CHAPTER IV

FIXED POINT THEOREMS FOR
MAPPINGS IN BIMETRIC SPACE
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4.1. The concept of Bimetric space has been introduced by Maia [131] and many authors like Iseki [88], Mishra [134], Rus [169], Rhoades [159] etc., have studied the contractive and contraction type mapping in Bimetric space. Maia [131] obtained the following result -

THEOREM A: Let \( X \) be a metric space with two metrics \( d_1 \) and \( d \) such that

\[(4.1.1)\] \( d_1(x,y) \leq d(x,y), \text{ for all } x,y \text{ in } X, \)

\[(4.1.2)\] \( X \) is complete with respect to \( d_1, \)

\[(4.1.3)\] the mapping \( f:X \rightarrow X \) is continuous with respect to \( d_1 \) and

\[(4.1.4)\] \( d(f(x),f(y)) \leq kd(x,y) \)

for all \( x,y \) in \( X \), where \( 0 \leq k < 1 \), then \( T \) has a unique fixed point.

Afterward Iseki [88] generalized the above result as follows -
THEOREM B : Let \( X \) be a metric space with two metrics \( d_1 \) and \( d \) such that (4.1.1) and (4.1.2) hold and

(4.1.5) two mappings \( f, g : X \rightarrow X \) are continuous with respect to \( d_1 \) and

(4.1.6) \( d(f(x), g(y)) \leq a_1 d(x, y) + a_2 [d(x, f(x)) + d(y, g(y))] + a_3 [d(x, g(y)) + d(y, f(x))] \)

for all \( x, y \) in \( X \), where \( a_1, a_2, a_3 \) are non-negative and \( a_1 + 2a_2 + 2a_3 < 1 \). Then \( f \) and \( g \) have a unique common fixed point.

4.2. In this section, we establish some fixed point theorems in Bimetric space.

THEOREM 1 : Let \( X \) be a metric space with two metrics \( d_1 \) and \( d \) such that (4.1.1), (4.1.2), (4.1.5) hold and

(4.2.1) \( d(f(x), g(y)) \leq a_1 d(x, y) + a_2 [d(x, f(x)) + d(y, g(y))] \)
\[ + a_3[\text{d}(x, g(y)) + \text{d}(y, f(x))] \]
\[ + a_4[ \frac{\text{d}(x, g(y)) \text{d}(x, y)}{\text{d}(x, y) + \text{d}(y, g(y))} ] \]

for all \( x, y \) in \( X \), where \( a_1, a_2, a_3, a_4 \) are non-negative and \( a_1 + 2a_2 + 2a_3 + a_4 < 1 \). Then \( f \) and \( g \) have a unique common fixed point.

**Proof**: Let \( x_0 \) be an arbitrary point of \( X \) and define a sequence as follows -

\[ x_1 = f(x_0), x_2 = g(x_1), \ldots \]

i.e.

\[ x_{2n+1} = f(x_{2n}), x_{2n+2} = g(x_{2n+1}), n = 0, 1, 2, \ldots \]

By the inequality (4.2.1), we have

\[ \text{d}(x_1, x_2) = \text{d}(f(x_0), g(x_1)) \]
\[ \leq a_1 \text{d}(x_0, x_1) \]
\[ + a_2[\text{d}(x_0, f(x_0)) + \text{d}(x_1, g(x_1))] \]
\[ + a_3[\text{d}(x_0, g(x_1)) + \text{d}(x_1, f(x_0))] \]
\[ + a_4[ \frac{\text{d}(x_0, g(x_1)) \text{d}(x_0, x_1)}{\text{d}(x_0, x_1) + \text{d}(x_1, g(x_1))} ] \]
i.e.

\[ d(x_1, x_2) \leq a_1 d(x_0, x_1) \]

\[ + a_2 [d(x_0, x_1) + d(x_1, x_2)] \]

\[ + a_3 [d(x_0, x_2) + d(x_1, x_1)] \]

\[ + a_4 \left[ \frac{d(x_0, x_2) d(x_0, x_1)}{d(x_0, x_1) + d(x_1, x_2)} \right]. \]

Or,

\[ d(x_1, x_2) \leq \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2 - a_3} d(x_0, x_1) \]

\[ = h \cdot d(x_0, x_1), \]

where \( h = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_2 - a_3} < 1. \)

In general,

\[ d(x_n, x_{n+1}) \leq h^n d(x_0, x_1). \]

Hence,

\[ d(x_n, x_m) \leq \frac{h^n}{1-h} d(x_0, x_1), \text{ for } m > n. \]

It follows that \( \{x_n\} \) is a Cauchy sequence with respect to \( d \) and therefore by condition (4.1.1), we have \( \{x_n\} \)
is a Cauchy sequence with respect to \( d_1 \). Since \( X \) is complete under condition (4.1.2), so \( \{x_n\} \) has a limit \( z \) in \( X \), i.e. \( \lim_{n} (x_{2n}) = z \).

