CHAPTER IV

RELATIVE PURE INJECTIVE MODULES
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Relative pure injective modules

The idea of purity has played from the beginning a basic role in the theory of abelian groups. In recent year as the study of rings and modules advanced, it was natural, and indeed inevitable to extend this abelian group theoretical concept.

In this chapter the notions of pure-M-injective, pure injective and pure noetherian modules are defined and proved the following results.

(1) A necessary and sufficient condition for an \( R \)-module to be pure \( M \)-injective is obtained.

(2) If \( Q \) is pure \( M \)-injective then the functor \( \text{Hom}_R (-, Q) \) preserves the exactness for all short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
& & f & & g & & \\
\end{array}
\]

where \( M' \) is pure sub module of \( M \).

(3) The class of pure \( M \)-injective module is closed under pure sub modules.

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(4) Established relation among pure submodule of \( M \), pure \( M \)-injective module and pure injective module.

(5) Finally we fined that in any pure short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \overset{f}{\longrightarrow} & M & \overset{g}{\longrightarrow} & M'' & \longrightarrow & 0 \\
\end{array}
\]

\( M \) is pure Noetherian if and only if \( M' \) and \( M'' \) are pure Noetherian module.

**Definition 1:**

An \( R \)-module \( Q \) is called pure \( M \)-injective module (pure injective relative to \( M \)) if any monomorphism

\[ f : M' \longrightarrow M \text{ with } \text{Im} f \text{ is pure in } M \text{ and homomorphism} \]

\[ v : M' \longrightarrow Q \text{ there is an homomorphism} \]

\[ h : M \longrightarrow Q \text{ such that} \]

\[ v = h \cdot f \]

\[
\begin{array}{cccccc}
0 & \longrightarrow & M' & \overset{f}{\longrightarrow} & M & \overset{g}{\longrightarrow} & M'' & \longrightarrow & 0 \\
& & \downarrow{v} & & & & \downarrow{} & & \downarrow{} \\
& & Q & & & & & & \\
\end{array}
\]
Definition 2 :-

We say that an $R$-module $Q$ is pure-injective if it is pure $M$-injective for all $m \in R \downarrow M$, where $R \downarrow M$ denote the category of left $R$-modules.

Definition 3 :-

Let $N$ be any submodule of a left $R$-module $M$, $N$ is said to be a pure sub module of $M$, if for all $m \in M$, $rm \in M$ implies there exists $n \in N$ such that $rm = rn$.

Example :- Every injective modules and $M$-injective modules are examples of pure $M$-injective module.

Proposition 1 [42] :-

If $N \subseteq M' \subseteq M$ are left $R$-modules, Then:

(i) $N$ is pure in $M'$ and $M'$ is pure in $M$ $\implies$ $N$ is pure in $M$.

(ii) $N$ is pure in $M$ $\implies$ $N$ is pure in $M'$.

(iii) $M'$ is pure in $M$ $\implies$ $M'/N$ is pure in $M/N$.

(iv) If $N$ is pure in $M$ then $M'/N$ is pure in $M/N$ $\implies$ $M'$ is pure in $M$. 

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(v) If \( N \) is pure in \( M \) then the one - one correspondence between the submodules of \( M \) containing \( N \) and the submodules of \( M/M' \) pure submodules correspond to pure submodules.

**Proposition 2 :-**

An \( R \)-modules \( Q \) is pure \( M \)-injective if and only if any diagram of \( R \)-module and \( R \)-homomorphism of the form

\[
\begin{array}{c}
N \xrightarrow{t} M' \xrightarrow{f} M \\
\downarrow{v} \\
Q
\end{array}
\]

in which row is exact \( v \cdot t = 0 \) and \( M/\ker f \) is pure in \( M \) there exist \( h : M \longrightarrow Q \) such that \( h \cdot f = v \).

**Proof :-**

Let \( Q \) is pure \( M \)-injective module.
Since \( v \cdot t = 0 \) it follows that 
\[
\ker f = \text{Im} t \subseteq \ker v
\]

Let \( f' : M/\ker f \longrightarrow M \) and \( v' : M/\ker v \longrightarrow Q \) be the homomorphism induced by \( f \) and \( v \) respectively, it is clear that
$f'$ is monomorphism.

Since $\ker f \subseteq \ker v$.

