Chapter III

ON WEAKLY PAIRWISE CONTINUOUS MAPPINGS *

In a topological space the notion of weakly continuous mappings owes to Levine [45]. It is defined as follows:

**DEFINITION (3.A)**: A mapping \( f : (X, T) \rightarrow (Y, \gamma) \) is said to be weakly continuous if for each point \( x \in X \) and each neighbourhood \( M \) of \( x \), there exists a neighbourhood \( N \) of \( x \) such that \( f(N) \subseteq \text{cl} \, M \).

These mappings have been studied later by Deb [12], Noiri [69, 69 a], Singh [98] and others. It is well known [12 a] that every continuous mapping is weakly continuous but the converse need not be true. The present chapter investigates an analogous generalization of pairwise continuity and presents its study.

**WEAK PAIRWISE CONTINUITY**

We introduce the new concept as follows:

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**DEFINITION (3.1)**: A mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is termed weakly pairwise continuous if for each $x \in X$ and each $T_i$-open set $V$ containing $f(x)$, there exists a $P_j$-open set $U$ containing $x$ such that $f(U) \subseteq T_j$-cl $V$, where $i, j = 1, 2$ and $i \neq j$.

**THEOREM (3.1)**: Every pairwise continuous mapping is weakly pairwise continuous.

**PROOF**: Let $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ be pairwise continuous. Let $x \in X$ and $V$ be a $T_i$-open set containing $f(x)$. Since $f$ is pairwise continuous, $f^{-1}(V)$ is $P_i$-open. Put $f^{-1}(V) = U$. Then $U$ is a $P_i$-open set such that $f(U) = f(f^{-1}(V)) \subseteq V \subseteq T_j$-cl $V$. //

**REMARK (3.1)**: A weakly pairwise continuous mapping may fail to be pairwise continuous, as is shown in the following example:

**EXAMPLE (3.1)**: Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let $P_1 = \{\emptyset, \{a\}, \{x\}\}$, $P_2 = \{\emptyset, \{b, c\}, \{x\}\}$ and $T_1 = \{\emptyset, \{x\}, \{y\}\}$, $T_2 = \{\emptyset, \{y\}, \{y\}\}$. 
Then, \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) defined by \( f(a) = x \), \( f(b) = y \), \( f(c) = z \) is weakly pairwise continuous but it is not pairwise continuous. Note that the space \((Y, T_1, T_2)\) is not pairwise regular.

The following theorem investigates a condition when weak pairwise continuity implies pairwise continuity:

**Theorem (3.2):** If \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) is weakly pairwise continuous and \((Y, T_1, T_2)\) is pairwise regular then \( f \) is pairwise continuous.

**Proof:** Let \( V \) be any \( T_1 \)-open set and let \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \((Y, T_1, T_2)\) is pairwise regular there exists a \( T_1 \)-open set \( M \) such that \( f(x) \in M \subseteq T_j\text{-cl } M \subseteq V \). Since \( f \) is weakly pairwise continuous and \( M \) is a \( T_1 \)-open set containing \( f(x) \), there exists a \( P_1 \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq T_j\text{-cl } M \), so that \( f(U) \subseteq V \). That is, \( x \in U \subseteq f^{-1}(V) \). This shows that \( f^{-1}(V) \) is a \( P_1 \)-neighbourhood of \( x \). Consequently, \( f^{-1}(V) \) is \( P_1 \)-open. Hence, \( f \) is pairwise continuous. //

Theorems (3.1) and (3.2) yield the following:
**Theorem (3.3)**: Let \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) and let \((Y, T_1, T_2)\), be pairwise regular. Then the mapping \( f \) is weakly pairwise continuous iff \( f \) is pairwise continuous.

**Theorem (3.4)**: If \( f_1 : (X_1^*, P_1^*, P_2^*) \rightarrow (Y_1^*, T_1^*, T_2^*) \) be weakly pairwise continuous then the mapping \( f : (X_1 \times X_1^* \times \prod_1 \times \prod_2) \rightarrow (Y_1 \times Y_1^* \times \prod_1^* \times \prod_2^*) \) defined by \( f((x_1, x_2)) = (f_1(x_1), f_2(x_2)) \) for each \((x_1, x_2) \in X_1 \times X_1^* \), is weakly pairwise continuous.

