ON THE \((H, 1) (E, q) (C, 1)\) SUMMABILITY OF THE SEQUENCE OF FOURIER COEFFICIENTS \(\{nB_n(x)\}\)
8.1 Let $f(x)$ be a function, integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and periodic with period $2\pi$ outside this range.

Let the Fourier series of $f(x)$ be given by

$$8.1.1 \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x).$$

The Conjugate series of (8.1.1) is

$$8.1.2 \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

Harmonic Summability of an infinite series has been defined as follows.

$(H, 1)$ Summability.

An infinite series $\sum u_n$ with partial sum $S_n = \sum_{k=0}^{n} u_k$ is said to be harmonic summable to the sum $S$, if

$$8.1.3 \quad \frac{1}{\log n} \sum_{k=0}^{n} \frac{S_{n-k}}{k+1} \text{ tends to } S \text{ as } n \to \infty.$$
(E, q) Summability\textsuperscript{1)}

An infinite series $\sum u_n$ with sequence of partial sum \{S_n\} is said to be (E, q) summable to the sum S, if

$$8.1.4 \quad \frac{1}{(1+q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} S_m \to S \quad \text{as} \quad n \to \infty$$

$$0 < q \leq 1$$

(C, 1) Summability.

An infinite series $\sum u_n$ with sequence of partial sum \{S_n\} is said to be (C, 1) summable to the sum S, if

$$8.1.5 \quad \lim_{n \to \infty} \frac{1}{k+1} \sum_{k=0}^{n} S_k = S$$

(E, q) (C, 1) Summability\textsuperscript{2)}

(E, q) (C, 1) summability is defined by superimposing the mean of (E, q) summability on the mean of (C, 1) summability.

Let (C, 1) mean of \{S_n\} is denoted by $\rho_n$ and $\sigma_n$ denotes the (E, q) (C, 1) mean of \{S_n\}.

Then

$$8.1.6 \quad \sigma_n = \frac{1}{(1+q)^n} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \rho_m$$

If $\sigma_n \to S$ as $n \to \infty$, we say that $\sum u_n$ is summable (E, q) (C, 1) to the sum S.

\textsuperscript{1)} Hardy, G.H. (1)
\textsuperscript{2)} Prakash, Ved (1) [Chapter III]
(H, 1) (E, q) (C, l) Summability.

We can define (H, 1) (E, q) (C, l) Summability by super-imposing the mean of (H, 1) summability on the mean of (E, q) (C, l) summability.

Let \((H, 1) (E, q) (C, l)\) mean of \(\{S_n\}\) is denoted by \(H_n\).

Then

\[
H_n = \frac{1}{\log n} \sum_{k=0}^{n} \frac{\sigma_{n-k}}{k+1}
\]

If \(\lim_{n \to \infty} H_n = S\). We say that \(\sum u_n\) is summable \((H, 1)(E, q)(C, l)\) to the sum \(S\).

Mohanty and Nanda\(^3\) proved the following theorem for \((C, 1)\) summability of \(\{nB_n(x)\}\).

**Theorem - A**

If

\[
f(x+t) - f(x-t) - \frac{1}{\pi} = \Psi(t) = 0((\log \frac{1}{t})^{-1}), \quad \text{as} \quad t \to 0
\]

and \(a_n\) and \(b_n\) are \(0(n^{-\delta})\), \(0 < \delta < 1\), then the sequence \(\{nB_n(x)\}\) is summable \((C, 1)\) to the value \(\frac{1}{\pi}\).

Varshney\(^4\) proved the following theorem for \((H, 1) (C, l)\) summability of \(\{nB_n(x)\}\).

\[^3\] Mohanty, R. and Nanda, M. (1)
\[^4\] Varshney, O.P. (1)
Theorem - B

If

8.1.9 \[ \int_0^t |\psi(u)| \, du = O(t (\log \frac{1}{t})^{-1}), \text{ as } t \to 0, \]
then the sequence \( \{ nB_n(x) \} \) is summable \( (H, 1) \) \( (C, 1) \) to the value \( \frac{1}{n} \).

Dixit and Shukla\(^5\) proved the theorem for \( B(C, 1) \) summability of \( \{ nB_n(x) \} \).