By the continuity of \( f \) with respect to metric \( d_1 \), we have

\[
f(z) = f \lim_{n} (x_{2n}) = \lim_{n} f(x_{2n}) = \lim_{n} (x_{2n+1}) = z.
\]

Similarly by the continuity of \( g \) with respect to metric \( d_1 \), it can be easily shown that \( g(z) = z \). Hence \( z \) is a common fixed point of \( f \) and \( g \).

Now to show the uniqueness of \( z \), let \( w \) be another common fixed point of \( f \) and \( g \), if possible different from \( z \). Then by condition (4.2.1), we have

\[
d(z, w) = d(f z, g w)
\]

\[
\leq a_1 d(z, w)
\]

\[
+ a_2 [d(z, f(z)) + d(w, g(w))]
\]

\[
+ a_3 [d(z, g(w)) + d(w, f(z))]
\]

\[
+ a_4 \left[ \frac{d(z, g(w)) d(z, w)}{d(z, w) + d(w, g(w))} \right]
\]
i.e.
\[ d(z, w) \leq (a_1 + 2a_3 + a_4) d(z, w), \]
leading a contradiction, since \( a_1 + 2a_3 + a_4 < a_1 + 2a_2 + 2a_3 + a_4 < 1. \)
Thus \( z = w. \)

This completes the proof of the Theorem 1.

**THEOREM 2**: Let \( X \) be a metric space with two metrics \( d_1 \) and \( d \) and \( T_i \) \((i = 1, 2, 3, \ldots, k)\) a finite family of continuous mappings of \( X \) into itself. Suppose that (4.1.1), (4.1.2) hold and

(4.2.2) \( T_i T_j = T_j T_i \) \((i, j = 1, 2, 3, \ldots, k)\),

(4.2.3) there are two systems of positive integers \((m_1, m_2, \ldots, m_k)\) and \((n_1, n_2, \ldots, n_k)\) such that

(4.2.4) \[
\begin{align*}
& d(T_1^{m_1} T_2^{m_2} \cdots T_k^{m_k} (x), T_1^{n_1} T_2^{n_2} \cdots T_k^{n_k} (y)) \\
& \leq a_1 d(x, y) \\
& + a_2 [d(x, T_1^{m_1} T_2^{m_2} \cdots T_k^{m_k} (x)) + d(y, T_1^{n_1} T_2^{n_2} \cdots T_k^{n_k} (y))] 
\end{align*}
\]
for all \( x, y \) in \( X \), where \( a_1, a_2, a_3, a_4 \) are non-negative and
\( a_1 + 2a_2 + 2a_3 + a_4 < 1 \). Then \( T_i (i = 1, 2, \ldots, k) \) has a
unique common fixed point.

**PROOF.** Let \( f = T_1^{m_1}T_2^{m_2} \cdots T_k^{m_k} \), \( g = T_1^{n_1}T_2^{n_2} \cdots T_k^{n_k} \).

Then \( f \) and \( g \) are continuous. Therefore by Theorem 1, \( f \) and
\( g \) have a unique common fixed point \( z \) in \( X \).

\[ f(z) = g(z) = z. \]

Then for each \( i \), \( T_i (f(z)) = T_i (g(z)) = T_i (z) \).

By the condition (4.2.2), we have

\[ f(T_i(z)) = g(T_i(z)) = T_i(z). \]

Therefore \( T_i(z) (i = 1, 2, \ldots, k) \) are common fixed point
of \( f \) and \( g \). From the uniqueness of the common fixed point
of \( f \) and \( g \), we have

\[ T_i(z) = z (i = 1, 2, \ldots, k). \]
This completes the proof of the Theorem 2.

**REMARKS** :

(1) If we put \( f = g \) and \( a_2 = a_3 = a_4 = 0 \) in Theorem 1, we obtain Theorem A.

(2) If we put \( a_4 = 0 \) in Theorem 1, we obtain Theorem B.

**THEOREM 3** : Let \( X \) be a metric space with two metrics \( d_1 \) and \( d \) and \( f_n (n = 1, 2, 3, \ldots) \) be a sequence of mappings of \( X \) into itself, such that (4.1.1), (4.1.2) hold and

(4.2.5) \( f_0 \) is continuous with respect to \( d_1 \).