We get an induced homomorphism $v'' : M / \ker f \longrightarrow Q$

\[
\begin{array}{ccc}
0 & \longrightarrow & M / \ker f \longrightarrow f' \longrightarrow M \\
\downarrow v'' & & \downarrow h \\
Q & & \end{array}
\]

in which row is exact. By pure $M$-injectivity,

we find $h : M \longrightarrow Q$ such that.

$v'' = h.f'$

The gives

$h.f'.\mu = v''.\mu$

where $\mu : M' \longrightarrow M' / \ker f$ is an epimorphism

Therefore

$v = h.f$ \quad \therefore \quad f'.\mu = f \quad \text{and} \quad v''.\mu = v$

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Conversely:

Taking $N = 0$, we find the result.

**Proposition 3 i-**

Let $M$ and $Q$ are left $R$-modules. Then the following statements are equivalent:

(i) $Q$ is pure $M$-injective module.

(ii) For any pure submodule $M' \leq M$ each $k \in \text{Hom}_R(M', Q)$ can be extended to homomorphism $\overline{f} \in \text{Hom}_R(M, Q)$

(iii) For each short exact sequence in $\text{mod}-R$ with the middle term $M$ and $M'$ is pure submodule of $M$,

\[
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0
\]

then the induced sequence

\[
0 \longrightarrow \text{Hom}_R(M'', Q) \longrightarrow \text{Hom}_R(M, Q) \longrightarrow \text{Hom}_R(M', Q) \longrightarrow 0
\]

is an exact.

**Proof :-**

(i) $\implies$ (ii): obviously
(ii) \( \implies \) (iii)

Now we show that given induced sequence

\[
0 \longrightarrow \text{Hom}_R(M'', Q) \longrightarrow g^* \longrightarrow \text{Hom}_R(M, Q) \longrightarrow f^* \longrightarrow \text{Hom}_R(M', Q) \longrightarrow 0
\]

is an exact.

We first note that

\[
f^* . g^* = g.f = 0
\]

clearly \( f^* \) is a surjective for if \( h \in \text{Hom}_R(M'', Q) \).

Then

\[
f^* . g^* (h) = 0
\]

as \( h.g.f = 0 \).

Finally suppose that \( p \in \text{Hom}_R(M, Q) \), we have

\[
f^*(p) = p.f = 0
\]

Thus

\[
\text{ker} p \supset \text{Im} f = \text{ker} g
\]

and we define
1 : M" \rightarrow Q \text{ by the operation }

l(a) = p(m)

where \( g(m_1) = m_2 \)

We observe that for every \( a \in M" \) and \( m_1, m_2 \in M \).

The above assumption comes from the fact that

\[ \ker p \supseteq \ker g \]

Hence the required sequence is exact.

(iii) \( \implies \) (i)

Let \( k \in \text{Hom}_R (M', Q) \), there exists

\[ \mu \in \text{Hom}_R (M, Q) \]

such that \( f^* \mu = \mu f = k \).

Then \( Q \) is pure \( M \)-injective.
Proposition 4-

Let Q be an R-module, if

\[ 0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0 \]

is exact sequence of R-modules with M' is pure sub module of M, and Q is pure M-injective module.
Then Q is pure M'-injective.

proof :-

Let Q be a pure M-injective module, let any sequence

\[ 0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0 \]

with M' is pure submodule of M. If \[ 0 \rightarrow K' \xrightarrow{h} M' \]
with K' is pure submodule of M'.

So \[ f \cdot h : K' \rightarrow M \]
is monomorphism and K' being pure in M, then the induced sequence

\[ 0 \rightarrow \text{Hom}_R(M'', Q) \xrightarrow{g^*} \text{Hom}_R(M, Q) \xrightarrow{f^*} \text{Hom}_R(M', Q) \rightarrow 0 \]
is an exact.
Thus

\[ h^* . f^* = (f. h)^* \] is an epimorphism.

Therefore \( h^* \) is an epimorphism, then \( Q \) is pure \( M' \)-injective.

**Theorem 1:**

The following conditions are equivalent, for a projective module \( M \):

(i) Every pure submodule of \( M \) is \( A \)-projective, where \( A \) is pure \( M \)-injective.

(ii) Every quotient of pure \( M \)-injective module is pure \( M \)-injective module.

(iii) Every quotient of pure injective module is pure \( M \)-injective module.

