CHAPTER I

ON ALMOST PAIRWISE CONTINUOUS MAPPINGS

The notion of continuous mappings is fundamental in topology. In fact, the study of a topological space centres round the study of continuous mappings defined on it. It is defined as follows [36, 66, 73, 92]:

**DEFINITION (1.1) :** A mapping \( f : (X, \tau') \to (Y, \tau) \) is said to be continuous if the inverse image under \( f \) of each open subset of \( Y \) is open in \( X \).

Due to its importance several characterisations of this concept have been obtained and it has been generalised from time to time by several workers in topology. One such generalisation owes to Singal and Singal [95], also studied later by Noiri [69] and Long and Herrington [48]. It is given below:

**DEFINITION (1.2) :** A mapping \( f : (X, \tau') \to (Y, \tau) \) is said to be almost continuous at a point \( x \in X \), if for every open set \( U \) containing \( f(x) \) there is an open set \( V \) containing \( x \) such that \( f(V) \subseteq \text{int cl } U \), where 'int' and 'cl' stand for the 'interior' and the 'closure' operators respectively.
The study of a topological space itself has been extended to a wider setting when Kelly [37] in 1963 conceived the concept of a bitopological space. It is interesting to note that every topological space can be considered as a bitopological space whereas every bitopological study can be thought of as a generalised topological study. Consequently, the topological studies in the new setting have been resumed by several mathematicians in topology from time to time. While extending the concept of connectedness in topological spaces to bitopological spaces, Pervin [76] was led to extend the concept of continuous mappings in topological spaces to bitopological spaces. It was introduced as follows:

**DEFINITION (1, G)**: A mapping \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \)
is said to be pairwise continuous if the mappings \( f : (X, P_1) \rightarrow (Y, T_1) \) and \( f : (X, P_2) \rightarrow (Y, T_2) \) are continuous.

Few generalisations, cited in the introductory chapter of pairwise continuity have been introduced and studied in the recent past. However, its generalisation analogous to almost continuity due to Singal and Singal
has not yet appeared in the literature. The aim of this chapter is to investigate a generalisation of pairwise continuity in this direction and to present its study.

**ALMOST PAIRWISE CONTINUITY**

The new concept is introduced as follows:

**DEFINITION (1.1)**: A mapping \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) is said to be almost pairwise continuous at a point \( x \in X \) if for every \( T_1 \)-open set \( M \) containing \( f(x) \) there is a \( P_i \)-open set \( N \) containing \( x \) such that \( f(N) \subseteq T_1 \text{-int } T_j \text{-cl } M \), where \( i, j = 1, 2, i \neq j \). It is termed almost pairwise continuous if it is almost pairwise continuous at each \( x \in X \).

We have the following theorem:

**THEOREM (1.1)**: Every pairwise continuous mapping \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) is almost pairwise continuous.

**PROOF**: Let \( x \in X \) and \( M \) be any \( T_1 \)-open set containing \( f(x) \). Then \( f^{-1}(M) \subseteq f^{-1}(T_1 \text{-int } T_j \text{-cl } M) \). Put \( f^{-1}(M) = N \). Since \( f \) is pairwise continuous therefore \( N \) is \( P_i \)-open and contains \( x \).
It is clear that \( f(\mathbb{N}) \subseteq T_1 \text{-int} \ T_j \text{-cl} M \). Hence, \( f \) is almost pairwise continuous. //

**REMARK (1.1)**: An almost pairwise continuous mapping may fail to be pairwise continuous. For,

**EXAMPLE (1.1)**: Let \( X = \{ x, y, z \} \), \( Y = \{ a, b \} \),

\[
P_1 = \{ \emptyset, \{ x, y \}, x \}, \quad P_2 = \{ \emptyset, \{ y \}, x \},
\]

\[
T_1 = \{ \emptyset, \{ a \}, x \} \quad \text{and} \quad T_2 = \{ \emptyset, \{ y \} \}.
\]

Then the mapping \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \)
defined by \( f(x) = a, \ f(y) = f(z) = b \), is almost pairwise continuous but it is not pairwise continuous.

**REMARK (1.2)**: Theorem (1.1) and Example (1.1) shows that the concept of almost pairwise continuity generalises the notion of pairwise continuity. However, the two concepts become equivalent if the codomain space is pairwise semiregular. The concept of pairwise semiregular spaces owes to Singal [93]. It is defined as follows:
DEFINITION (1.D) : A space \((X, P_1, P_2)\) is pairwise semiregular if for every \(P_1\)-open set \(U\) containing \(x \in X\), there exists a \(P_1\)-open set \(V\) such that \(x \in V \subseteq P_1\)-int \(P_j\)-cl \(V \subseteq U\), where \(i, j = 1, 2\) such that \(i \neq j\).

We have the following theorem:

THEOREM (1.2) : If \(f\) is an almost pairwise continuous mapping of a space \((X, P_1, P_2)\) into a pairwise semiregular space \((Y, T_1, T_2)\) then \(f\) is pairwise continuous.