**Proof**: Let \((x_1, x_2) \in X_1 \times X_1^* \) and let \( V \) be any \( \prod_1^* \)-open set containing \( f((x_1, x_2)) \). Now there exist a \( T_1^* \)-open set \( U_1 \) and a \( T_1^* \)-open set \( U_2 \) such that \( f((x_1, x_2)) \in U_1 \times U_2 \subseteq V \).

Since \( f_1(x_1) \in U_1 \) and \( f_1 \) is weakly pairwise continuous there exists a \( P_1^* \)-open set \( V_1 \) such that \( x_1 \in V_1 \) and \( f_1(V_1) \subseteq T_1^* \)-cl \( U_1 \).

Similarly, there is a \( P_1^* \)-open set \( V_2 \) such that \( x_2 \in V_2 \) and \( f_2(V_2) \subseteq T_2^* \)-cl \( U_2 \). Now, \( f(V_1 \times V_2) = f_1(V_1) \times f_2(V_2) \subseteq T_1^* \)-cl \( U_1 \times U_2 \subseteq \prod_1^* \)-cl \( V \). Since \( V_1 \times V_2 \) is \( \prod_1^* \)-open and contains \((x_1, x_2)\), it follows that \( f \) is weakly pairwise continuous. \( \blacksquare \)
THEOREM (3.5) : Let \( f : (X_1, P_1, P_2) \rightarrow (Y_1, T_1, T_2) \) and
let \( g : (X_1, P_1, P_2) \rightarrow (X \times Y, \prod_1, \prod_2) \) be given by \( g(x) = (x, f(x)) \). Then, \( f \) is weakly pairwise continuous iff \( g \) is weakly pairwise continuous.

PROOF : Suppose that \( f \) is weakly pairwise continuous. Let \( x \in X \) and \( W \) be any \( \prod_1 \)-open set in \( X \times Y \) containing \( g(x) \).
Then there exist a \( P_1 \)-open set \( R \) and a \( T_1 \)-open set \( V \) such that \( g(x) = (x, f(x)) \in R \times V \subseteq W \). Since \( f \) is weakly pairwise continuous there exists a \( P_1 \)-open set \( U \) containing \( x \) such that \( U \subseteq R \) and \( f(U) \subseteq T_j \)-cl \( V \). Therefore, \( g(U) \subseteq R \times T_j \)-cl \( V \subseteq P_j \)-cl \( R \times T_j \)-cl \( V = \prod_j \)-cl \( (R \times V) \subseteq \prod_j \)-cl \( W \). Hence \( g \) is weakly pairwise continuous.

Conversely, suppose that \( g \) is weakly pairwise continuous. Let \( x \in X \) and \( V \) be any \( T_1 \)-open set containing \( f(x) \). Then \( X \times V \) is \( \prod_1 \)-open in \( X \times Y \) and contains \( g(x) \).
Since \( g \) is weakly pairwise continuous, there exists a \( P_1 \)-open set \( U \) in \( X \) containing \( x \) such that \( g(U) \subseteq \prod_j \)-cl \( (X \times V) = X \times T_j \)-cl \( V \). Thus if \( a \in U \) then \( g(a) = (a, f(a)) \in X \times T_j \)-cl \( V \), so that \( f(a) \in T_j \)-cl \( V \), showing that, \( f(U) \subseteq T_j \)-cl \( V \). Hence, \( f \) is weakly pairwise continuous. //
**THEOREM (3.6):** Any restriction of a weakly pairwise continuous mapping is weakly pairwise continuous.

