CHAPTER III

SOME FIXED POINT FOR
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3.1. In a recent paper Pachpatte [144] proved the following result for pair of mappings in metric space.

**THEOREM A**: If E and F are mappings of a complete metric space \((X,d)\) into itself satisfying the inequality -

\[
(3.1.1) \quad \left[ d(Ex, Fy) \right]^2 \leq \alpha \left[ d(x, Ex) d(y, Fy) + d(x, Fy) d(y, Ex) \right] + \beta \left[ d(x, Ex) d(y, Ex) + d(x, Fy) d(y, Fy) \right]
\]

for all \(x, y\) in \(X\), where \(\alpha, \beta \geq 0\) and \(\alpha + 2\beta < 1\). Then \(E\) and \(F\) have a unique common fixed point.

3.2. In this section, we generalize the above result of Pachpatte [144] for three mappings in complete metric space.

We prove the following theorems -

**THEOREM 1**: Let \(E, F\) and \(T\) be three self-mappings of a complete metric space \((X,d)\) such that \(T\) is continuous and
E, F, T satisfy the following conditions -

\[(3.2.1) \ ET = TE, \ FT = TF, \ E(X) \subset T(X) \text{ and } F(X) \subset T(X).\]

\[(3.2.2) \ [d(Ex,Fy)]^2 \leq \alpha [d(Tx,Ex) \cdot d(Ty,Fy) + d(Tx,Fy) \cdot d(Ty,Ex)]
+ \beta [d(Tx,Ex) \cdot d(Ty,Ex) + d(Tx,Fy) \cdot d(Ty,Fy)]
+ \gamma \left[ \frac{d(Tx,Fy) \cdot d(Ex,Fy) \cdot d(Tx,Ty)}{d(Tx,Ty) + d(Ty,Fy)} \right] \]

for all x, y in X, where \( \alpha, \beta, \gamma \) are non-negative and \( \alpha + 2\beta + \gamma < 1 \). Then E, F and T have a unique common fixed point.

**Proof** : Let \( x_0 \) be an arbitrary point of X and define a sequence \( \{Tx_n\} \) as follows -

\[(3.2.3) \ Tx_{2n+1} = Ex_{2n}, \ Tx_{2n+2} = Fx_{2n+1}, \ n = 0, 1, 2, \ldots \]

We can do this since \( E(X) \) and \( F(X) \) are sub-sets of \( T(X) \).

By the inequality (3.2.2), we have

\[
[d(Tx_{2n+1},Tx_{2n+2})]^2 = [d(Ex_{2n},Fx_{2n+1})]^2
\leq \alpha [d(Tx_{2n},Ex_{2n}) \cdot d(Tx_{2n+1},Fx_{2n+1})
+ d(Tx_{2n},Fx_{2n}) \cdot d(Tx_{2n+1},Ex_{2n})]
\]
\[ + \beta [ d(T_{x2n}, E_{x2n}) \ d(T_{x2n+1}, E_{x2n}) + d(T_{x2n}, F_{x2n+1}) \ d(T_{x2n+1}, F_{x2n+1}) ] \]

\[ + \gamma \left[ \frac{d(T_{x2n}, F_{x2n+1}) \ d(E_{x2n}, F_{x2n+1}) \ d(T_{x2n}, T_{x2n+1})}{d(T_{x2n}, T_{x2n+1}) + d(T_{x2n+1}, F_{x2n+1})} \right] \]

\[ [d(T_{x2n+1}, T_{x2n+2})]^{2} \leq \alpha [d(T_{x2n}, T_{x2n+1}) \ d(T_{x2n+1}, T_{x2n+2}) + d(T_{x2n}, T_{x2n+2}) \ d(T_{x2n+1}, T_{x2n+1}) ] \]

\[ + \beta [d(T_{x2n}, T_{x2n+2}) \ d(T_{x2n+1}, T_{x2n+1}) + d(T_{x2n}, T_{x2n+1}) \ d(T_{x2n+2}, T_{x2n+2}) ] \]

\[ + \frac{d(T_{x2n}, T_{x2n+2}) \ d(T_{x2n+1}, T_{x2n+2}) \ d(T_{x2n}, T_{x2n+1})}{d(T_{x2n}, T_{x2n+1}) + d(T_{x2n+1}, T_{x2n+2})} \]

Or,

\[ d(T_{x2n+1}, T_{x2n+2}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta} \ d(T_{x2n}, T_{x2n+1}) \]

\[ d(T_{x2n+1}, T_{x2n+2}) \leq h \ d(T_{x2n}, T_{x2n+1}) \]

where \( h = \frac{\alpha + \beta + \gamma}{1 - \beta} \ < 1 \).

