \xi, T^m \xi)} + d(T^m \xi, T^{2m} \xi) \]
K.C. Srivastava & P.K. Dubey

\[ f(T\eta) < (c_1 + c_4 + c_6 + c_8) f(\xi) + (c_3 + c_4 + c_6 + c_8 + c_9) f(T\eta) \]

Therefore,

\[ f(T\eta) \frac{c_1 + c_4 + c_6 + c_9}{1 - c_3 - c_4 - c_6 - c_8 - c_9} f(\xi) \]

but this contradicts the definition of \( f(\xi) \). Thus \( f(\xi) = 0 \) and which gives \( \xi = T\eta \). To prove the uniqueness, let \( \eta \) be another fixed point of \( T^m \). Then

\[ d(\xi, \eta) = d(T\eta, T\eta) \]
\[ < c_1 d(\xi, \eta) + c_4 d(\xi, T\eta) + c_6 d(\xi, T\eta) \]
\[ + c_9 \frac{d(\xi, T\eta) + d(T\xi, T\eta)}{d(\xi, T\xi) + d(T\xi, T\eta)} \]
\[ < (c_1 + c_4 + c_6 + c_9) d(\xi, \eta) \]
\[ d(\xi, \eta) < d(\xi, \eta) \]

which is a contradiction. Since \( c_1 + c_4 + c_6 + c_9 \leq 1 \), thus \( \xi \) is a unique fixed point of \( T^m \). Therefore complete the proof of the theorem.

References

3. Iseki, K.: A fixed point theorems for densifying mappings, Math. Seminar Notes 2 (1974) 70-74 (Kobe Univ.)
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