**PROOF:** Let \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) be weakly pairwise continuous and let \( A \) be a nonempty subset of \( X \). Let \( x \in A \) and \( V \) be any \( T_1 \)-open set containing \( f(x) \). Since \( f \) is weakly pairwise continuous there exists a \( P_1 \)-open set \( U \) such that \( x \in U \subseteq f^{-1}(T_1 \text{-cl } V) \). Therefore, \( x \in U \cap A \subseteq f^{-1}(T_1 \text{-cl } V) \cap A = (f/A)^{-1}(T_1 \text{-cl } V) \), so that \( (f/A)(U \cap A) \subseteq T_1 \text{-cl } V \). Since \( U \cap A \) is \( P_1 \)-open in \( A \), it follows that \( f/A \) is weakly pairwise continuous. \( \Box \)

**THEOREM (3.7):** Let \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) be weakly pairwise continuous and let \( A \) be a nonempty subset of \( X \). Then, \( g \equiv f/A : A \rightarrow f(A) \) is weakly pairwise continuous.

**PROOF:** Let \( x \in A \) and \( U \) be any \( T_{1f(A)} \)-open set in \( f(A) \) containing \( g(x) \). Then there exists a \( T_1 \)-open set \( G \) in \( Y \) such that \( U = G \cap f(A) \). Since \( f \) is weakly pairwise continuous there exists a \( P_1 \)-open set \( H \) in \( X \) such that \( x \in H \subseteq f^{-1}(T_1 \text{-cl } G) \). Thus, \( x \in H \cap A \subseteq f^{-1}(T_1 \text{-cl } G) \cap A = g^{-1}(T_1 f(A) \cap U) \). Since \( H \cap A \) is \( P_1 \)-open in \( A \), it follows that \( g \) is weakly pairwise continuous. \( \Box \)
REMARK (3.2) : The composition of two weakly pairwise continuous mappings may fail to be weakly pairwise continuous as is asserted by the following example:

EXAMPLE (3.2) : Let $X = \{a, b, c\} = Z$, and $Y = \{a, b, c, d\}$. Let $P_1 = \{\emptyset, X\}$, $P_2 = \{\emptyset, \{a, c\} \cup X\}$,

$T_1 = \{\emptyset, \{a, c\} \cup Y\}$, $T_2 = \{\emptyset, \{c, d\} \cup Y\}$,

and $Q_1 = \{\emptyset, \{a\} \cup Z\}$, $Q_2 = \{\emptyset, \{c\} \cup Z\}$.

Then, the mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ defined by $f(a) = d$, $f(b) = b$, $f(c) = a$ and the mapping $g : (Y, T_1, T_2) \rightarrow (Z, Q_1, Q_2)$ defined by $g(a) = a$, $g(b) = g(c) = b$, $g(d) = c$ are weakly pairwise continuous but their composite mapping $g \circ f$ is not weakly pairwise continuous. However, we have the following theorem:

THEOREM (3.3) : If $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is weakly pairwise continuous and $g : (Y, T_1, T_2) \rightarrow (Z, Q_1, Q_2)$ is pairwise continuous then $g \circ f$ is weakly pairwise continuous.

PROOF : Let $x \in X$ and $V$ be any $Q_1$-open set in $Z$ such that $g(f(x)) \in V$. Then, $f(x) \in g^{-1}(V)$ and $g$ being pairwise
continuous $g^{-1}(V)$ is $T_1$-open. Now, if being weakly pairwise continuous there exists a $P_1$-open set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq T_j$-$cl$ $g^{-1}(V)$. But, $T_j$-$cl$ $g^{-1}(V) \subseteq g^{-1}(Q_j$-$cl$ $V)$ for $g$ is pairwise continuous. Therefore, $x \in U \subseteq f^{-1}(g^{-1}(Q_j$-$cl$ $V))$, that is, $(g \circ f)(U) \subseteq Q_j$-$cl$ $V$. Hence, $g \circ f$ is weakly pairwise continuous. //

**THEOREM (3.9)**: If $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is pairwise continuous and $g^1 : (Y, T_1, T_2) \rightarrow (Z, Q_1, Q_2)$ is weakly pairwise continuous then $g \circ f$ is weakly pairwise continuous.