Similarly we can show that

\[ d(T_{x2n}, T_{x2n+1}) \leq h \ d(T_{x2n-1}, T_{x2n}) \].
Proceeding in this way, we get

\[ d(T_{2n+1}, T_{2n+2}) \leq h^{2n+1} d(T_0, T_1). \]

By routine calculations, for \( k > n \), we have

\[
    d(T_n, T_{n+k}) \leq \sum_{i=1}^{k} d(T_{n+i-1}, T_{n+i}) \\
    \leq \sum_{i=1}^{k} h^{n+i-1} d(T_0, T_1) \\
    \leq \frac{h^n}{1-h} d(T_0, T_1) \\
    \xrightarrow{n \to \infty} 0,
\]

Since \( h < 1 \). Hence \( \{T_n\} \) is a Cauchy sequence. By the completeness of \( X \), \( \{T_n\} \) converges to a point \( z \) in \( X \). It follows from (3.2.3) that \( \{E_{2n}\} \) and \( \{F_{2n+1}\} \) also converges to \( z \). Since \( T \) is continuous and (3.2.1) hold, we have

(3.2.4) \( E(T_{2n}) = T(E_{2n}) \to Tz \),

(3.2.5) \( F(T_{2n+1}) = T(F_{2n+1}) \to Tz \),

(3.2.6) \( T(T_{2n}) \to Tz \).
Now it is to be shown that $E(Tx_{2n}) \rightarrow Ez$.

By (3.2.2), we have

$$d(E(Tx_{2n}), Ez) \leq d(E(Tx_{2n}), F(Tx_{2n+1})) + d(F(Tx_{2n+1}), Ez)$$

$$\leq \alpha[d(T(Tx_{2n}),E(Tx_{2n})), d(T(Tx_{2n+1}), F(Tx_{2n+1}))$$

$$+ d(T(Tx_{2n}), F(Tx_{2n+1})) d(T(Tx_{2n+1}), E(Tx_{2n}))$$

$$+ \beta[d(T(Tx_{2n}), E(Tx_{2n})), d(T(Tx_{2n+1}), E(Tx_{2n}))$$

$$+ d(T(Tx_{2n}), F(Tx_{2n+1})) d(T(Tx_{2n+1}), F(Tx_{2n+1}))$$

$$d(T(Tx_{2n}), F(Tx_{2n+1})) d(E(Tx_{2n}), F(Tx_{2n+1}))$$

$$+ \gamma\left[\frac{d(T(Tx_{2n}), T(Tx_{2n+1}))}{d(T(Tx_{2n}), T(Tx_{2n+1}))+d(T(Tx_{2n+1}), F(Tx_{2n+1}))}\right]^{1/2}$$

$$+ \alpha[d(Tz, Ez)d(T(Tx_{2n+1})), F(Tx_{2n+1}))$$

$$+ d(Tz, F(Tx_{2n+1})) d(T(Tx_{2n+1}), Ez))$$

$$+ \beta[d(Tz, Ez) d(T(Tx_{2n+1}), Ez)$$

$$+ d(Tz, F(Tx_{2n+1})) d(T(Tx_{2n+1}), F(Tx_{2n+1})))]$$

$$- \gamma\left[\frac{d(Tz, F(Tx_{2n+1}))d(Ez, F(Tx_{2n+1}))d(Tz, T(Tx_{2n+1}))}{d(Tz, T(Tx_{2n+1}) + d(T(Tx_{2n+1}), F(Tx_{2n+1}))}\right]^{1/2}$$

On letting $n \rightarrow \infty$ and applying (3.2.4), (3.2.5), (3.2.6),

we have

$$d(Tz, Ez) = 0,$$
which implies that $Tz = Ez$.