PROOF : Let \(x \in X\) and let \(A\) be a \(T_1\)-open set containing \(f(x)\). Since \(Y\) is pairwise semiregular, there is a \(T_1\)-open set \(M\) in \(Y\) such that \(f(x) \in M \subseteq T_1\)-int \(T_j\)-cl \(M \subseteq A\). Now since \(f\) is almost pairwise continuous, there is a \(P_1\)-open set \(U\) in \(X\) containing \(x\) such that \(f(x) \in f(U) \subseteq T_1\)-int \(T_j\)-cl \(M\). Thus \(U\) is a \(P_1\)-open set containing \(x\) such that \(f(U) \subseteq A\). Hence, \(f\) is pairwise continuous. //

The term \((i, j)\)-regular open is due to Singel and Arya [94]. It is defined as follows:

DEFINITION (1.E) : A subset \(A\) of a space \((X, P_1, P_2)\) is termed \((i, j)\)-regular open if \(A = P_1\)-int \(P_j\)-cl \(A\), \(i, j = 1, 2, i \neq j\).
**Remark (1.1):** Every \((i,j)\)-regular open set in \((X, P_1, P_2)\) is \(P_i\)-open but not conversely. The complement of an \((i,j)\)-regular open set is called \((i,j)\)-regular closed [94].

Let us introduce the following:

**Definition (1.2):** A not \(e : (D, \leq) \rightarrow (X, P_1, P_2)\) is said to be convergent at \(x \in X\) if it is eventually in every \(P_i\)-open (resp. \(P_2\)-open) set containing \(x\).

We utilize these notions to obtain several characterizations of almost pairwise continuous mappings as follows:

**Theorem (1.3):** For a mapping \(f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)\) the following statements are equivalent where \(i,j = 1,2\) and \(i \neq j\):

(a) \(f\) is almost pairwise continuous.

(b) Inverse image of every \((i,j)\)-regularly open subset of \(Y\) is a \(P_i\)-open subset of \(X\).

(c) Inverse image of every \((i,j)\)-regularly closed subset of \(Y\) is a \(P_i\)-closed subset of \(X\).
(d) For each point \( x \in X \) and each \((i,j)\)-regularly open set \( M \) containing \( f(x) \) there is a \( P_1 \)-open set \( N \) containing \( x \) such that \( f(N) \subseteq M \).

(e) For every \( T_1 \)-open subset \( A \) of \( Y \),
\[
f^{-1}(A) \subseteq P_1 \text{-int } \left[ f^{-1}(T_1 \text{-int } T_j \text{-cl } A) \right]
\]

(f) For every \( T_1 \)-closed subset \( B \) of \( Y \),
\[
P_1 \text{-cl } \left[ f^{-1}(T_1 \text{-cl } T_j \text{-int } B) \right] \subseteq f^{-1}(B).
\]

(g) For any point \( x \in X \) and for any net \( \{ s_\lambda, \lambda \in D, \geq \} \)
which converges to \( x \), the net \( \{ f(s_\lambda), \lambda \in D, \geq \} \)
is eventually in each \((i,j)\)-regularly open set containing \( f(x) \).

**PROOF:** (a) \(\Rightarrow\) (b): Let \( U \) be any \((i,j)\)-regularly open subset of \( Y \) and let \( x \in f^{-1}(U) \). Then \( f(x) \in U \). Since \( U \) is \( T_1 \)-open there exists a \( P_1 \)-open set \( V \) containing \( x \) such that \( f(V) \subseteq T_1 \text{-int } T_j \text{-cl } U = U \). Thus, \( x \in V \subseteq f^{-1}(U) \) and so \( f^{-1}(U) \) is \( P_1 \)-open.

(b) \(\Rightarrow\) (c): Let \( A \) be any \((i,j)\)-regularly closed subset of \( Y \), then \( Y - A \) is \((i,j)\)-regularly open. And so, \( f^{-1}(Y - A) = X - f^{-1}(A) \) is \( P_1 \)-open. Hence, \( f^{-1}(A) \) is \( P_1 \)-closed.
(c) \implies (d): Since \( M \) is \((i,j)\)-regularly open and contains \( f(x) \), therefore \( Y \sim M \) is \((i,j)\)-regularly closed and so, \( f^{-1}(Y \sim M) = X \sim f^{-1}(M) \) is \( P_1 \)-closed. That is \( f^{-1}(M) \) is \( P_1 \)-open. Let \( f^{-1}(M) = N \). Then \( N \) is a \( P_1 \)-open set containing \( x \) such that \( f(N) \subseteq M \).

(d) \implies (e): Let \( x \in f^{-1}(A) \). Then \( T_j \)-int \( T_j \)-cl \( A \) is an \((i,j)\)-regularly open set containing \( f(x) \), since \( A \) is \( T_1 \)-open. Then there exists a \( P_1 \)-open set \( N \) containing \( x \) such that \( f(N) \subseteq T_1 \)-int \( T_j \)-cl \( A \). Thus, \( x \in N \subseteq f^{-1}(T_1 \)-int \( T_j \)-cl \( A \)). This means that \( x \in P_1 \)-int \[ f^{-1}(T_1 \)-int \( T_j \)-cl \( A \)).

(e) \implies (f): Since \( Y \sim B \) is \( T_1 \)-open therefore, \( f^{-1}(Y \sim B) \subseteq P_1 \)-int \[ f^{-1}(T_1 \)-int \( T_j \)-cl \( (Y \sim B) \)). This implies that, \( P_1 \)-cl \[ f^{-1}(T_1 \)-cl \( T_j \)-int \( B)) \subseteq f^{-1}(B) \).