**PROOF**: Let $x \in X$ and let $V$ be a $Q_1$-open set such that $g(f(x)) \in V$. Since $g$ is weakly pairwise continuous there exists a $T_1$-open set $G$ such that $f(x) \in G$ and $g(G) \subseteq Q_j$-$cl$ $V$. This implies that $G \subseteq g^{-1}(Q_j$-$cl$ $V)$. Now by pairwise continuity of $f$, there exists a $P_1$-open set $U$ containing $x$ such that $f(U) \subseteq G$. That is, $U \subseteq f^{-1}(G)$. Thus, $U \subseteq f^{-1}(G) \subseteq f^{-1}g^{-1}(Q_j$-$cl$ $V) = (g \circ f)^{-1}(Q_j$-$cl$ $V)$. This shows that $(g \circ f)(U) \subseteq Q_j$-$cl$ $V$. Hence, $g \circ f$ is weakly pairwise continuous. //
Let us introduce the terms utilised later:

**DEFINITION (3.2)**: A subset $A$ of a space $(X, P_1, P_2)$ is termed a $P_1(\alpha)$-set if $A \subseteq P_1$-int $P_j$-cl $P_1$-int $A$, $i, j = 1, 2$, such that $i \neq j$.

**DEFINITION (3.3)**: A subset $A$ of a space $(X, P_1, P_2)$ is termed a $P_1(\cos)$-set if $X - A$ is a $P_1(\alpha)$-set, $i = 1, 2$. Equivalently, $A$ is a $P_1(\cos)$-set iff $P_1$-cl $P_j$-int $P_1$-cl $A$ $\subseteq A$, $i, j = 1, 2$, such that $i \neq j$.

The following theorem explores several characterizations of the concept of weak pairwise continuity:

**THEOREM (3.10)**: If $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$, then the following statements are equivalent:

(a) $f$ is weakly pairwise continuous.

(b) $f^{-1}(V) \subseteq P_1$-int $f^{-1}(T_j$-cl $V)$, for each $T_1$-open set $V$.

(c) $P_j$-cl $f^{-1}(V) \subseteq f^{-1}(T_j$-cl $V)$, for each $T_1$-open set $V$.

(d) $f^{-1}(T_j$-int $B) \subseteq P_j$-int $f^{-1}(B)$, for each $T_1$-closed set $B$. 
(e) For each point \( p \in X \) and each net
\[
\{ p_n \colon n \in D, 2 \} \text{ converging to } p,
\]
the image net \( f(p_n) \) is eventually in every \( T_j \)-closed \( T_1 \)-neighbourhood of \( f(p) \),
\( i, j = 1, 2, i \neq j \).

(f) \( f^{-1}(A) \subseteq P_1 \text{-int } f^{-1}(T_j \text{-cl } A) \) for every
\( T_1(\alpha) \)-set \( A \).

(g) \( P_j \text{-cl } f^{-1}(A) \subseteq f^{-1}(T_j \text{-cl } A) \) for every
\( T_1(\alpha) \)-set \( A \).

(h) \( f^{-1}(T_j \text{-int } M) \subseteq P_j \text{-int } f^{-1}(M) \) for every
\( T_1(\text{co}x) \)-set \( M \).

where \( i, j = 1, 2, \) such that \( i \neq j \).

**Proof:**

(e) \( \Rightarrow \) (b): Let \( x \in f^{-1}(V) \). Then \( f(x) \in V \).

Since \( V \) is \( T_1 \)-open, by (a) there exists a \( P_1 \)-open set \( U \)
containing \( x \) such that \( f(U) \subseteq T_j \text{-cl } V \). It follows that
\( x \in U \subseteq P_1 \text{-int } f^{-1}(T_j \text{-cl } V) \). Thus, (b) holds.

(b) \( \Rightarrow \) (e): Let \( x \in X \) and \( V \) be a \( T_1 \)-open set containing
\( f(x) \). Then, \( x \in f^{-1}(V) \subseteq P_1 \text{-int } f^{-1}(T_j \text{-cl } V) = 0 \), say. Then,
0 is a $P_j$-open set containing $x$ such that $f(0) = f(P_j$-int $f^{-1}(T_j-cl V)) \subseteq f^{-1}(T_j-cl V) \subseteq T_j-cl V$. Thus, (a) holds.