**Proof:**

\( (i) \Rightarrow (ii) \)

Let \( A \) be pure \( M \)-injective and consider the following diagram with row exact,
Let $N$ be any pure submodule of $M$.

Since $N$ is $A$-projective. There exists a homomorphism $h : N \rightarrow A$ such that $\eta h = f$.

By $M$-injectivity of $A$.

We obtain $l : M \rightarrow A$ such that $l g = h$.

Then $k = \eta l : M \rightarrow B$ gives,

$$k g = \eta l g = \eta h = f.$$

Hence $B$ is pure $M$-injective module.
(ii) \implies (iii)

It is clear.

(iii) \implies (i)

Let \( \text{O} \to \text{N} \to \text{M} \) be any exact sequence with \( \text{N} \) is a pure submodule of \( \text{M} \).

To show that \( \text{N} \) is \( \text{A} \)-projective. It is suffices to consider the following diagram:

\[
\begin{array}{ccc}
\text{O} & \to & \text{N} \\
\downarrow f & & \downarrow t \\
\text{A} & \to & \text{B} \\
\downarrow \kappa & & \\
& \to & \text{M}
\end{array}
\]

in which rows are exact with \( \text{A} \) is pure injective since \( \text{B} \) is pure \( \text{M} \)-injective.

\[\therefore \text{We obtain } g : \text{M} \to \text{B} \text{ such that } g \cdot t = f\]

since \( \text{M} \) is projective, there exists \( h : \text{M} \to \text{A} \) such that
\[ \pi \cdot h = g \]

Then

\[ k = h \cdot t : N \longrightarrow A \quad \text{gives} \]

\[ = \pi \cdot h \cdot t \]

\[ = g \cdot t \]

\[ = f \]

\[ \implies N \text{ is projective module.} \]

**Proposition 5:**

Let \( M \) be an \( R \)-module. Then following conditions are equivalent:

(i) Any non empty collection of pure submodule of \( M \) has a maximal element.

(ii) For any increasing sequence of pure submodules of \( M \),

\[ N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_n \subseteq \cdots \]

There exists some integer \( m \) such that \( N_k = N_m \) for all \( m \leq k \).

(iii) Every pure submodule of \( M \) is a finitely generated.
Proof 1-

(i) \implies (ii)

Let any increasing sequence of pure submodules of $M$.

say $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_n \subseteq \cdots$

be given consider the collection $\Sigma = \{N_n\}$.

The collection has a maximal element $N_n$. Then

$N_k = N_m$ for all $k \geq m$.

(ii) \implies (iii)

Let $\Lambda$ is non empty collection of pure submodules of $M$. To show that $\Lambda$ contain maximal element. Assume contrary $\Lambda$ is not containing maximal element. Choose $N_1 \in \Lambda$. Then since $N_1$ is not maximal element in $\Lambda$, we can find $N_2 \in \Lambda$ such that $N_2 \subseteq N_2$.

But $N_2$ is not maximal element in $\Lambda$, consequently there exists $N_3 \in \Lambda$ for which $N_2 \subseteq N_3$ and so on.

In this way a strictly increasing sequence

$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_n \subseteq \cdots$

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is generated contrary to the assumption (ii).

Hence any non empty collection of pure submodules of M has a maximal element.

(ii) $\implies$ (iii)

Let $N$ be a pure submodule of $M$, choose any $x_1 \in N$

If $R_{x_1} = N$, choose $x_2 \in N$, $x_2 \notin R_{x_1}$.

If $R_{x_1} + R_{x_2}$ is not equal to $N$. Continue this process.

By assumption (ii) after a finite number of steps we have

$$R_{x_1} + R_{x_2} + \ldots + R_{x_t} = N$$

Thus $N$ is finitely generated.

(iii) $\implies$ (ii)

Consider the increasing sequence of $M$, for each $N \subseteq M$. The increasing sequence

$$N_1 \subseteq N_2 \subseteq \ldots \subseteq N_m \subseteq \ldots$$
Denoted by $N$, the union of all $N_m$ and $N$ is also pure submodule of $M$, which is finitely generated by (iii) $N$ can by generated a finite number of elements.

Let $u_1, u_2, \ldots, u_p$ generate $n$ and, for each $i (1 \leq i \leq p)$, choose $n_i$ so that $u_i \in N_{n_i}$. If now $\mu = \max (n_1, n_2, \ldots, n_p)$, then all the $u_i \in N_{\mu}$ and therefore the module, which they generate, namely $N$, is contained in $N_{\mu}$.