Similarly we can prove that $Tz = Fz$. Thus we have

(3.2.7) $Ez = Tz = Fz$, and

(3.2.8) $T(Tz) = T(Ez) = E(Tz) = E(Fz) = E(Fz)$

$= F(Tz) = F(Ez) = F(Fz)$.

By (3.2.2), (3.2.7) and (3.2.8), if $Ez \neq F(Ez)$, we have

$$[d(Ez, F(Ez))]^2 \leq \alpha [d(Tz, Ez) d(T(Ez), F(Ez))$$

$$+ d(Tz, F(Ez)) d(T(Ez), Ez)]$$

$$+ \beta [d(Tz, Ez) d(T(Ez), Ez)$$

$$+ d(Tz, F(Ez)) d(T(Ez), F(Ez))]$$

$$+ \gamma [\frac{d(Tz, F(Ez)) d(Ez, F(Ez)) d(Tz, T(Ez))}{d(Tz, T(Ez)) + d(T(Ez), F(Ez))}].$$

Or,

$$d(Ez, F(Ez)) \leq (\alpha + \gamma) d(Ez, F(Ez)),$$

which is a contradiction, since $\alpha + \gamma < 1$. 

Hence,

\[(3.2.9) \quad Ez = F(Ez).\]

By (3.2.8) and (3.2.9), we have

\[Ez = F(Ez) = E(Ez) = T(Ez),\]

which implies that \(Ez\) is common fixed point of \(E, F\) and \(T\).

Now to show the uniqueness of \(z\), let \(w\) be another common fixed point of \(X\) different from \(z\) such that

\[Ez = Tw = Fz = z \text{ and } Ew = Tw = Fw = w.\]

Applying (3.2.2), we have

\[\left[ d(z,w) \right]^2 = \left[ d(Ez,Fw) \right]^2 \text{ leads to a contradiction, since } \alpha + \gamma < 1. \text{ Hence } z = w.\]
**Theorem 2**: Let $E$, $F$ and $T$ be three self-mappings of a complete metric space $(X, d)$ such that $T$ is continuous and $E$, $F$, $T$ satisfy (3.2.1) and following condition -

$$(3.2.10) \ [d(E^m x, F^n y)]^2 \leq \alpha [d(Tx, E^m x) \ d(Ty, F^n y)]$$

$$+ d(Tx, F^n y) \ d(Ty, E^m x)]$$

$$+ \beta [d(Tx, E^m x) \ d(Ty, E^m x)]$$

$$+ d(Tx, F^n y) \ d(Ty, F^n y)]$$

$$+ \gamma [\frac{d(Tx, F^n y) \ d(E^m x, F^n y) \ d(Tx, Ty)}{d(Tx, Ty) + d(Ty, F^n y)}]$$

for all $x, y$ in $X$, where $\alpha$, $\beta$, $\gamma$ are non-negative such that $\alpha + 2\beta + \gamma < 1$. Then $E$, $F$ and $T$ have a unique common fixed point.

**Proof**: It follows from (3.2.1) that

$$E^m T = TE^m, \ F^n T = TF^n, \ E^m (X) C E (X) C T(X) \ and$$

$$F^n (X) C F(X) C T(X).$$

Therefore, by Theorem 1, there exists a unique fixed point $z$ in $X$ such that $z = Tz = E^m z = F^n z$. 
Also,

\[ T(Ez) = E(Tz) = Ez = E(E^m z) = E^m(Ez). \]

This means that \( Ez \) is a common fixed point of \( T \) and \( E^m \).

Similarly, \( Fz \) is a common fixed point of \( T \) and \( F^n \). The uniqueness of \( z \) implies

\[ Ez = Fz = Tz = z. \]

This completes the proof of the Theorem 2.

**Remarks**:

(i) If we put \( T = Ix \) (the identity map on \( X \)) and \( \beta = \gamma = 0 \) in Theorem 1, then we get result due to Fisher [50].