(b) $\implies$ (c) : Let $x \in P_j-cl f^{-1}(V)$ and $V$ be $T_1$-open. Let $M$ be a $T_1$-open set containing $f(x)$. Then, $x \in f^{-1}(M) \subseteq P_j$-int $f^{-1}(T_j-cl M) \subseteq f^{-1}(T_j-cl M)$. Therefore, $f^{-1}(T_j-cl M)$ is a $P_j$-neighbourhood of $x$ and it follows that $f^{-1}(T_j-cl M) \cap f^{-1}(V) \neq \emptyset$. That is, $(T_j-cl M) \cap V \neq \emptyset$. Since $V$ is $T_1$-open, $(T_j-cl M) \cap V \subseteq T_j-cl (M \cap V)$, so that $M \cap V \neq \emptyset$. Therefore, $f(x) \in T_j-cl V$, and so $x \in f^{-1}(T_j-cl V)$. Thus, (c) holds.

(c) $\implies$ (d) : If $B$ is $T_1$-closed then $Y - B$ is $T_1$-open. So that by (c) we have, $P_j-cl f^{-1}(Y - B) \subseteq f^{-1}(T_j-cl (Y - B))$. By taking complements we get, $f^{-1}(T_j-int B) \subseteq P_j$-int $f^{-1}(B)$.

(d) $\implies$ (b) : Let $V$ be $T_1$-open. Then, $T_j-cl V$ is $T_j$-closed. Now by (d), $f^{-1}(T_j-int T_j-cl V) \subseteq P_j$-int $f^{-1}(T_j-cl V)$. Since $V$ is $T_1$-open it follows that $V \subseteq T_j-int T_j-cl V$. Hence, $f^{-1}(V) \subseteq f^{-1}(T_j-int T_j-cl V) \subseteq P_j$-int $f^{-1}(T_j-cl V)$. Thus, (b) holds.
(d) \implies (e): Let \( p \in X \) and let \( \{ p_n, n \in D, \omega \} \) be a net converging to \( p \). Let \( B \) be any \( T_j \)-closed \( T_j \)-neighbourhood of \( f(p) \). There exists a \( T_j \)-open set \( G \) such that \( f(p) \in G \subseteq T_j \)-cl \( G \subseteq B \). By the hypothesis, \( f^{-1}(T_j \text{-int } T_j \text{-cl } G) \subseteq P_j \text{-int } f^{-1}(T_j \text{-cl } G) \). Since \( G \) is \( T_j \)-open therefore \( G \subseteq T_j \text{-int } T_j \text{-cl } G \). Thus, \( p \in f^{-1}(G) \subseteq P_j \text{-int } f^{-1}(T_j \text{-cl } G) \subseteq f^{-1}(T_j \text{-cl } G) \). That is, \( f(p) \in T_j \text{-cl } G \subseteq B \). This shows that the net \( \{ f(p_n), n \in D, \omega \} \) is eventually in \( B \).

(e) \implies (a): Let \( p \in X \) and let \( G \) be any \( T_j \)-neighbourhood of \( f(p) \). If for no \( P_j \)-neighbourhood \( N \) of \( p \), \( f(N) \subseteq T_j \text{-cl } G \) then \( N \cap (X \setminus f^{-1}(T_j \text{-cl } G)) \neq \emptyset \), for every \( P_j \)-neighbourhood \( N \) of \( p \). For each \( P_j \)-neighbourhood \( N \) of \( p \), let \( p_N \in N \cap (X \setminus f^{-1}(T_j \text{-cl } G)) \). Then, \( \{ p_N, N \in \mathcal{U}_j(p), \subseteq \} \), where \( \mathcal{U}_j(p) \) is the \( P_j \)-neighbourhood system of the point \( p \), is a net in \( X \) which converges to \( p \). But, the image net \( \{ f(p_N), N \in \mathcal{U}_j(p), \subseteq \} \) is not eventually in the \( P_j \)-closed \( P_j \)-neighbourhood \( T_j \text{-cl } G \) of \( f(p) \). This is a contradiction. Hence, there exists a \( P_j \)-open neighbourhood \( N \) of \( p \) such that \( f(N) \subseteq T_j \text{-cl } G \). Hence, \( f \) is weakly pairwise continuous.
(a) $\implies$ (f) : Let $A$ be a $T_1(a)$-set. Let $x \in f^{-1}(A)$. Then, $f(x) \in A \subseteq T_1\text{-}int T_j\text{-}cl T_1\text{-}int A \subseteq T_1\text{-}int T_j\text{-}cl A$. Therefore, there exists a $T_1$-open set $G$ such that $f(x) \in A \subseteq G \subseteq T_j\text{-}cl G \subseteq T_j\text{-}cl A$. But $f$ is weakly pairwise continuous at $x$, therefore there exists a $P_1$-open set $V$ containing $x$ such that $f(V) \subseteq T_j\text{-}cl G$. This shows that $x \in V \subseteq f^{-1}(f(V)) \subseteq f^{-1}(T_j\text{-}cl G) \subseteq f^{-1}(T_j\text{-}cl A)$. That is, $f^{-1}(T_j\text{-}cl A)$ is a $P_1$-neighbourhood of $x$. Hence, $x \in P_1\text{-}int f^{-1}(T_j\text{-}cl A)$. Consequently, $f^{-1}(A) \subseteq P_1\text{-}int f^{-1}(T_j\text{-}cl A)$.