(f) \implies (g): Let \( N \) be any \((i,j)\)-regularly open set containing \( f(x) \). Then, \( Y \sim N \) being \( T_1 \)-closed, \( P_1 \)-cl \( f^{-1}(T_1 \)-cl \( T_j \)-int \( (Y \sim N) \)) \subseteq f^{-1}(Y \sim N) \). Since \( Y \sim N \) is \((i,j)\)-regularly closed, therefore, \( P_1 \)-cl \( f^{-1}(Y \sim N) \) \subseteq X \sim f^{-1}(N) \). This implies that \( f^{-1}(N) \) is a \( P_1 \)-open set containing \( x \). Since the net \( \{e_\lambda\}_{\lambda \in D} \) converges to \( x \) there exists \( \lambda_0 \in D \) such that for
all \ \lambda \geq \lambda_0 \ , \ \exists \lambda \in f^{-1}(N). \ This \ means \ that \ f(\varepsilon_\lambda) \in N \\
for \ \lambda \geq \lambda_0. \ That \ is \ the \ net \ \{f(\varepsilon_\lambda)\}_\lambda \in D \ is \ eventually \ in \ N.

(g) \Rightarrow \ (a): \ Suppose \ that \ f \ is \ not \ almost \ pairwise \ continuous \ at \ x. \ Then \ there \ is \ a \ T^*_1-open \ set \ V \ containing 
\ f(x) \ such \ that \ for \ every \ P^*_1-open \ set \ U \ containing \ x, f(U) \cap (Y-T^*_1-inter \ T^*_j-cl \ V) \neq \emptyset. \ This \ implies \ that \ U \cap f^{-1}(Y-T^*_1-inter 
T^*_j-cl \ V) \neq \emptyset, \ for \ every \ P^*_1-open \ set \ containing \ x. \ The \ family 
\mathcal{U} \ of \ all \ P^*_1-open \ sets \ U \ containing \ x \ is \ directed \ by \ \subseteq.
For \ each \ U \in \mathcal{U}, \ choose \ a \ point \ x_u \in U \cap f^{-1}(Y-T^*_1-inter \ T^*_j-cl V). 
Then \ \{x_u\}_u \in \mathcal{U} \ is \ a \ net \ in \ X \ which \ converges \ to \ x \ and \ such 
that \ no \ f(x_u) \ belongs \ to \ T^*_1-inter \ T^*_j-cl \ V. \ Thus, \ the \ net 
\{f(x_u)\}_u \in \mathcal{U} \ is \ not \ eventually \ in \ the \ (1,j)-regularly \ open 
set \ T^*_1-inter \ T^*_j-cl \ V. //

Remark (1.3): Composition of two almost pairwise continuous 
mapping may not be almost pairwise continuous. For,

Example (1.2): Let \(X = Y = \mathbb{Z} = \{a, b, c\}\). Let
\[P_1 = \{\emptyset, \{a\}, \{a, b, c\}, \mathbb{Z}\}, \quad P_2 = \{\emptyset, \{b, c\}, \mathbb{Z}\}, \]
\[T_1 = \{\emptyset, \{a\}, \mathbb{Z}\}, \quad T_2 = \{\emptyset, \{b\}, \{b, c\}, \mathbb{Z}\}, \]
and \[Q_1 = \{\emptyset, \{a\}, 2\}, \quad Q_2 = \{\emptyset, \{b\}, 2\}.\]
Let $f: (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ and $g: (Y, T_1, T_2) \rightarrow (Z, Q_1, Q_2)$ be the identity mappings. Then $f$ and $g$ are almost pairwise continuous mappings but $g \circ f$ fails to be almost pairwise continuous.

However, we have the following theorem:

**Theorem 1.4**: If $f: (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is pairwise continuous and $g: (Y, T_1, T_2) \rightarrow (Z, Q_1, Q_2)$ is almost pairwise continuous, then $g \circ f$ is almost pairwise continuous.

**Proof**: Let $x \in X$ and $M$ be a $Q_1$-open set containing $g(f(x))$. Since $g$ is almost pairwise continuous there exists a $T_1$-open set $N$ containing $f(x)$ such that $g(N) \subseteq Q_1 \text{-int} Q_j \text{-cl} M$. Since $f$ is pairwise continuous there is a $P_1$-open set $U$ containing $x$ such that $f(U) \subseteq N$. It follows that $g(f(U)) \subseteq Q_1 \text{-int} Q_j \text{-cl} M$. Hence, $g \circ f$ is almost pairwise continuous. //

The concept of pairwise open mappings owes to Reilly [82] and Popa [78]. It is independent of the concept of pairwise continuity and is defined as follows:

**Definition 1.5**: A mapping $f: (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is termed pairwise open, if the mappings $f: (X, P_1) \rightarrow (Y, T_1)$ and $f: (X, P_2) \rightarrow (Y, T_2)$ are open.
We have,

**Theorem (1.9):** Let \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) be surjective, pairwise open and pairwise continuous and let \( g : (Y, T_1, T_2) \to (Z, Q_1, Q_2) \). Then, \( g \circ f \) is almost pairwise continuous iff \( g \) is almost pairwise continuous.

**Proof:** Necessity: Let \( g \circ f \) be almost pairwise continuous. If \( A \) be any \((1,1)\)-regular open set in \( Z \), then \((g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \) is \( P_1 \)-open by Theorem (1.3 b). Therefore, \( f(f^{-1}(g^{-1}(A))) \) is \( T_1 \)-open for \( f \) is pairwise open and equals \( g^{-1}(A) \) since \( f \) is surjective. Hence, \( g \) is almost pairwise continuous.

The sufficiency follows from Theorem (1.4). //

**Theorem (1.6):** Any restriction of an almost pairwise continuous mapping is almost pairwise continuous.

**Proof:** Let \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) be almost pairwise continuous and let \( A \) be any nonempty subset of \( X \). For any \((1,1)\)-regularly open subset \( S \) of \( Y \), \((f/A)^{-1}(S) = A \cap f^{-1}(S) \) which is \( P_1 \)-relatively open set in \( A \), for in view of Theorem (1.3 b), \( f^{-1}(S) \) is \( P_1 \)-open. Hence, \( f/A \) is almost pairwise continuous. //
THEOREM (1.7): Let \( f: (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) and \( x \in X \). If there exists a bi-open set \( N \) containing \( x \) such that \( f/N \) is almost pairwise continuous at \( x \), then \( f \) is almost pairwise continuous at \( x \).