(ii) If we put \( T = Ix \) and \( \gamma = 0 \) in Theorem 1, then we get Theorem A.

(iii) If we put \( m = n = 1 \), in Theorem 2, then Theorem 2, reduces to Theorem 1.
3.3. Bhagwat and Singh [10] proved the following theorem -

**THEOREM B**: Let $T_1$ and $T_2$ be two self-mappings of a metric space $(X,d)$ such that

$$d(T_1x, T_2y) \leq \frac{d(x, T_1x) + d(y, T_2y)}{d(x, T_2y) + d(y, T_1x)}$$

for all $x, y$ in $X$. If for some $x_0 \in X$ the sequence $\{x_n\}$ of elements $x_n$ where $x_{2n+1} = T_1x_{2n}$, $x_{2n+2} = T_2x_{2n+1}$, has a convergent sub-sequence $\{x_{n_k}\}$ converges to point $u \in X$, then $u$ is a unique fixed point of $T_1$ and $T_2$.

3.4. Now we prove the following generalization -

**THEOREM 3**: Let $E$, $F$ and $T$ be three self-mappings of a complete metric space $(X,d)$ such that $T$ is continuous and $E$, $F$, $T$ satisfy (3.2.1) and following condition -

$$d(Ex, Fy) \leq \alpha \frac{d(Tx, Ex) + d(Ty, Fy)}{d(Tx, Fy) + d(Ty, Ex)}$$

(3.4.1)
for all $x, y$ in $X$ with $Tx \neq Ty$, $0 < \alpha < 1$ and

$d(Tx, Fy) + d(Ty, Ex) \neq 0$. Then $E, F$ and $T$ have common fixed point. Further if $d(Tx, Fy) + d(Ty, Ex) = 0$ implies $d(Ex, Fy) = 0$.

Then $E, F$ and $T$ have a unique common fixed point.

**Proof**: Let $x_0$ be an arbitrary point of $X$ and we define a sequence $\{T^n x_0\}$ as in Theorem 1. We can do this since $E(X)$ and $F(X)$ are subset of $T(X)$.

By the inequality (3.4.1), we have

$$d(T_{2n+1}, T_{2n+2}) = d(Ex_{2n}, Fx_{2n+1})$$

$$\leq \alpha \frac{d(T_{2n+1}, Ex_{2n})d(T_{2n}, Fx_{2n+1}) + d(T_{2n+1}, Fx_{2n+1})}{d(T_{2n+1}, Fx_{2n+1}) + d(T_{2n+1}, Ex_{2n})}$$

$$d(T_{2n+1}, T_{2n+1}) \leq \alpha \frac{d(T_{2n+1}, T_{2n+1})}{d(T_{2n+1}, T_{2n+2}) + d(T_{2n+1}, T_{2n+1})}$$

$$d(T_{2n+1}, T_{2n+2}) \leq \alpha d(T_{2n}, T_{2n+1}).$$

Proceeding in this way, we get

$$d(T_{2n+1}, T_{2n+2}) \leq \alpha^{2n+1}d(T_0, T_1).$$
By routine calculations, for \( k > n \), we have

\[
d(T_{x_n}, T_{x_{n+k}}) \leq \sum_{i=1}^{k} d(T_{x_{n+i-1}}, T_{x_{n+i}})
\]

\[
\leq \sum_{i=1}^{k} h^{n+i-1}d(T_{x_0}, T_{x_1})
\]

\[
\leq \frac{h^n}{1-h} d(T_{x_0}, T_{x_1}).
\]

\[\longrightarrow 0 \text{ as } n \longrightarrow \infty.\]

Since \( h < 1 \), hence \( \{T_{x_n}\} \) is a Cauchy sequence. By the completeness of \( X \), \( \{T_{x_n}\} \) converges to a point \( z \) in \( X \).

It follows from (3.2.3) that \( \{E_{2n}\} \) and \( \{F_{2n+1}\} \) also converges to \( z \). Since \( T \) is continuous also (3.2.1) hold, we have

(3.4.2) \( E(T_{2n}) = T(E_{2n}) \longrightarrow Tz \),

(3.4.3) \( F(T_{2n+1}) = T(F_{2n+1}) \longrightarrow Tz \),

(3.4.4) \( T(T_{2n}) \longrightarrow Tz \).