(4.2.6) \[
[d(f_0x,f_ny)]^2 \leq a_1[d(x,f_0x)d(y,f_ny)+d(x,f_ny)d(y,f_0x)] \\
+ a_2[d(x,f_0x)d(y,f_0x)+d(x,f_ny)d(y,f_ny)] \\
+ a_3[d(x,y)d(f_0x,f_ny)]
\]

for all \( x, y \) in \( X \), where \( a_1, a_2, a_3 \) are non-negative and

\( a_1 + 2a_2 + a_3 < 1, a_2 < 1, a_1 + a_3 < 1 \). Then there exists a unique common fixed point of \( f_n (n = 0, 1, 2, \ldots) \) in \( X \).
**Proof**: Let \( x_0 \) be an arbitrary point of \( X \) and we define a sequence as follows -

\[
\begin{align*}
   x_0, x_1 &= f_0 x_0, x_2 = f_1 x_1, \ldots, x_{2n-1} = f_0 x_{2n-2}, x_{2n} = f_n x_{2n-1}, \\
   x_{2n+1} &= f_0 x_{2n}.
\end{align*}
\]

Applying (4.2.6), we have

\[
[d(x_1, x_2)]^2 = [d(f_0 x_0, f_1 x_1)]^2 
\]

\[
\leq a_1 [d(x_0, f_0 x_0) d(x_1, f_1 x_1) + d(x_0, f_1 x_1) d(x_1, f_0 x_0)] 
\]

\[
+ a_2 [d(x_0, f_0 x_0) d(x_1, f_0 x_0) + d(x_0, f_1 x_1) d(x_1, f_1 x_1)] 
\]

\[
+ a_3 [d(x_0, x_1) d(f_0 x_0, f_1 x_1)] 
\]

i.e.

\[
[d(x_1, x_2)]^2 \leq a_1 [d(x_0, x_1) d(x_1, x_2) + d(x_0, x_2) d(x_1, x_1)] 
\]

\[
+ a_2 [d(x_0, x_1) d(x_1, x_1) + d(x_0, x_2) d(x_1, x_2)] 
\]

\[
+ a_3 [d(x_0, x_1) d(x_1, x_2)].
\]

Thus,

\[
d(x_0, x_1) \leq \frac{a_1 + a_2 + a_3}{1 - a_2} d(x_0, x_1).\]
\[ d(x_0, x_1) \leq q \cdot d(x_0, x_1), \]

where \( q = \frac{a_1 + a_2 + a_3}{1 - a_2} < 1. \)

In general,

\[ d(x_n, x_{n+1}) \leq q^n d(x_0, x_1). \]

Now,

\[ d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \]

\[ \leq (q^n + q^{n+1} + \ldots + q^{n+p-1}) d(x_0, x_1) \]

\[ \leq \frac{q^n}{1 - q} d(x_0, x_1) \]

\[ \longrightarrow 0 \text{ as } n \longrightarrow \infty. \]

Thus \( \{x_n\} \) is a cauchy sequence. Since \( X \) is complete, \( \{x_n\} \) converges to a point \( z \) in \( X \).

i.e. \( \lim_{n \to \infty} x_n = z \).

Since \( f_0 \) is continuous, so

\[ f_0 z = \lim_{n \to \infty} f_0(x_{2n}) = z. \]
Thus $z$ is a fixed point of $f_0$.

We wish to show that $z$ is a fixed point of $f_n$ for each $n = 1, 2, \ldots$. Put $x = y = z$ in (4.2.6), then

$$[d(z, f_n z)]^2 = [d(f_0 z, f_n z)]^2$$

$$\leq a_1[d(z, f_0 z)d(z, f_n z) + d(z, f_0 z)d(z, f_0 z)] + a_2[d(z, f_0 z)d(z, f_0 z) + d(z, f_n z)d(z, f_n z)] + a_3[d(z, z)d(f_0 z, f_n z)].$$

i.e.

$$[d(z, f_n z)]^2 \leq a_2[d(z, f_n z)]^2,$$

leading to a contradiction, since $a_2 < 1$. Thus $z = f_n z$.

Hence $z$ is a common fixed point of $f_n (n = 0, 1, 2, \ldots)$.

Now to claim its uniqueness, let $w$ be another common fixed point of $f_n$, if possible different form $z$.

Then by (4.2.6), we have

$$[d(z, w)]^2 = [d(f_0 z, f_n w)]^2$$

$$\leq a_1[d(z, f_0 z)d(w, f_n w) + d(z, f_n w)d(w, f_0 z)].$$
+ \alpha_2 [d(z, f_0 z) d(w, f_0 z) + d(z, f_n w) d(w, f_n w)]
+ \alpha_3 [d(z, w) d(f_0 z, f_n w)]

[d(z, w)]^2 \leq (\alpha_1 + \alpha_3) [d(z, w)]^2,

again leading to a contradiction, since \alpha_1 + \alpha_3 < 1. Thus

z = w. Hence z is a unique common fixed point of f_n.