Theorem - C

If

8.1.10 \[ \int_0^t |\psi(u)| \, du = O(t), \text{ as } t \to 0 \]
and

8.1.11 \[ \int_0^t \frac{|\psi(u)|}{u} \exp (p \cos u) \, du = O(\exp (p \cos t)), \text{ as } t \to 0, \]
then the sequence \( \{ nB_n(x) \} \) is summable \( B(C, 1) \) to the value \( \frac{1}{n} \).

Ved Prakash\(^6\) proved the following theorem for \( (E, q) \) \( (C, 1) \) summability of \( \{ nB_n(x) \} \).

Theorem - D

If

8.1.12 \[ f(x+t) - f(x-t) - \frac{1}{n} = \psi(t) \]

8.1.13 \[ \int_0^t |\psi(u)| \, du = O(t) \text{ as } t \to 0 \]
and

\[ \text{---------------------------------------------------------------------} \]

5) Dixit, S.S. and Shukla, Y.B. (1)
6) Prakash, Ved (1) [Chapter III]
8.1.14 \[ \int_{t}^{6} \frac{|\Psi(u)|}{u} (1+2q \cos u + q^2)^{n/2} \, du = O(1+q)^n \text{ as } t \to 0 \]

Then the sequence \( \{nR_n(x)\} \) is summable \((E, q)\) \((C, l)\) to the value \(\frac{1}{n}\).

Now we will prove the following theorem.

**Theorem - 1**

If

8.1.15 \[ \int_{0}^{t} |\Psi(u)| \, du = O(t) \text{ as } t \to 0 \]

and

8.1.16 \[ \int_{t}^{6} \frac{|\Psi(u)|}{u} \sum_{k=1}^{n} (1+q)^{-k} \frac{(1+2q \cos u + q^2)^{k/2}}{(n-k+1)} \, du = O(\log \frac{1}{t}) \text{ as } t \to 0 \]

then the sequence \( \{nR_n(x)\} \) is \((H, l)\) \((E, q)\) \((C, l)\) summable to the value \(\frac{1}{n}\).

8.2 For the proof of our theorem we shall require the following lemma.

**Lemma - 1**

If

\[ T(n, t) = \frac{1}{\log n} \sum_{k=0}^{n} (1+q)^{-k} g(k, t) \]

and

\[ g(n, t) = \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin(m+1)t}{(m+1) t^2} \frac{\cos(m+1)t}{t} \]
Then,
\[ T(n, t) = O(n) \quad \text{for} \quad 0 < t < \frac{1}{n} \]

**Proof**

Since \( g(n, t) = O[n(1+q)^n] \)  \(^7\)

Therefore,
\[
T(n, t) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{(1+q)^{-k} \cdot g(k, t)}{(n-k+1)}
= \frac{1}{\log n} \sum_{k=1}^{n} \frac{(1+q)^{-k} \cdot 0 \cdot (k(1+q)^k)}{(n-k+1)}
= \frac{1}{\log n} \sum_{k=1}^{n} 0 \cdot \left(\frac{k}{n-k+1}\right)
= 0(\frac{1}{\log n} \cdot n \log n)
= O(n)
\]

For \( \frac{1}{n} < t < \pi \)
\[ g(n, t) = O[\frac{(1+2q \cos t + q^2)^{n/2}}{t}] \]  \(^8\)

**8.3 PROOF OF THE THEOREM - 1.**

If we denote the \((C, 1)\) transform of the sequence \( \{ n R_n(x) \} \) by \( \rho_n \),

We have,

[ C.F. Mohanty and Nanda \(^9\) ]

\[ 8.3.1 \quad \rho_n - \frac{1}{n} = \frac{1}{n} \int_{0}^{\pi} \Psi(t) \left[ \frac{\sin(n+1)t}{(n+1)t^2} - \frac{\cos(n+1)t}{t} \right] dt + O(1) \]