Now it is to be shown that \( E(T_{2n}) \longrightarrow Ez \). Applying (3.4.1), we have
\[ d(E(T_{2n}), Ez) \leq d(E(T_{2n}), F(T_{2n+1})) + d(F(T_{2n+1}), Ez) \]

\[ d(T(T_{2n}), E(T_{2n})d(T(T_{2n}), F(T_{2n+1})) + \]

\[ \leq \left\{ a \frac{d(T(T_{2n+1}), F(T_{2n+1}))d(T(T_{2n+1}), E(T_{2n}))}{d(T(T_{2n}), F(T_{2n+1}))+d(T(T_{2n+1}), E(T_{2n}))} \right\} \]

\[ d(Tz, Ez)d(Tz, F(T_{2n+1}))+d(T(T_{2n+1}), F(T_{2n+1})) \]

\[ + \left\{ a \frac{d(T(T_{2n+1}), Ez)}{d(Tz, F(T_{2n+1}))+d(T(T_{2n+1}), Ez)} \right\} \]

On letting \( n \to \infty \) and applying (3.4.2), (3.4.3), (3.4.4),

we have

\[ d(Tz, Ez) = 0 \]

which implies that \( Tz = Ez \).

Similarly we can prove that \( Tz = Fz \). Thus we have

(3.2.5) \( Ez = Tz = Fz \) and

(3.2.6) \( T(Tz) = T(Ez) = E(Tz) = E(Ez) = E(Fz) = T(Fz) = F(Tz) = F(Ez) = F(Fz) \).

If we assume \( Ez \neq F(Ez) \), then \( d(Ez, F(Ez)) > 0 \).

By (3.4.1), (3.4.5) and (3.4.6), we have
\[ d(Ez, F(Ez)) \leq \alpha \frac{d(Tz, Ez)d(Tz, F(Ez)) + d(T(Ez), F(Ez))d(T(Ez), Ez)}{d(Tz, F(Ez)) + d(T(Ez), Ez)} \]

\[ d(Ez, F(Ez)) \leq 0, \]

a contradiction. Hence,

(3.4.7) \( Ez = F(Ez) \).

By (3.4.6) and (3.4.7), we have

\[ Ez = F(Ez) = E(Ez) = T(Ez). \]

Which implies that \( Ez \) is the common fixed point of \( E, F \) and \( T \).

Now to show that the uniqueness of \( z \). Let us consider that \( d(Tx, Fy) + d(Ty, Ez) = 0 \) implies \( d(Ez, Fy) = 0 \), and let \( w \) be another fixed point of \( X \). Then

\[ d(Tz, Fw) + d(Tw, Ez) = 0 \] which implies \( z = w \).

This completes the proof of the Theorem 3.

**Remark :**

If we put \( T = Ix \) (the indentity map on \( X \)) and \( \alpha = 1 \) in Theorem 3 then we get Theorem B.
3.5. In this section, we obtained some result for three mappings in complete metric space. The obtained results are the generalizations of Banach Contraction Principle [8], Kannan [102] and Fisher [54].

**Theorem 4**: Let E, F and T be three continuous mappings of a complete metric space X into itself satisfying (3.2.1) and

\[
(3.5.1) \quad d(Ex,Fy) \leq \alpha_1 \left[ \frac{d(Tx,Fy)}{d(Tx,Ty) + d(Ty,Fy)} \right] \\
+ \alpha_2 \left[ d(Tx,Ex) + d(Ty,Fy) \right] \\
+ \alpha_3 \left[ d(Tx,Fy) + d(Ty,Ex) \right] \\
+ \alpha_4 \ d(Tx,Ty)
\]

for all \( x, y \) in X with \( Tx \neq Ty \) and \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are non-negative such that \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1 \). Then E, F and T have a unique common fixed point.