Thus $N \subseteq N_{\mu} \subseteq N_m \subseteq N$, for every $m \geq \mu$

whence

$$N_{\mu} = N_m$$

This completes the proof.

Above proposition generalises the well known result

[ 80 page 32 ].

Definition 4 :-

An R - module $M$ is called pure Noetherian module if it satisfies any one of above equivalent conditions.
**Definition 5 :-**

An exact sequence

\[ A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} A_n \xrightarrow{\phi_n} A_{n+1} \]  

is called pure exact if the image of \( \phi_i \) is a pure sub module of \( A_{i+1} \) for each \( i \).

**Proposition 6 :-**

Let any sequence of \( R \)-module

\[ 0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0 \]

is exact with \( M' \) is pure sub module of \( M \).

Then \( M \) is pure Noetherian module, if and only if both \( M' \) and \( M'' \) are pure Noetherian modules.

**Proof :-**

\( \Rightarrow \) Let \( M \) is pure Noetherian module. Then show that \( M' \) and \( M'' \) are pure Noetherian module. Since \( M' \) is pure submodule of \( M \). The every pure submodule of \( M \) is pure Noetherian module.

Hence \( M' \) is pure Noetherian module.
Now every pure submodule $L$ of $M''$ is isomorphic to $M_1/M'$
($L \cong M_1/M'$) where $M_1$ is pure submodule of $M$. Since $M_1/M'$ is
pure submodule of $M/M'$, this implies that $M'$ is pure
submodule of $M$ [42]. Hence $M'$ is finitely generated.
Therefore $M''$ is pure Noetherian module.

Let $M'$ and $M''$ are pure Noetherian, we have that $M''$ is
isomorphic to $M/M'$ ($M'' \cong M/M'*$).

Let any pure submodule $N$ of $M$. We have

$$N/N \cap M' \cong N + M/M \text{ is pure submodule of } M''$$.

Since $M''$ is pure Noetherian, $N + M'/M'$ is finitely
generated and $M'$ is pure Noetherian, $N \cap M'$ is finitely
generated, this implies that $N$ is finitely generated.

Hence $M$ is pure Noetherian.

**Corollary 1**

Let $M$ be an $R$-module. If $M_1$ and $M_2$ are pure
submodules of $M$ such that $M/M_1$ and $M/M_2$ are pure Noetherian,
then $M/M_1 \cap M_2$ is pure Noetherian.
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Introduction:

In this paper we have shown that any $R$-module $U$ is $M$-projective if and only if given any exact sequence $M \rightarrow N \rightarrow L$ with $\nu : U \rightarrow N$ and $\beta \nu = 0$, there exist $\theta : U \rightarrow M$ such that $\alpha \theta = \nu$. If $M$ is finite generated, we proved that $U$ is $M$-projective iff $U$ is $Rm_i$ projective ($i=1...n$), where $(m_1, m_2...m_n)$ are generators of $M$. We also find that $\text{Hom}_R (X/A, Y/B)$ is isomorphic to $\triangle_{A,B}/\triangle_{X,B}$ where $\triangle_{A,B} \subset \text{Hom}_R (X, Y)$ is a submodule such that $f(A) \subseteq B$ for all $f \in \triangle_{A,B}$ and $X$ is $Y$-projective.

The $U(M, N)$ of those $f \in \text{Hom}_R (M, N)$ which can be factorised through $U$ i.e. $f=h\circ k$ where $k : M \rightarrow U$ and $h : U \rightarrow N$, then $U(M, N)$ will be subgroup of $\text{Hom}_R (M, N)$, if $U$ is $M$-projective.

Preliminaries: Through out this paper $R$-denots a Ring with unity and all the modules considered are left unity modules over $R$.

Definition 1: An $R$-module $U$ is called $M$-projective, if given a diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & O \\
\downarrow \phi & & \\
M & \xrightarrow{\phi} & N & \xrightarrow{} & O
\end{array}
\]

of $R$ homomorphism and of $R$-Modules with the horizontal sequence exact there exists a homomorphism $h : U \rightarrow M$ such that $\phi \circ h = f$. 
Example: Projective modules are examples of M-projective modules. Also we have Q is a Z-projective modules but not projective modules over Z.