This completes the proof of the Theorem 3.

**Remarks**:

(1) If we put \( f_0 = E, \ f_n = F \) and \( \alpha_2 = \alpha_3 = 0 \) in Theorem 3
then we get result of Fisher [50] in complete metric space.

(2) If we put \( f_0 = E, \ f_n = F \) and \( \alpha_3 = 0 \) in Theorem 3,
then we get result of Pachpatte [144] in complete metric space.

4.3. In this section, we generalize the result of Maia [131]
and Iseki [88] for a pair of mappings involving six point of
a Bimetric space.
THEOREM 4: Let $X$ be a metric space with two metrics $d_1$ and $d$. If $f$ and $g$ are two mappings of $X$ into itself such that $(4.1.1), (4.1.2)$ and hold

$$(4.3.1) \quad d(fu_1, gu_2) \leq a_1d(u_1, u_2) + a_2[d(u_1, fu_3) + d(u_2, gu_4)] + a_3[d(u_4, fu_5) + d(u_3, gu_6)]$$

for all $u_1, u_2, u_3, u_4, u_5, u_6$ in $X$ and $a_1, a_2, a_3$ are positive real numbers such that $a_1 + 2a_2 < 1, a_2 + a_3 < 1, a_1 + 2a_3 < 1$. Then $f$ and $g$ have a unique common fixed point.

PROOF: Let $x, y$ in $X$ and put $u_1 = gfx, u_2 = fgy, u_3 = gy, u_4 = fx, u_5 = x, u_6 = y$ in $(4.3.1)$, we have

$$d(fgf(x), gfg(y)) \leq a_1d(gf(x), fg(y)) + a_2[d(gf(x), fg(y)) + d(fg(y), gf(x))] + a_3[d(f(x), f(x)) + d(g(y), g(y))].$$

$$(4.3.2) \quad d(fgf(x), gfg(y)) \leq (a_1 + 2a_2)d(gf(x), fg(y)) = h d(gf(x), fg(y)), $$
where \( h = a_1 + 2a_2 < 1 \).

Now let \( x_0 \) be an arbitrary element of \( X \). Define the sequence as follows -

\[
x_n = f(x_{n-1}), \quad x_{n+1} = g(x_n), \text{ for odd natural numbers.}
\]

Now for \( x = x_{n-3} \) and \( y = x_{n-2} \), where \( n \) is odd \( \geq 3 \), by (4.3.2), we have

\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n).
\]

Proceeding in this manner, we have

\[
d(x_n, x_m) < \frac{h^n}{1-h} d(x_0, x_1), \quad m > n \geq 3, \quad n \text{ is odd.}
\]

Also, when \( n \) is even, a similar discussion will help to get the same strict inequality. Since \( h < 1 \), so \( \lim_{n \to \infty} h^n = 0 \). It follows that \( \{x_n\} \) is a Cauchy sequence with respect to \( d \).

Further by condition (4.1.1), sequence \( \{x_n\} \) is a Cauchy sequence under metric \( d_1 \). Since \( X \) is complete with respect to \( d_1 \), there exists a point \( z \) in \( X \) such that

\[
\lim_{n \to \infty} (x_n) = z.
\]
We shall show that \( f(z) = z \). Suppose that \( f(z) \neq z \).

On putting \( u_1 = u_3 = u_5 = z \) and \( u_2 = u_4 = u_6 = x_n \) (when \( n \) is odd) in (4.3.2), we have

\[
\begin{align*}
d(f(z), x_{n+1}) &= d(f(z), g(x_n)) \\
&\leq a_1 d(z, x_n) \\
&\quad + a_2 [d(z, f(z)) + d(x_n, g(x_n))] \\
&\quad + a_3 [d(x_n, f(z)) + d(z, g(x_n))]
\end{align*}
\]

\[
\begin{align*}
d(f(z), x_{n+1}) &\leq a_1 d(z, x_n) \\
&\quad + a_2 [d(z, f(z)) + d(z, x_{n+1})] \\
&\quad + a_3 [d(x_n, f(z)) + d(z, x_{n+1})].
\end{align*}
\]

Taking limit as \( n \to \infty \) and noting that \( \{x_n\} \) is Cauchy, we have,

\[
\begin{align*}
d(f(z), z) &\leq a_1 d(z, z) \\
&\quad + a_2 [d(z, f(z)) + d(z, z)] \\
&\quad + a_3 [d(z, f(z)) + d(z, z)] \\
&= (a_2 + a_3) d(z, f(z)),
\end{align*}
\]

leading to a contradiction, since \( a_2 + a_3 < 1 \). Thus \( z \) is a
fixed point of $T$. Similarly we can show that $z$ is a fixed point of $g$. Hence $z$ is a common fixed point of $f$ and $g$.