**Proof**: Let \( x_0 \) be an arbitrary point of X and we define a sequence \( \{Tx_n\} \) as in Theorem 1.

We can do this since \( E(X) \subset T(X) \) and \( F(X) \subset T(X) \).
By the inequality (3.5.1), we have

\[ d(T_{2n+1}, T_{2n+2}) = d(E_{2n}, F_{2n+1}) \]

\[ \leq \alpha_1 \left[ \frac{d(T_{2n}, F_{2n+1})d(T_{2n}, T_{2n+1})}{d(T_{2n}, T_{2n+1}) + d(T_{2n+1}, F_{2n+1})} \right] \]

\[ + \alpha_2 \left[ d(T_{2n}, E_{2n}) + d(T_{2n+1}, F_{2n+1}) \right] \]

\[ + \alpha_3 \left[ d(T_{2n}, F_{2n+1}) + d(T_{2n+1}, E_{2n}) \right] \]

\[ + \alpha_4 \left[ d(T_{2n}, T_{2n+1}) \right] \]

\[ d(T_{2n+1}, T_{2n+2}) \leq \alpha_1 \left[ \frac{d(T_{2n}, T_{2n+2})d(T_{2n}, T_{2n+1})}{d(T_{2n}, T_{2n+1}) + d(T_{2n+1}, T_{2n+2})} \right] \]

\[ + \alpha_2 \left[ d(T_{2n}, T_{2n+2}) + d(T_{2n+1}, T_{2n+2}) \right] \]

\[ + \alpha_3 \left[ d(T_{2n}, T_{2n+2}) + d(T_{2n+1}, T_{2n+1}) \right] \]

\[ + \alpha_4 \left[ d(T_{2n}, T_{2n+1}) \right]. \]

Or,

\[ d(T_{2n+1}, T_{2n+2}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3} d(T_{2n}, T_{2n+1}) \]

\[ = h d(T_{2n}, T_{2n+1}). \]
where \( h = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3} < 1. \)

Similarly we can show that,

\[
d(T_{2n}, T_{2n+1}) \leq h \cdot d(T_{2n-1}, T_{2n}).
\]

Proceeding in this way, we get

\[
d(T_{2n+1}, T_{2n+2}) \leq h^{2n+1}d(T_0, T_1).
\]

By routine calculations, for \( k > n \), we have

\[
d(T_n, T_{n+k}) \leq \sum_{i=1}^{k} d(T_{n+i-1}, T_{n+i-1})
\leq \sum_{i=1}^{k} h^{n+i-1}d(T_0, T_1)
\leq \frac{h^n}{1 - h} d(T_0, T_1)
\rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Since \( h < 1 \). Hence \( \{T_n\} \) is a Cauchy sequence. By the completeness of \( X \), \( \{T_n\} \) converges to a point \( z \) in \( X \). It follows from (3.2.3) that \( \{E_{2n}\} \) and \( \{F_{2n+1}\} \) also converges to \( z \).
Since $E$, $F$ and $T$ are continuous, we have

\[(3.5.2) \quad E(Tx_{2n}) \rightarrow Ez, \quad F(Tx_{2n+1}) \rightarrow Fz.\]

From (3.2.1), $T$ commutes with $E$ and $F$, we have

\[E(Tx_{2n}) = T(Ex_{2n}), \quad F(Tx_{2n+1}) = T(Fx_{2n+1}), \quad \text{for all } n = 0, 1, 2, \ldots\]

On letting $n \rightarrow \infty$, we have

\[(3.5.3) \quad Ez = Tz = Fz, \quad \text{and}\]

\[(3.5.4) \quad T(Tz) = T(Ez) = E(Tz) = E(Tz) = E(Fz) = T(Fz) = F(Tz) = F(Ez) = F(Fz).\]

By (3.5.1), (3.5.3) and (3.5.4), if $Ez \neq F(Ez)$, we have

\[d(Ez, F(Ez)) \leq \alpha_1 \left[ \frac{d(Tz, F(Ez)) \cdot d(Tz, T(Ez))}{d(Tz, T(Ez)) + d(T(Ez), F(Ez))} \right] \]

\[+ \alpha_2 \left[ d(Tz, Ez) + d(T(Ez), F(Ez)) \right] \]

\[+ \alpha_3 \left[ d(Tz, F(Ez)) + d(T(Ez), Ez) \right] \]

\[+ \alpha_4 \cdot d(Tz, T(Ez)) \]
\[ d(Ez,F(Ez)) \leq (\alpha_1 + 2\alpha_3 + \alpha_4) \cdot d(Ez,F(Ez)), \]

which is a contradiction, since \( \alpha_1 + 2\alpha_3 + \alpha_4 < 1 \).