(f) $\implies$ (g) : Let $x \in P_j\text{-}cl f^{-1}(A)$, where $A$ is a $T_1(a)$-set. Let $G$ be a $T_j$-open neighbourhood of $f(x)$. Then $G$ is a $T_1(a)$-set. So $x \in f^{-1}(G) \subseteq P_j\text{-}int f^{-1}(T_j\text{-}cl G) \subseteq f^{-1}(T_j\text{-}cl G)$. Thus, $f^{-1}(T_j\text{-}cl G)$ is a $P_j$-neighbourhood of $x$. Then there exists a $P_j$-open set $V$ in $X$ such that $x \in V \subseteq f^{-1}(T_j\text{-}cl G)$. Since, $x \in P_j\text{-}cl f^{-1}(A)$ therefore $V \cap f^{-1}(A) \neq \emptyset$. This implies that, $f(V) \cap A \neq \emptyset$. And so, $T_j\text{-}cl G \cap A \neq \emptyset$. Since, $A \subseteq T_1\text{-}int T_j\text{-}cl T_1\text{-}int A$, therefore, $T_j\text{-}cl G \cap T_1\text{-}int T_j\text{-}cl T_1\text{-}int A \neq \emptyset$. Therefore, $G \cap T_1\text{-}int T_j\text{-}cl T_1\text{-}int A \neq \emptyset$. And so, $G \cap T_j\text{-}cl T_1\text{-}int A \neq \emptyset$. Since $G$ is $T_j$-open, this yields $G \cap T_1\text{-}int A \neq \emptyset$. That is, $G \cap A \neq \emptyset$. 

Hence \( f(x) \in T_j \text{-} cl \ A \). And so, \( x \in f^{-1}(T_j \text{-} cl \ A) \).

\[(g) \implies (h) : \] Let \( M \) be a \( T_4(\text{coa}) \)-set, then \( Y - M \) is a \( T_4(\alpha) \)-set. Therefore by hypothesis we have, \( P_j \text{-} cl f^{-1}(Y - M) \subseteq f^{-1}(T_j \text{-} cl (Y - M)) \). Taking complements we get, \( f^{-1}(T_j \text{-} int M) \subseteq P_j \text{-} int f^{-1}(M) \).

\[(h) \implies (b) : \] Let \( V \) be \( T_4 \) -open. Then \( T_j \text{-} cl \ V \) is \( T_j \) -closed.

Now, \( T_j \text{-} cl \ V \) is a \( T_j(\text{coa}) \)-set for every \( T_j \) -closed set is a \( T_j(\text{coa}) \)-set. Therefore by hypothesis, \( f^{-1}(T_j \text{-} int T_j \text{-} cl V) \subseteq P_j \text{-} int f^{-1}(T_j \text{-} cl V) \). Since \( V \) is \( T_4 \)-open, therefore \( f^{-1}(V) \subseteq f^{-1}(T_j \text{-} int T_j \text{-} cl V) \subseteq P_j \text{-} int f^{-1}(T_j \text{-} cl V) \). //

The notion of pairwise Urysohn is due to Singal and Arya [94]. It is introduced as follows:

**DEFINITION (3.3)**: A space \( (X, P_1, P_2) \) is said to be pairwise Urysohn if for every pair of points \( x, y \in X \), \( x \neq y \), there exists a \( P_1 \)-open set \( U \) and a \( P_2 \)-open set \( V \) containing \( y \) such that \( P_2 \text{-} cl U \cap P_1 \text{-} cl V = \emptyset \).