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7) Prakash, Ved (1) [Chapter III]
8) Prakash, Ved (1) [Chapter III]
9) Mohanty, R. and Nanda, M. (1)
If $\sigma_n$ is the $(E, q)$ transform of (8.3.1) then the sequence $\{nB_n(x)\}$ is summable $(E, q) (C, 1)$ to the value $\frac{1}{n}$. If $\sigma_n = O(1)$ as $n \to \infty$

\[
8.3.2 \quad \sigma_n = \frac{1}{\pi} (1+q)^{-n} \int_0^\pi \psi(t) \sum_{m=0}^n \binom{n}{m} q^{n-m} \left[ \frac{\sin(m+1)t}{(m+1)t^2} - \frac{\cos(m+1)t}{t} \right] dt
\]

\[
= \frac{1}{\pi} (1+q)^{-n} \int_0^\pi \psi(t) g(n, t) dt
\]

where

\[
g(n,t) = \sum_{m=0}^n \binom{n}{m} q^{n-m} \left[ \frac{\sin(m+1)t}{(m+1)t^2} - \frac{\cos(m+1)t}{t} \right]
\]

Let $(H, 1) (E, q) (C, 1)$ Summability of the sequence $\{nB_n(x)\}$ be denoted by $H_n$.

Then

\[
8.3.3 \quad H_n = \frac{1}{\pi \log n} \int_0^\pi \psi(t) \sum_{k=1}^n \frac{(1+q)^{-k} g(k, t)}{(n-k+1)} dt
\]

On account of regularity of Harmonic Summation, we need only to prove that

\[
\lim_{n \to \infty} H_n = O(1)
\]

Then $\{nB_n(x)\}$ is $(H, 1) (E, q) (C, 1)$ summable to the value $\frac{1}{n}$.

\[
8.3.4 \quad H_n = \frac{1}{n} \int_0^\pi \psi(t) T(n, t) dt
\]

10) Prakash, Ved (1) [Chapter III]
where
\[ T(n, t) = \frac{1}{\log n} \sum_{k=1}^{n} \frac{(1+q)^{-k} g(k, t)}{(n-k+1)} \]

\[ H_n = \frac{1}{n} \left[ \int_{0}^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{n} \right] \psi(t) \ T(n, t) \ dt \]

8.3.5 \[ = I_1 + I_2 + I_3, \] say

\[ I_1 = \frac{1}{n} \int_{0}^{1/n} \psi(t) \ T(n, t) \ dt \]

\[ = \frac{1}{n} \int_{0}^{1/n} \psi(t) \cdot 0(n) \ dt \]

\[ = 0(n) \int_{0}^{1/n} \psi(t) \ dt \]

[ by Lemma - 1 ]

Therefore
\[ |I_1| \leq 0(n) \int_{0}^{1/n} |\psi(t)| \ dt \]

\[ = 0(n) \cdot 0\left(\frac{1}{n}\right) \]

[ by hypothesis (8.1.16) ]

8.3.6 \[ = O(1). \]

\[ I_2 = \frac{1}{n} \int_{1/n}^{\delta} \psi(t) \ T(n, t) \ dt \]

\[ = \frac{1}{n} \int_{1/n}^{\delta} \psi(t) \frac{1}{\log n} \sum_{k=1}^{n} \frac{(1+q)^{-k} g(k, t)}{(n-k+1)} \ dt \]

\[ = \frac{1}{n} \int_{1/n}^{\delta} \psi(t) \frac{1}{\log n} \sum_{k=1}^{n} \frac{(1+q)^{-k} 0(1+2q \cos t + q^2)^{k/2}}{t(n-k+1)} \ dt \]

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11) Prakash, Ved (1) [Chapter III]
\[
\begin{align*}
|I_2| &\leq O\left(\frac{1}{\log n}\right) \left(\frac{1}{\log n}\right) \int_{1/n}^\delta \frac{\sum_{k=1}^n (1+q)^{-k} (1+2q \cos t + q^2)^{k/2}}{t} dt
\end{align*}
\]

\[
|I_2|\leq O\left(\frac{1}{\log n}\right) \int_{1/n}^\delta \left|\frac{\sum_{k=1}^n (1+q)^{-k} (1+2q \cos t + q^2)^{k/2}}{t}\right| dt
\]

\[
= O\left(\frac{1}{\log n}\right) O(\log n)
\]

[by hypothesis (8.1.17)]

\[
8.3.7 = O(1)
\]

Since Harmonic Summability method is regular.

We have

\[
I_3 = \frac{1}{n} \int_{6}^{n} \Psi(t) \tau(n, t) dt
\]

\[
8.3.8 = O(1)
\]

by Riemann Lebesgue theorem.

Collection of (8.3.6), (8.3.7) and (8.3.8) completes the proof of the theorem - 1.