Hence

\[ (3.5.5) \quad Ez = F(Ez). \]

By (3.5.4) and (3.5.5), we have

\[ Ez = F(Ez) = T(Ez) = E(Ez), \]

which implies that \( Ez \) is the common fixed point of \( E, F \) and \( T \).

Now to show that the uniqueness of \( z \), let \( w \) be another common fixed point of \( X \) different from \( z \) such that

\[ Ez = Tz = Fz = z \quad \text{and} \quad Fw = Tw = Fw = w. \]

Applying (3.5.2), we have

\[ d(z,w) = d(Ez,Fw) \]

\[ \leq \alpha_1 \left[ \frac{d(Tz,Fw) \cdot d(Tz,Tw)}{d(Tz,Tw) + d(Tw,Fw)} \right] + \alpha_2 [d(Tz,Ez) + d(Tw,Fw)] + \alpha_3 [d(Tz,Fw) + d(Tw,Ez)] + \alpha_4 \cdot d(Tz,Tw) \]
\[ d(z, w) \leq (\alpha_1 + 2\alpha_3 + \alpha_4) d(z, w), \]
again leading to a contradiction, since \( \alpha_1 + 2\alpha_3 + \alpha_4 < 1 \).

Hence \( z = w \).

This completes the proof of the Theorem 4.

**REMARKS** :

(1) If we put \( E = F, \; T = Ix \) (The indentify map on \( X \)) and \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) in Theorem 4, then we get Banach contraction Principle [8].

(2) If we put \( E = F, \; T = Ix \) and \( \alpha_1 = \alpha_3 = \alpha_4 = 0 \) in Theorem 4, then we get Kannon [102].

(3) If we put \( E = F, \; T = Ix \) and \( \alpha_1 = \alpha_2 = \alpha_4 = 0 \) in Theorem 4, then we get Fisher [54].

**THEOREM 5** : Let \( E, F \) and \( T \) be three continuous mapping of a complete metric space \( X \) into itself satisfying (3.2.1) and

\[
(3.5.6) \quad d(Ex, Fy) \leq \frac{\alpha_1 d(Tx, Fy) [1 + d(Tx, Ex)]}{[1 + d(Tx, Ty)]} + \alpha_2 [d(Tx, Ex) + d(Ty, Fy)]
\]
+ \alpha_3 [d(Tx,Fy) + d(Ty,Ex)]
+ \alpha_4 d(Tx,Ty)

for all \( x, y \) in \( X \) with \( Tx \neq Ty \) and \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are non negative such that \( 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1 \). Then \( E, F \) and \( T \) have a common fixed point.

**Proof**: Let \( x_0 \) be an arbitrary point of \( X \) and define a sequence \( \{Tx_n\} \) as in Theorem 1. We can do this since \( E(X) \subseteq T(X) \) and \( F(X) \subseteq T(X) \). By the inequality (3.5.6), we have

\[
d(Tx_{2n+1},Tx_{2n+2}) = d(Ex_{2n},Fx_{2n+1})
\leq \alpha_1 \frac{d(Tx_{2n},Fx_{2n+1})[1 + d(Tx_{2n},Ex_{2n})]}{[1 + d(Tx_{2n},Tx_{2n+1})]}
+ \alpha_2 [d(Tx_{2n},Ex_{2n}) + d(Tx_{2n+1},Fx_{2n+1})]
+ \alpha_3 [d(Tx_{2n},Fx_{2n+1}) + d(Tx_{2n+1},Ex_{2n})]
+ \alpha_4 d(Tx_{2n},Tx_{2n+1})
\]

\[
d(Tx_{2n+1},Tx_{2n+2}) \leq \alpha_1 \frac{d(Tx_{2n},Tx_{2n+2})[1+d(Tx_{2n},Tx_{2n+1})]}{[1 + d(Tx_{2n},Tx_{2n+1})]}
\]
+ \alpha_2 [d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n+2})] \\
+ \alpha_3 [d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})] \\
+ \alpha_4 d(Tx_{2n}, Tx_{2n+1}).