PROOF: Let \( U \) be any \((i,j)\)-regular open set in \( Y \) such that \( f(x) \in U \). Since \( f/N \) is almost pairwise continuous at \( x \), in view of Theorem (1.3 d) there is a \( P_1 \)-open set \( V \) such that \( x \in N \cap V \) and \( f(N \cap V) \subseteq U \). Evidently, \( N \cap V \) is \( P_1 \)-open since \( N \) is biopen. Hence the result follows. //

As a direct consequence of Theorem (1.7), we obtain,

COROLLARY (1.1): Let \( f: (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) and let \( \{ G_\lambda \mid \lambda \in \Lambda \} \) be a biopen cover of \( X \) such that for each \( \lambda \in \Lambda \), \( f/G_\lambda \) is almost pairwise continuous. Then, \( f \) is almost pairwise continuous.

THEOREM (1.8): Let \( f: (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) and let \( X = X_1 \cup X_2 \), where \( X_1 \) and \( X_2 \) are bi-closed in \( X \). If \( f/X_1 \) and \( f/X_2 \) are almost pairwise continuous, then so is \( f \).

PROOF: Let \( A \) be any \((i,j)\)-regular closed set in \( Y \).

Since \( f/X_1 \) and \( f/X_2 \) are almost pairwise continuous, therefore by Theorem (1.3 c), \( (f/X_1)^{-1}(A) \) and \( (f/X_2)^{-1}(A) \) are \( P_1 \)-relatively closed in \( X_1 \) and \( P_1 \)-relatively closed in \( X_2 \).
respectively. Since \( X_1 \) and \( X_2 \) are \( P_1 \)-closed by hypothesis, 
\( (f/X_1)^{-1}(A) \) and \( (f/X_2)^{-1}(A) \) are both \( P_1 \)-closed. As, \( f^{-1}(A) = (f/X_1)^{-1}(A) \cup (f/X_2)^{-1}(A) \), it follows that \( f^{-1}(A) \) is \( P_1 \)-closed. Consequently, by Theorem (1.3 c), \( f \) is almost pairwise continuous. //

**Theorem (1.9)**: Let \( f : (X, P_1 \times P_2) \rightarrow (Y, T_1 \times T_2) \) and let \( X = X_1 \cup X_2 \). If \( f/X_1 \) and \( f/X_2 \) are both almost pairwise continuous at a point \( x \in X_1 \cap X_2 \), then \( f \) is almost pairwise continuous at \( x \).

**Proof**: Let \( U \) be any \((i,j)\)-regular open set in \( Y \) containing \( f(x) \). Since \( x \in X_1 \cap X_2 \) and \( f/X_1 \), \( f/X_2 \) are both almost pairwise continuous at \( x \) therefore in view of Theorem (1.3 d) there exist \( P_1 \)-open sets \( V_1 \) and \( V_2 \) such that \( x \in X_1 \cap V_1 \), 
\( f(X_1 \cap V_1) \subseteq U \) and \( x \in X_2 \cap V_2 \), \( f(X_2 \cap V_2) \subseteq U \). Since \( X = X_1 \cup X_2 \), therefore \( f(V_1 \cap V_2) = f(X_1 \cap V_1 \cap V_2) \cup f(X_2 \cap V_1 \cap V_2) \subseteq U \). Since \( V_1 \cap V_2 \) is a \( P_1 \)-open set containing \( x \), it follows by Theorem (1.3 d) that \( f \) is almost pairwise continuous at \( x \). //

Consider the following example:

**Example (1.3)**: Let \( X = \{ a, b, c \} \) and \( Y = \{ x, y, z \} \). Let
\[
P_1 = \{ \emptyset, \{ a, b \}, X \}, \quad P_2 = \{ \emptyset, X \},
\]
and
\[
T_1 = \{ \emptyset, \{ z \}, \{ x, z \}, Y \}, \quad T_2 = \{ \emptyset, \{ y, z \}, Y \}.
\]
Define \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) by \( f(a) = x, f(b) = f(c) = y \). Then \( f \) is almost pairwise continuous but \( f : (X, P_1, P_2) \rightarrow (f(x), T_1, f(x), T_2, f(x)) \) is not so, where \( T_1f(x) \) (resp. \( T_2f(x) \)) stands for the topology induced on \( f(x) \) by the topology \( T_1 \) (resp. \( T_2 \)) on \( Y \). The following lemma leads to a condition when such restrictions are almost pairwise continuous.

**Lemma (1.1)**: Let \( Y \) be a \( P_1 \)-open subspace of the space \((X, P_1, P_2)\). If \( U \subseteq Y \) is an \((i, j)\)-regular open subset of \( Y \), then \( U = Y \cap P_1\text{-int } P_j\text{-cl } U, i, j = 1, 2, i \neq j \).