Proposition. Any R-Module U will be X-projective if and only if given any diagram of R-modules and R-homomorphisms of the form:

\[ \begin{array}{ccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\
\downarrow{\alpha} & & \downarrow{\beta} & & \\
U & & v & & \\
\end{array} \]

in which the row is exact and \( \beta \circ v = 0 \), then there exists \( \xi : U \rightarrow X \) such that \( \alpha \circ \xi = v \).

Proof. Let U is X-projective. Since \( \beta \circ v = 0 \).

it follows that \( \text{Im} \, \beta = \text{Im} \, \alpha \).

Let \( \alpha' : X \rightarrow \text{Im} \) and \( v' : U \rightarrow \text{Im} \alpha \) be the R-homomorphism induced by \( \alpha \) and \( v \).

Then \( \alpha' \) is an epimorphism and we have the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{\alpha'} & \text{Im} \alpha & \xrightarrow{v'} & O \\
\downarrow{\alpha} & & \downarrow{v'} & & \\
U & & & & \\
\end{array} \]

in which the row is exact by X-projectivity of U, we find \( \xi : U \rightarrow X \) such that \( \alpha' \circ \beta = v' \) it is clear that

\[ i \circ \alpha' \circ \xi = i \circ v' \] where \( i : \text{Im} \alpha \rightarrow Y \) be the canonical induction map.

Therefore \( \alpha \circ \xi = v \).

Conversely: Putting \( z = 0 \) in the given diagram we find that condition are satisfied. Hence U is X-projective.
Proposition 2. If the diagram of $R$-modules and $R$-homomorphisms be such

\[
\begin{array}{ccccccc}
\cdots U_3 & \overset{g_3}{\longrightarrow} & U_2 & \overset{g_2}{\longrightarrow} & U_1 & \overset{g_1}{\longrightarrow} & A \\
\downarrow f_1 & & \downarrow & & \downarrow f_0 \\
M_3 & \overset{h_3}{\longrightarrow} & M_2 & \overset{h_2}{\longrightarrow} & M_1 & \overset{h_1}{\longrightarrow} & B \\
\end{array}
\]

both rows are exact and each $U_i$ is $M_i$-projective for all $i \in N$, then for every positive integer $n$. There is an $R$-homomorphism $f_n : U_n \rightarrow M_n$ such that $h_n f_n = f_{n-1} \circ g_n$.

**Proof.** Since $U_1$ is $M_1$ projective therefore there exists a homomorphism $f_1 : U_1 \rightarrow M_1$ such that

\[ h_1 \circ f_1 = f_0 \circ g_1 \]

Suppose proposition is true for $n-1$ i.e. there exist

\[ f_{n-1} : U_{n-1} \rightarrow M_{n-1} \text{ such that } h_{n-1} \circ f_{n-1} = f_{n-2} \circ g_{n-1} \tag{1} \]

Consider the diagram

\[
\begin{array}{ccccccc}
U_n & \overset{g_n}{\longrightarrow} & U_{n-1} & \overset{g_{n-1}}{\longrightarrow} & U_{n-2} \\
\downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2} \\
M_n & \overset{h_n}{\longrightarrow} & M_{n-1} & \overset{h_{n-1}}{\longrightarrow} & M_{n-2} \\
\end{array}
\]

in which each of the rows is exact we have

\[
h_{n-1} \circ f_{n-1} \circ g_n = (h_{n-1} \circ f_{n-1}) \circ g_n
\]

\[
= (f_{n-2} \circ g_{n-1}) \circ g_n
\]

\[
= f_{n-2} \circ g_{n-1} \circ g_n
\]

\[
= f_{n-2} \circ 0
\]

\[
= 0.
\]
Since $U_n$ being $M_n$-projective, by the proposition 1.

The existence of $R$-homomorphism.

\[ f_n : U_n \rightarrow M_n \text{ such that } \]
\[ h_n \circ f_n = f_{n-1} \circ g_n. \]

Thus by induction proposition is true for all $n$.

**Definition 2.** Let $T(M) = \{U/U \text{ is } M\text{-projective modules}\}$.

\[ T^{-1}(U) = \{M/U \text{ is } M\text{-projective modules}\}. \]

**Lemma 1.** $T(M)$ is closed under direct sums and direct summands [1].

$T^{-1}(U)$ is closed under submodules, homomorphic image and finite direct sums [1].