**THEOREM (3.11)**: If \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) is a weakly pairwise continuous injection and \( (Y, T_1, T_2) \) is pairwise Urysohn, then \( (X, P_1, P_2) \) is pairwise Hausdorff.
PROOF : Let $x, y \in X$ and $x \neq y$. Since $f$ is injective therefore, $f(x) \neq f(y)$. Since $Y$ is pairwise Urysohn there exists a $T_1$-open set $U$ containing $f(x)$ and a $T_2$-open set $V$ containing $f(y)$ such that $T_2$-cl $U \cap T_1$-cl $V = \emptyset$. So that, $f^{-1}(T_2$-cl $U) \cap f^{-1}(T_1$-cl $V) = \emptyset$. This implies that $P_1$-int $f^{-1}(T_2$-cl $U) \cap P_2$-int $f^{-1}(T_1$-cl $V) = \emptyset$. Since $f$ is weakly pairwise continuous, by Theorem (3.10 b), $f(x) \in U$ implies that $x \in f^{-1}(U) \subseteq P_1$-int $f^{-1}(T_2$-cl $U)$. Similarly, $y \in f^{-1}(V) \subseteq P_2$-int $f^{-1}(T_1$-cl $V)$. Since $P_1$-int $f^{-1}(T_2$-cl $U)$ is $P_1$-open and $P_2$-int $f^{-1}(T_1$-cl $V)$ is $P_2$-open, it follows that, $(X, P_1, P_2)$ is pairwise Hausdorff. \\

THEOREM (3.12): If $f : (X, P_1, P_2) \to (Y, T_1, T_2)$ is a weakly pairwise continuous surjection and $(X, P_1, P_2)$ is pairwise connected then $(Y, T_1, T_2)$ is pairwise connected.

PROOF : Suppose that $Y$ is not pairwise connected. Then there exists a nonempty proper subset $A$ of $Y$ which is both $T_1$-open and $T_2$-closed. Since $A$ is $T_1$-open, by Theorem (3.10c), $P_2$-cl $f^{-1}(A) \subseteq f^{-1}(T_2$-cl $A) = f^{-1}(A)$ since $A$ is $T_2$-closed. Therefore, $f^{-1}(A)$ is $P_2$-closed. Again since $A$ is $T_2$-closed,
by Theorem (3.10 d), $f^{-1}(T_1 \text{-int } A) \subseteq P_1 \text{-int } f^{-1}(A)$. This reduces to $f^{-1}(A) \subseteq P_1 \text{-int } f^{-1}(A)$, since $A$ is $P_1$-open.

Therefore, $f^{-1}(A)$ is $P_1$-open. Since $f$ is surjective, $f^{-1}(A)$ is nonempty. Thus, $f^{-1}(A)$ is a nonempty proper subset of $X$ which is both $P_1$-open and $P_2$-closed. This leads to a contradiction that $(X, P_1, P_2)$ is not pairwise connected.

Consequently, $(Y, T_1, T_2)$ is pairwise connected. //

**Remark (3.3)**: Theorem (3.12) may fail if $f$ is not surjective, as the following example asserts:

**Example (3.3)**: Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let

$$P_1 = \emptyset, \{a\}, X \} \quad P_2 = \emptyset, \{b\}, X \} ,$$

and

$$T_1 = \emptyset, \{x, z\}, Y \} \quad T_2 = \emptyset, \{y, z\}, Y \} .$$

Then the mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ defined by $f(a) = x$, $f(b) = f(c) = y$ is weakly pairwise continuous, but it is not surjective. Note that though the space $(X, P_1, P_2)$ is pairwise connected but $f(X) = \{x, y\}$ is not a pairwise connected subspace of the space $(Y, T_1, T_2)$.

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