**Proof**: Since \( U \) is \( P_1\text{-Y} \) - open in \( Y \) and \( Y \) is \( P_1 \)-open in \( X \), \( U \) is \( P_1 \)-open where \( P_1\text{-Y} \) is the topology induced on \( Y \) with respect to the topology \( P_1 \). Noting that \( P_j\text{-cl } U = Y \cap P_j\text{-cl } U \) and that \( U \) is \((i, j)\)-regular open in \( Y \), we have, \( U = P_1\text{-int } P_j\text{-Y} - cl U = U(\alpha \cap Y) \), where \( \alpha \) is \( P_1 \)-open in \( X \) and \( \alpha \cap Y \subseteq Y \cap P_j\text{-cl } U \). Now, \( P_1\text{-int } P_j\text{-cl } U \) is \( P_1 \)-open in \( X \) and is a subset of \( P_j\text{-cl } U \), hence, \( Y \cap P_1\text{-int } P_j\text{-cl } U \subseteq U \cap (\alpha \cap Y) \). Further, \( U \) is \( P_1 \)-open in \( X \), so that \( U \subseteq P_1\text{-int } P_j\text{-cl } U \), giving that \( U = U \cap Y \subseteq (P_1\text{-int } P_j\text{-cl } U) \cap Y \). Therefore, \( U \subseteq Y \cap P_1\text{-int } P_j\text{-cl } U \subseteq U \). //
**Theorem (1.10)**: Let \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) be almost pairwise continuous and let \( A \subseteq X \) be such that \( f(A) \) is bi-open in \( Y \). Then \( f/A : (A, P_{1A}, P_{2A}) \to (f(A), T_{1f(A)}, T_{2f(A)}) \) is almost pairwise continuous.

**Proof**: Let \( U \) be \((i,j)\)-regular open in \( f(A) \). Since \( f(A) \) is \( T_i \)-open, by Lemma (1.1), \( U = f(A) \cap T_i \text{-int} T_j \text{-cl } U \).

Thus,

\[
(f/A)^{-1}(U) = (f/A)^{-1}(f(A) \cap T_i \text{-int} T_j \text{-cl } U) = (f/A)^{-1}(f(A)) \cap (f/A)^{-1}(T_i \text{-int} T_j \text{-cl } U) = A \cap f^{-1}(T_i \text{-int} T_j \text{-cl } U).
\]

Since \( T_i \text{-int} T_j \text{-cl } U \) is \((i,j)\)-regular open in \( Y \) and \( f \) is almost pairwise continuous, it follows by Theorem (1.3 b) that \( f^{-1}(T_i \text{-int} T_j \text{-cl } U) \) is \( P_1 \)-open. This implies that \( (f/A)^{-1}(U) \) is \( P_{1A} \)-open in \( A \). This leads to the required result in view of Theorem (1.3 b). //

The product space of the spaces \((X_\alpha, P_\alpha, P_\alpha^*)\), \(\alpha \in \Lambda\) is the space \((\prod X_\alpha, \Pi_1, \Pi_2)\) where \(\Pi_1\) and \(\Pi_2\) are the product topologies of the topologies \(P_\alpha\) and \(P_\alpha^*\) resp. [78].
We have,

**Theorem (1.11)**: For each \( \alpha \in \Lambda \), let \( f_\alpha : (x_\alpha, P_1(\alpha)) \rightarrow (y_\alpha, T_1(\alpha), T_2(\alpha)) \) be given and let \( f : (\prod x_\alpha, \prod_1, \prod_2) \rightarrow (\prod y_\alpha, \prod_1^*, \prod_2^*) \) be defined by \( f((x_\alpha)) = (f_\alpha(x_\alpha)) \). Then \( f_\alpha \) is almost pairwise continuous, for each \( \alpha \in \Lambda \) iff \( f \) is almost pairwise continuous.

**Proof**: Necessity. Let \( (x_\alpha) \in \prod x_\alpha \) and let \( O \) be an \((1,1)\)-regular open set in \( \prod y_\alpha \) containing \( f((x_\alpha)) \). Since \( O \) is \( \prod_1^* \)-open, there is a member \( \prod O_\alpha \) of the defining base of the product topology on \( \prod y_\alpha \) such that \( f((x_\alpha)) \in \prod O_\alpha \subseteq O \) and \( O_\alpha = y_\alpha \) for all \( \alpha \in \Lambda \) except for a finite number of indices \( \alpha_k \), \( k = 1, 2, \ldots, n \), say, and \( O_\alpha \) is \( T_1(\alpha_k) \)-open set in \( y_\alpha \). Now since \( O \) is \((1,1)\)-regular open in \( \prod y_\alpha \), therefore \( \prod_1^* \)-int \( \prod_1^* \)-cl \( \prod O_\alpha \subseteq O \). Thus, for each \( \alpha_k \), \( f_\alpha(x_\alpha) \in O_\alpha \subseteq T_1(\alpha_k) \)-int \( T_1(\alpha_k) \)-cl \( O_\alpha \), and \( f_\alpha \) being almost pairwise continuous by Theorem (1.3 d), there is a \( P_1(\alpha_k) \)-open subset \( U_\alpha \) of \( x_\alpha \) such that \( x_\alpha \in U_\alpha \) and \( f_\alpha(x_\alpha) \in f_\alpha(U_\alpha) \subseteq T_1(\alpha_k) \)-int \( T_1(\alpha_k) \)-cl \( O_\alpha \).
Thus, $\prod U_\alpha$ where $U_\alpha = X_\alpha$ when $\alpha \neq \alpha_k$, $k = 1, 2, \ldots, n$, is a $\prod_1$-open set containing $(x_\alpha)$ such that $f(\prod U_\alpha) \subseteq 0$. Hence, by Theorem (1.3 d), $f$ is almost pairwise continuous.