**Proposition 3.** Any diagram of $R$-modules and $R$-homomorphism of the form

\[
\begin{array}{c}
O \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow O \\
\downarrow \alpha \quad \quad \quad \quad \downarrow \beta \\
U' \quad \quad \quad \quad U''
\end{array}
\]

in which the row is exact and $U'$, $U''$ are $M$-projectives, can be extended to the following commutative diagram

\[
\begin{array}{c}
O \rightarrow U' \xrightarrow{i} U \xrightarrow{\pi} U'' \rightarrow O \\
\downarrow \alpha \quad \quad \quad \downarrow \gamma \quad \quad \downarrow \beta \\
O \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow O
\end{array}
\]

in which above row is exact and $U$ is $M$-projective module.

**Proof.** Since $U'$, $U''$ is $M$-projective therefore $U' \oplus U''$ is $M$-projective as $T(M)$ is closed under direct sum.

The sequence

\[
\begin{array}{c}
O \rightarrow U' \xrightarrow{i} U' \oplus U'' \xrightarrow{\pi} U'' \rightarrow O
\end{array}
\]

where $(i \cdot u') = (u', 0)$ and $(u' \oplus u'') = u''$, is then split exact.
Since $U''$ is $M$-projective, there exists $\beta$ such that $g \circ \beta = \beta$. Now we can extended the above diagram to the following

\[
\begin{array}{ccccccc}
\text{O} & \xrightarrow{\alpha} & U' & \xrightarrow{i} & U' \oplus U'' & \xrightarrow{\pi} & U'' & \xrightarrow{\beta} & \text{O} \\
\text{O} & \xrightarrow{f} & M' & \xrightarrow{j} & U' \oplus M & \xrightarrow{\gamma} & U'' & \xrightarrow{\beta} & \text{O} \\
\end{array}
\]

consider the mapping $\gamma : U' \oplus U'' \to M$ given by

$\gamma = f \circ \alpha + \beta \circ \pi$ where $j : U' \oplus U'' \to U'$ is a left hand splitting homomorphism. Certainly $\gamma$ will be $K$-homomorphism. It makes the diagram commutative as

$= (fo \circ \alpha j + \beta \circ \pi f) \circ i$  
$= fo \circ \alpha j i + \beta \circ \pi f o i$

$= fo \circ \alpha + 0$  
and  
$\pi \circ i = 0$

$= fo \alpha$  
and similarly we can prove $g \circ \alpha = \beta \circ \pi$.

write $U$ in place of $U' \oplus U''$ in the above diagram, we get the required result.

**Proposition 4.** Let $R$ be a commutative ring with unit element and let $X, Y$ be $R$-modules with $X$ is $Y$-projective. If $A, B$ are sub-modules of $X, Y$ respectively. Then

$\Delta_{A,B} = \{ f \in \text{Hom}_R (X, Y) \mid f(A) \subseteq B \}$

is submodule of the $R$-module $\text{Hom}_R (X, Y)$ and $\text{Hom}_R (X|A, Y/B) \cong \Delta_{A,B}|\Delta_{X,B}$.

**Proof.** It is clear that

$\Delta_{A,B} = \{ f \in \text{Hom}_R (X, Y) \mid f(A) \subseteq B \}$

is sub module of the $R$-module $\text{Hom}_R (X, Y)$, as

1. $\Delta_{A,B} \neq \phi$ since it contain the zero homomorphism
(126)

(ii) \( f, g \in \Delta_{A,B} \Rightarrow (f + g) (A) C B \Rightarrow f + g \in \Delta_{A,B} \)

(iii) for any \( f \in \Delta_{A,B} \) \( \lambda f \in \Delta_{A,B} \) for all \( \lambda \in R \)

Suppose now that \( f \in \Delta_{A,B} \)

then since \( f(A) \subseteq B \), there is a unique \( R \)-homomorphism,

\[ f_* : X/A \rightarrow Y/B \text{ such that the diagram} \]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
X/A & \xrightarrow{f_*} & Y/B \\
\end{array}
\]

is commutative. We can therefore define a map

\[ \xi : \Delta_{A,B} \rightarrow \text{Hom}_R (X/A, Y/B) \]

by \( \xi(f) = f_* \). It is readily verified that is an \( R \)-homomorphism. Due to uniqueness of \( f_* \). We have \( (f + g)_* = f_* + g_* \) and \( (\lambda f)_* = \lambda f_* \). To see that \( \xi \) is surjective, let \( t \in \text{Hom}_R (X/A, Y/B) \), and consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{\phi_1} & & \downarrow{\phi_2} \\
X/A & \xrightarrow{t} & Y/B \\
\downarrow & & \downarrow \\
Q & & Q \\
\end{array}
\]