Now to claim the uniqueness of $z$, let $w$ be another fixed point of $f$ and $g$, if possible different from $z$. Putting $u_1 = u_3 = u_5 = z$ and $u_2 = u_4 = u_6 = w$ in $(4.3.3)$, we have

$$d(z, w) = d(f(z), g(w))$$

$$\leq a_1 d(z, w)$$

$$+ a_2 [d(z, f(z)) + d(w, g(w))]$$

$$+ a_3 [d(w, f(z)) + d(z, g(w))]$$

$$= (a_1 + 2a_3) d(z, w),$$

again leading to a contradiction, since $a_1 + 2a_3 < 1$.

Thus $z = w$.

This completes the proof of the Theorem 4.

**Theorem 5**: Let $X$ be a metric space with two metrics $d_1$ and $d$ and $f_k (k = 1, 2, 3, \ldots, n)$ be a finite family of
of mappings of \(X\) into itself. Such that (4.1.1), (4.1.2)

hold and

(4.3.3) \(f_1, f_2, \ldots, f_n\) commutes with every \(f_k\).

(4.3.4) \(d(f_1 f_2 \ldots f_n(u_1), f_n f_{n-1} \ldots f_1(u_2)) \leq a_1 d(u_1, u_2)\)

\(< a_1 d(u_1, u_2)\)

\(+ a_2[d(u_1, f_1 f_2 \ldots f_n(u_3)) + d(u_2, f_n f_{n-1} \ldots f_1(u_4))]\)

\(+ a_3[d(u_4, f_1 f_2 \ldots f_n(u_5)) + d(u_3, f_n f_{n-1} \ldots f_1(u_6))]|\)

for all \(u_1, u_2, u_3, u_4, u_5, u_6, u_6\) in \(X\) and \(a_1, a_2, a_3\) are

positive real numbers such that \(a_1 + 2a_2 < 1, a_2 + a_3 < 1,\)

\(a_1 + 2a_3 < 1\). Then \(f_k (k = 1, 2, 3, \ldots, n)\) have a unique

common fixed point in \(X\).

**PROOF**: Let \(f = f_1 f_2 \ldots f_n, g = f_n f_{n-1} \ldots f_1\).

Then by (4.3.3), we have

(4.3.5) \(d(f(u_1), g(u_2)) \leq a_1 d(u_1, u_2)\)

\(< a_1 d(u_1, u_2)\)

\(+ a_2[d(u_1, f(u_3)) + d(u_2, g(u_4))]\)

\(+ a_3[d(u_4, f(u_5)) + d(u_3, g(u_6))].\)
By Theorem 4, f and g have a unique common fixed point z.

i.e. \( f(z) = z = g(z) \).

For any \( f_k \), \( f_k(f(z)) = f_k(z) \).

Then by (4.3.3), we have

\[ f(f_k(z)) = f_k(z). \]

So \( f_k(z) \) is a fixed point of f and g.

By putting \( u_1 = u_3 = u_5 = f_k(z) \) and \( u_2 = u_4 = u_6 = z \) in (4.3.5), we have

\[ d(f_k(z), z) = d(ff_k(z), g(z)) \]

\[ \leq a_1 d(f_k(z), z) \]

\[ + a_2 [d(f_k(z), ff_k(z)) + d(z, g(z))] \]

\[ + a_3 [d(z, ff_k(z)) + d(f_k(z), g(z))] \]

\[ = (a_1 + 2a_3) d(f_k(z), z), \]

leading to a contradiction, since \( a_1 + 2a_3 < 1 \).

i.e. \( f_k(z) = z \) (\( k = 1, 2, 3, -\ldots, n \)). Hence z is a common fixed point of \( f_k \).
The uniqueness of fixed point can be proved easily.

This completes the proof of the Theorem 5.

**REMARKS** :

1. If we put \( f = g, u_1 = u_3 = u_5 = x, u_2 = u_4 = u_6 = y \) and \( a_2 = a_3 = 0 \) in Theorem 4, then we get Theorem A.

2. If we put \( u_1 = u_3 = u_5 = x, u_2 = u_4 = u_6 = y \) in Theorem 4, then we get Theorem B.