**Sufficiency:** Let $\beta \in \Lambda$ be fixed and let $x_\beta \in X_\beta$. Let $V_\beta$ be an $(i,j)$-regular open set in $Y_\beta$ containing $f_\beta(x_\beta)$. Now consider any point $(x_\alpha) \in \prod X_\alpha$ whose $\beta$th coordinate is $x_\beta$.

Since $V_\beta$ is $T_1(\beta)$-open, the set $V = \prod Y_\alpha \times V_\beta$ is a $\prod_1$-open set in $\prod Y_\alpha$ containing $(f_\alpha(x_\alpha))$, and since $\prod_1\text{-cl} V = \prod_{\alpha \neq \beta} Y_\alpha \times T_1(\beta)\text{-cl} V_\beta \times \prod_1\text{-int} \prod_j\text{-cl} V = \prod_{\alpha \neq \beta} T_1(\alpha)\text{-int} Y_\alpha \times T_1(\beta)\text{-int} T_j(\beta)\text{-cl} V_\beta = \prod_{\alpha \neq \beta} Y_\alpha \times V_\beta = V$, which shows that $V$ is $(i,j)$-regular open in $\prod Y_\alpha$. Since $f$ is almost pairwise continuous by Theorem (1.3 d), there exists a $\prod_1$-open set $U$ which contains $(x_\alpha)$ such that $f(U) \subseteq V = \prod Y_\alpha \times V_\beta$. Now there exists a basic open set $G = \prod X_\alpha \times G_{\alpha_1} \times G_{\alpha_2} \times \ldots \times G_\beta \times \ldots \times G_{\alpha_n} \subseteq U$ containing $(x_\alpha)$ such that $f(G) \subseteq V$, which implies that $f_\beta(G_\beta) \subseteq V_\beta$. This shows by Theorem (1.3 d) that $f_\beta$ is almost pairwise continuous. Hence, $f_\alpha$ is almost pairwise continuous for each $\alpha \in \Lambda$. //
THEOREM (1.12) Let \( \{ Y_\alpha : T_1(\alpha) : T_2(\alpha) \mid \alpha \in \Lambda \} \) be a family of spaces and let \( f : (X, P_1, P_2) \rightarrow (\prod Y_\alpha, \prod P_1, \prod P_2) \). Then, 
\( f \) is almost pairwise continuous iff \( P_\alpha \circ f \) is almost pairwise continuous for each \( \alpha \in \Lambda \).

PROOF: Necessity: Suppose that \( f \) is almost pairwise continuous. Let \( x \in X \) and let \( V_\beta \) be an \((i,j)\)-regular open set in \( Y_\beta \) containing \( (P_\beta \circ f) (x) = P_\beta (f(x)) = P_\beta ((y_\alpha)) \). Then,
\[
\prod_{\alpha=\beta} \supseteq \prod_{\alpha=\beta} T_1(\alpha) = \prod_{\alpha=\beta} T_1(\alpha) \supseteq \prod_{\alpha=\beta} Y_\alpha x
\]
\[
T_1(\beta) \supseteq \prod_{\alpha=\beta} Y_\alpha x T_1(\beta) \supseteq \prod_{\alpha=\beta} Y_\alpha x V_\beta = \prod_{\alpha=\beta} Y_\alpha x V_\beta
\]
\[
P_\beta^{-1} (V_\beta) \text{ is } (i,j) \text{-regular open in } \prod Y_\alpha \text{ and contains } f(x) \text{.}
\]
Now since \( f \) is almost pairwise continuous, by Theorem (1.3 d), there exists a \( P_1 \)-open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq P_\beta^{-1} (V_\beta) \). Consequently,
\[
(P_\beta \circ f) (U) = P_\beta (f(U)) \subseteq P_\beta (P_\beta^{-1} (V_\beta)) = V_\beta \text{, showing that}
\]
\( P_\beta \circ f \) is almost pairwise continuous. Hence, \( P_\alpha \circ f \) is almost pairwise continuous for each \( \alpha \in \Lambda \).

Sufficiency: Suppose that \( P_\alpha \circ f \) is almost pairwise continuous for each \( \alpha \in \Lambda \). Let \( x \in X \) and let \( V \) be an \((i,j)\)-regular open set in \( \prod Y_\alpha \) containing \( f(x) \). Since \( V \) is \( P_1 \)-open, there exists
a basic open set \( \prod_{a \neq a_k} Y_a \times V_{a_1} \times V_{a_2} \times \ldots \times V_{a_n} \subseteq V \)
and contains \( f(x) \), where \( k = 1, 2, \ldots, n \). Thus, \( f(x) \in \prod_{a \neq a_k} Y_a \times \text{int} T_{j(a_1)} \times \ldots \times \text{int} T_{j(a_n)} \times T_{j(a_n)} = \text{cl} V = V \). Since each of the sets \( T_{j(a_k)} \times \text{cl} V_{a_k} \), \( k = 1, 2, \ldots, n \), \( P_{a_k} \circ f \) is almost pairwise continuous by Theorem (1.3 d).

There exists a \( P_1 \)-open set \( U_k \) in \( X \) which contains \( x \) and such that \( (P_{a_k} \circ f)(U_k) \subseteq \text{int} T_{j(a_k)} \times \text{cl} V_{a_k} \).

Let \( H = \bigcap_{k=1}^n U_k \). Then, \( f(h) \subseteq \prod_{a \neq a_k} Y_a \times \text{int} T_{j(a_1)} \times \ldots \times \text{int} T_{j(a_n)} \times \text{cl} V_{a_n} \subseteq V \).