Since \( X \) is \( Y \)-projective, there exist an \( R \)-homomorphism

\[ g : X \rightarrow Y \text{ such that } \phi_2 \circ g = t \circ \phi_1 \]

from this we get \( g(A) \subseteq B \) and \( \xi(g) = t \)
Now to find Kernal of $\xi$,

let $f \in \ker \xi$ iff $f_* = 0$ then $f_* (x + A) = B$

and hence $f_* \phi_1 (x) = B$ or $\phi_2 f(x) = B$ which implise

$f(x) \in B$ i. e. $f \in \Delta x \cdot B$.

Thus by the first isomorphism theorem

$$\text{Hom}_R (X/A, Y/B) = \text{Im} \xi \cong \Delta_A B / \ker \xi = \Delta A B / \Delta x B$$

Proposition 5. For Rmodules $M$ and $N$, let $U(M, N) \subset \text{Hom}_R (M, N)$ such that any $f \in U(M, N)$, factor through $U$, in the sense that there is a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{k} & & \downarrow{h} \\
U & \xrightarrow{i} & U
\end{array}
$$

Then let $U(M, N)$ denote the set of all $f: M \to N$ which factors through a $M$-projective module $U$. Then $U(M, N)$ is a sub-group of the group $
\text{Hom}_R (M, N)$.

Proof. If $U_1$ and $U_2$ are $M$-projective. Then $U_1 \oplus U_2$ also $M$-projective. We first show $U(M, N) \neq \phi$ as $\phi \in U(M, N)$. Now given $f, g \in U(M, N)$, we have to show that $f - g \in U(M, N)$.

Let $f$ factor through the $M$-projective $U_1$ and $g$ factor through the $M$-proje-

$$
\begin{array}{ccc}
U_1 & \xrightarrow{\pi_1} & U_1 \oplus U_2 \\
\downarrow{k_1} & & \downarrow{v} \\
U_1 & \xrightarrow{h_1} & M \\
\downarrow{f} & & \downarrow{g} \\
N & \xrightarrow{h_2} & U_2
\end{array}
$$

Consider the diagram (A). Here $\pi_1$ and $\pi_2$ are the projection mapping of $U_1 \oplus U_2$ on to $U_1, U_2$ respectively.
it following from the definition of co-product that there is a unique $R$-homomorphism.

$$\nu : M \rightarrow U_1 \oplus U_2$$

such that

$$\pi_1 \circ \nu = k_1 \text{ and } \pi_2 \circ \nu = k_2$$

then

$$h_1 \circ \pi_1 \circ \nu = h_1 \circ k_1 = f \text{ and } h_2 \circ \pi_2 \circ \nu = h_2 \circ k_2 = g$$

we see that

$$f - g = (h_1 \circ \pi_1 - h_2 \circ \pi_2) \circ \nu$$

i.e. $f - g$ factors through the $M$-projective module $U_1 \oplus U_2$.

Thus we have that $f - g \in \text{Hom}_R(M, N)$.

Theorem 1. Following condition for a finitely generated module $M$ are equivalent.

(i) $U$ is $M$-projective

(ii) $U$ is $R_m$-projective $\forall m \in M$

(iii) $U$ is $R_{m_i}$-projective $i = 1, 2, \ldots n$

where $m_1, m_2 \ldots m_n$ are generators of $M$

Proof. (i) $\Rightarrow$ (ii)

As $T^{-1}(U)$ is closed under submodules, we have $U$ is $R_m$-projective for all $m \in M$.

(ii) $\Rightarrow$ (iii) obvious

(iii) $\Rightarrow$ (i)
Since $T^{-1}(U)$ is closed under finite direct sum this implies that $U$ is $\bigoplus_{i=1}^{n} R_{m_i}$-projective. As $M$ is homomorphic image of $\bigoplus_{i=1}^{n} R_{m_i}$ and $T^{-1}(U)$ is closed under homomorphic image, we have $U$ is $M$-projective.

References