Or,

d(Tx_{2n+1}, Tx_{2n+2}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_1 - \alpha_2 - \alpha_3} d(Tx_{2n}, Tx_{2n+1})

= h d(Tx_{2n}, Tx_{2n+1}),

where \quad h = \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_1 - \alpha_2 - \alpha_3} < 1.

Similarly we can show that,

d(Tx_{2n},Tx_{2n+1}) \leq h d(Tx_{2n-1}, Tx_{2n}).

Proceeding in this way, we get

d(Tx_{2n+1},Tx_{2n+2}) \leq h^{2n+1} d(Tx_0, Tx_1).

By routine calculations, for \( k > n \), we have

d(Tx_n, Tx_{n+k}) \leq \sum_{i=1}^{k} d(Tx_{n+i-1}, Tx_{n+i-1}).
\[ \leq \sum_{i=1}^{k} h^{n+1-i} d(Tx_0, Tx_1) \]
\[ \leq \frac{h^n}{1-h} d(Tx_0, Tx_1) \]
\[ \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Since \( h < 1 \), Hence \( \{Tx_n\} \) is a Cauchy sequence. By the completeness of \( X \), \( \{Tx_n\} \) converges to a point \( z \) in \( X \). It follows from (3.2.3) that \( \{Ex_{2n}\} \) and \( \{Tx_{2n+1}\} \) also converges to \( z \).

Since \( E, F \) and \( T \) are continuous, we have

\[ (3.5.7) \ E(Tx_{2n}) \rightarrow Ez, \ F(Tx_{2n+1}) \rightarrow Fz. \]

By (3.2.1), \( T \) commutes with \( E \) and \( F \), we have

\[ E(Tx_{2n}) = T(Ex_{2n}), \ F(Tx_{2n+1}) = T(Fx_{2n+1}), \text{ for } n = 0, 1, 2, \ldots. \]

On letting \( n \rightarrow \infty \), we have

\[ (3.5.8) \ Ez = Tz = Fz \text{ and } \]

\[ (3.5.9) \ T(Tz) = T(Ez) = E(Tz) = E(Ez) = E(Fz) = T(Fz) = F(Tz) = F(Ez) = E(Fz). \]
By (3.5.6), (3.5.8) and (3.5.9), if $Ez \neq F(Ez)$,

we have

$$d(Ez, F(Ez)) \leq \alpha_1 \frac{d(Tz, F(Ez))[1 + d(Tz, Ez)]}{[1 + d(Tz, T(Ez))]}$$

$$+ \alpha_2 [d(Tz, Ez) + d(T(Ez), F(Ez))]$$

$$+ \alpha_3 [d(Tz, F(Ez)) + d(T(Ez), Ez)]$$

$$+ \alpha_4 d(Tz, T(Ez))$$

$$d(Ez, F(Ez)) \leq (2\alpha_3 + \alpha_4) d(Ez, F(Ez)),$$

which is a contradiction, since $2\alpha_3 + \alpha_4 < 1$. Hence,

(3.5.10) $Ez = F(Ez)$.

By (3.5.9) and (3.5.10), we have

$Ez = F(Ez) = E(Ez)$,

which show that $Ez$ is a common fixed point of $E$, $F$ and $T$.

This completes the proof of the Theorem 5.
REMARKS:

(1) If we put $E = F$, $T = Tx$ (the identity map on $X$) and $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in Theorem 5, then we get Banach contraction principle [8].

(2) If we put $E = F$, $T = Tx$ and $\alpha_1 = \alpha_3 = \alpha_4 = 0$ in Theorem 5, then we get Kanan [102].

(3) If we put $E = F$, $T = Tx$ and $\alpha_1 = \alpha_2 = \alpha_4 = 0$ in Theorem 5, then we get Fisher [54].