This shows by Theorem (1.3 d) that \( f \) is almost pairwise continuous since \( H \) is \( P_1 \)-open and contains \( x \). //

**COROLLARY (1.2):** Let \( \{(Y_\alpha, T_1(\alpha), T_2(\alpha)) \mid \alpha \in \Lambda \} \) be a family of spaces and let \( f_\alpha : (X, P_1, P_2) \rightarrow (Y_\alpha, T_1(\alpha), T_2(\alpha)) \), \( \alpha \in \Lambda \), be given. Define \( f : (X, P_1, P_2) \rightarrow (\prod Y_\alpha, \prod T_1, \prod T_2) \) by \( f(x) = (f_\alpha(x)) \). Then, \( f \) is almost pairwise continuous for each \( \alpha \in \Lambda \).
PROOF: For each $\alpha \in \Lambda$ we may write $f\alpha = P\alpha \circ f$. Now the result follows from Theorem (1.12).

The following theorems reveal some other properties of almost pairwise continuous mappings:

**Theorem (1.13):** Let $f: (X, P_1, P_2) \to (Y, T_1, T_2)$ be almost pairwise continuous and let $V \subseteq Y$ be $T_1$-open. If $x \notin f^{-1}(V)$ but $x \in P_j\text{-cl} f^{-1}(V)$ then $f(x) \in T_j\text{-cl} V$.

**Proof:** Let $x \in X$ be such that $x \notin f^{-1}(V)$ but $x \in P_j\text{-cl} f^{-1}(V)$. Suppose $f(x) \notin T_j\text{-cl} V$. Then there exists a $T_j$-open set $W$ such that $f(x) \in W$ and $W \cap V = \emptyset$. Thus, $T_1\text{-cl} W \cap V = \emptyset$ and $T_j\text{-int} T_1\text{-cl} W \cap V = \emptyset$. Since $f$ is almost pairwise continuous there exists a $P_j$-open set $U$ such that $x \in U$ and $f(U) \subseteq T_j\text{-int} T_1\text{-cl} W$. As a consequence, $f(U) \cap V = \emptyset$. However, since $x \in P_j\text{-cl} f^{-1}(V)$ we have, $U \cap f^{-1}(V) \neq \emptyset$. So that, $f(U) \cap V \neq \emptyset$. Thus, we have a contradiction. Consequently, $f(x) \in T_j\text{-cl} V$.

**Theorem (1.14):** Let $f: (X, P_1, P_2) \to (Y, T_1, T_2)$ be almost pairwise continuous. Then, for each $T_1$-open set $V \subseteq Y$, $P_j\text{-cl} f^{-1}(V) \subseteq f^{-1}(T_j\text{-cl} V)$.
PROOF: Let $V$ be $T_1$-open. Now by Theorem (1.13), we have, $f(P_j\text{-cl } f^{-1}(V)) \subseteq T_j\text{-cl } V$. And so, $P_j\text{-cl } f^{-1}(V) \subseteq f^{-1}(f(P_j\text{-cl } f^{-1}(V))) \subseteq f^{-1}(T_j\text{-cl } V)$. //

**Theorem (1.15):** Let $f : (X, P_1, P_2) \to (Y, T_1, T_2)$ be pairwise open and almost pairwise continuous. Then for each $T_1$-open subset $V \subseteq Y$, $P_j\text{-cl } f^{-1}(V) = f^{-1}(T_j\text{-cl } V)$.

**Proof:** Let $x \in f^{-1}(T_j\text{-cl } V)$. Then $f(x) \in T_j\text{-cl } V$. Let $U$ be any $P_j$-open set containing $x$. Since $f$ is pairwise open, $f(U)$ is $T_j$-open and contains $f(x)$. Therefore, $f(U) \cap V \neq \emptyset$. This implies that $U \cap f^{-1}(V) \neq \emptyset$. This yields that $x \in P_j\text{-cl } f^{-1}(V)$. Thus $f^{-1}(T_j\text{-cl } V) \subseteq P_j\text{-cl } f^{-1}(V)$.

The result follows now in view of Theorem (1.14). //

Kelly [37] introduced the concept of pairwise Hausdorff as follows:

**Definition (1.8):** A space $(X, P_1, P_2)$ is termed pairwise Hausdorff if for $x, y \in X$ and $x \neq y$, there exist a $P_1$-open set $U$ containing $x$ and a $P_2$-open set $V$ containing $y$ such that $U \cap V = \emptyset$.

**Theorem (1.16):** Let $f : (X, P_1, P_2) \to (Y, T_1, T_2)$ be an injective and almost pairwise continuous. If $Y$ is pairwise Hausdorff then so is $X$. 
**Proof:** Let \( a, b \in X \) such that \( a \neq b \). Since \( f \) is injective, \( f(a) \neq f(b) \), and so there exists say, a \( T_1 \)-open set \( U \) containing \( f(a) \) and a \( T_2 \)-open set \( V \) containing \( f(b) \) such that \( U \cap V = \emptyset \). This implies that \( f(a) \in T_1\text{-int } T_2\text{-cl } U = G \) say, and \( f(b) \in T_2\text{-int } T_1\text{-cl } V = H \) say, and that \( G \cap H = \emptyset \). Since \( G \) is \((1,2)\)-regular open and \( H \) is \((2,1)\)-regular open in \( Y \) and \( f \) is almost pairwise continuous it follows by Theorem (1.3b) that \( f^{-1}(G) \) is \( P_1 \)-open and contains \( a \), while \( f^{-1}(H) \) is \( P_2 \)-open and contains \( b \). Also, \( f^{-1}(G) \cap f^{-1}(H) = \emptyset \). Hence, \( X \) is pairwise Hausdorff. //

As a particular case of Theorem (1.16) we have,

**Corollary (1.3):** Let \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) be injective and pairwise continuous. If \( Y \) is pairwise Hausdorff then so is \( X \).

**Proof:** By Theorem (1.1), every pairwise continuous mapping is almost pairwise continuous and so Theorem (1.16) is applicable. //

The concept of pairwise connectedness is due to Pervin [76]. It is introduced as follows:

**Definition (1.4):** A space \((X, P_1, P_2)\) is termed pairwise connected if \( X \) can not be expressed as the union of two
nonempty disjoint sets $A$ and $B$ such that $A$ is $P_1$-open and $B$ is $P_2$-open (hence, $A$ is $P_2$-closed and $B$ is $P_1$-closed).

Pervin [76] has proved that if a mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is a pairwise continuous surjection and $X$ is pairwise connected then $f(X)$ is pairwise connected. We generalize this result as follows:

**Theorem (1.17)** Let $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ be surjective and almost pairwise continuous. If $X$ is pairwise connected then $f(X)$ is pairwise connected.

**Proof:** Assume that $(Y, T_1, T_2)$ is not pairwise connected. Therefore, $f(X) = Y$ can be expressed as the union of two nonempty disjoint sets $A$ and $B$ such that $A$ is say $T_1$-open and $B$ is $T_2$-open (and hence $A$ is $T_2$-closed and $B$ is $T_1$-closed).

And so, $A$ is $(1,2)$-regular open and $B$ is $(2,1)$-regular open in $Y$. Since $f$ is almost pairwise continuous, by Theorem (1.3 b), $f^{-1}(A)$ is $P_1$-open and $f^{-1}(B)$ is $P_2$-open in $X$. Since $A$ is nonempty and $f$ is surjective, $f^{-1}(A)$ is nonempty. Similarly $f^{-1}(B)$ is nonempty. Also, $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Moreover, $f^{-1}(A) \cup f^{-1}(B) = X$, for $Y = A \cup B$ and $f(X) = Y$. Consequently, $(X, P_1, P_2)$ is not pairwise connected, which is a contradiction. Hence, $(Y, T_1, T_2)$ is pairwise connected. //
REMARK (1.4): Theorem (1.17) fails if \( f \) is not surjective.

For, in Example (1.3), we observe that \((X, P_1, P_2)\) is pairwise connected and \( f \) is almost pairwise continuous but \( f(X) = \{x, y\} \) is not pairwise connected.

**THEOREM (1.18):** Let \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) and \( g : (X, P_1, P_2) \to (X \times Y, \bar{T}_1, \bar{T}_2) \) be given by \( g(x) = (x, f(x)) \). Then \( f \) is almost pairwise continuous iff \( g \) is almost pairwise continuous.

**PROOF:** Necessity: Let \( x \in X \) and let \( W \subseteq X \times Y \) be a \( \bar{T}_1 \)-open set containing the point \( g(x) = (x, f(x)) \). Then, there exists a \( P_1 \)-open set \( R \) and a \( T_1 \)-open set \( V \) such that \( x \in R, f(x) \in V \) such that \( R \times V \subseteq W \). Since \( f \) is almost pairwise continuous there is a \( P_1 \)-open set \( U \) such that \( U \subseteq R \) and \( x \in U \) such that \( f(U) \subseteq T_1 \)-int \( T_j \)-cl \( V \). Now observe that, \( U \times T_1 \)-int \( T_j \)-cl \( V \subseteq P_1 \)-cl \( U \times T_j \)-cl \( V = \bar{T}_j \)-cl \( (U \times V) \). So that \( \bar{T}_1 \)-int \( (U \times T_1 \)-int \( T_j \)-cl \( V) \subseteq \bar{T}_1 \)-int \( \bar{T}_j \)-cl \( (U \times V) \). However, \( \bar{T}_1 \)-int \( (U \times T_1 \)-int \( T_j \)-cl \( V) = P_1 \)-int \( U \times T_1 \)-int \( T_j \)-cl \( V \) = \( U \times T_1 \)-int \( T_j \)-cl \( V \). Therefore, \( U \times T_1 \)-int \( T_j \)-cl \( V \subseteq \bar{T}_1 \)-int \( \bar{T}_j \)-cl \( (U \times V) \). In view of the
fact that \( f(U) \subseteq T_4 \text{-int } T_j \text{-cl } V \), we see that \( g(U) \subseteq U \times T_4 \text{-int } T_j \text{-cl } V \subseteq \prod_4 \text{-int } \prod_j \text{-cl } (U \times V) \subseteq \prod_4 \text{-int } \prod_j \text{-cl } W \).

This shows that \( g \) is almost pairwise continuous.

**Sufficiency:** Let \( x \in X \) and let \( V \subseteq Y \) be a \( T_4 \)-open set containing \( f(x) \). Then \( X \times V \) is a \( \prod_4 \)-open set in \( X \times Y \) containing \( g(x) \). Since \( g \) is almost pairwise continuous there exists a \( P_4 \)-open set \( U \) in \( X \) containing \( x \) such that \( g(U) \subseteq \prod_4 \text{-int } \prod_j \text{-cl } (X \times V) = X \times T_4 \text{-int } T_j \text{-cl } V \). Thus, if \( z \in U \) then \( g(z) = (z, f(z)) \in X \times T_4 \text{-int } T_j \text{-cl } V \), so that \( f(z) \in T_4 \text{-int } T_j \text{-cl } V \). Consequently, \( f(U) \subseteq T_4 \text{-int } T_j \text{-cl } V \), which shows that \( f \) is almost pairwise continuous. //