INTRODUCTORY CHAPTER

(TOPOLOGY VS FUNCTIONS)

The first use of the term 'Topology' occurred in the title of the book 'Vorstudien Zur Topologie' by Listing in 1847. Among others, Riemann [88], Cantor [8] and Poincare[77] were the main who contributed to the development of topology during the later part of the nineteenth century. Topology during the course of its development split up into two main branches: General Topology (also called point set topology) and Algebraic Topology (also called combinatorial topology). General topology utilizes set-theoretic tools and algebraic topology uses algebra. In 1906, Riesz [89], Fréchet [18] and Moore [64] grappled with the problems of defining general spatial structure. This year may be considered as the year when general topology got its real start. With the publication of Poincare's paper in 1895, the combinatorial topology might have got an impetus for its start. In homology theory, dimension theory, manifolds, etc., the topological spaces are looked upon as generalised geometric configurations. In the theory of Banach spaces, Hilbert spaces,
Banach algebra, the modern theory of integration and abstract harmonic analysis on locally compact groups, continuous functions are the chief objects of interest and the topological spaces are regarded as carriers of such functions and as domains on which they can be integrated. In the early decades of the twentieth century, 'Topology' was only a conglomeration of loosely related theorems, but now it has become a systematic science and has established as one of the basic disciplines of pure mathematics.

Topology is a qualitative mathematics, that is, it is mathematics without numbers. It is a generalisation and abstraction of the set of real numbers. Topology on an abstract nonempty set means the introduction of the concepts of open set, closed set, neighbourhood, accumulation point and limit point. However, all these concepts are not independent for in terms of any one of them the other concepts can be easily explained. It is well known that Bourbaki [7] takes the open set, Kuratowski [41] the closure, Hausdorff [30] neighbourhood, Fréchet [18, 19] accumulation point and Hilbert [31] the convergence as basis. Experience convinced topologists that topology with open sets as basis leads to elegant results. The source of the axioms that
define the collection of all open sets, that is a topology, on an abstract nonempty set \( X \) is some of the properties of the collection of all open subsets of real numbers.

**Definition (O.A.)** A nonempty collection \( \mathcal{T} \) of subsets of a nonempty set \( X \) is said to be a topology (or a topological structure) on \( X \) if it satisfies the following axioms.

(I) The union of the members of each subfamily of \( \mathcal{T} \) is a member of \( \mathcal{T} \).

(II) The intersection of the members of each finite subfamily of \( \mathcal{T} \) is a member of \( \mathcal{T} \).

The pair \((X, \mathcal{T})\) is called a topological space.

The members of \( \mathcal{T} \) are said to be open and the members of \( X \) are called its points. When no confusion is likely to occur, one may forget, for the sake of simplicity, to mention the topology on the space and write that \( X \) is a topological space, or simply that \( X \) is a space.

Roughly speaking topology is concerned with those intrinsic qualitative properties of spatial configurations.
that are independent of size, location and shape. An intrinsic qualitative property is a property that does not change when the object under consideration is subjected to 'stretching and bending without tearing'. The phrase 'without tearing' means that points that are originally close together in the object remain close together throughout the stretching process. The notion of 'stretching and bending without tearing' can be expressed in terms of functions. It may be noted here that the terms 'function' and 'mapping' are synonymous in topology. A function stretches and/or bends its domain into its range. The concept of 'without tearing' is equivalent to bicontinuity of the functions. The stretching and bending can be both done and undone continuously. One to one functions that stretch and bend their domains into their ranges without tearing are homeomorphisms. If two spaces are homeomorphic, then from a topological point of view, they are carbon copies of each other. Their topological structure is the same. A topologist often tends to treat two homeomorphic spaces as the same. This has prompted a tongue-in-check definition of a topologist as a man who does not know the difference between the coffee cup and the doughnut.
The methods of constructing product and quotient spaces from a given space are motivated by making certain functions continuous.

Functions are thus important tools for studying properties of spaces and for constructing new spaces from the existing ones. The general concept of a function is too wide to be of interest in analysis or topology. A topological space can be thought of as a set from which has been swept away all structures irrelevant to the continuity of functions defined on it. The notion of continuous functions is certainly one of the most fundamental in the theory of topology. In that sense, one could say that the study of topology really begins with continuous functions. The study of topology is in fact the study of continuous functions. There is a constant illuminating interplay between the structure of the spaces and the properties of the continuous functions which they carry. The classical concept of continuity is that of metric continuity. This led to the notion of topological continuity.

The literature reveals that several generalisations of the concept of continuity have been introduced and studied
from time to time. For example, four types of almost continuity defined respectively by Frolik [20], Hussain [32], Singal and Singal [95] and Stalings [99], \( \eta \)-continuity [13], \( \alpha \)-continuity [98], \( \Theta \)-continuity [4, 17], \( c \)-continuity [23], \( s \)-continuity [40], \( H \)-continuity [47], weak continuity [45], \( p \)-continuous functions [81], near continuity [34], quasi continuity [60], semi-continuity [46], simple continuity [6], connected function [74], almost semi-continuity [35], feeble continuity [35], almost feeble continuity [49], etc. are certain generalizations of the notion of continuity. The notions of Frolik's almost continuity, quasi-continuity of Martin and semicontinuity due to Levine are equivalent. Moreover, semi-continuity is independent of Hussain's almost continuity and weak continuity due to Levine as well. Hussain's and Stalings's almost continuities are found to be independent of each other. Almost continuity due to Singal and Singal is seen to be independent of both these almost continuities and also with \( H \)-continuity, \( c \)-continuity and \( s \)-continuity [79], but Papp[72] and Kim[39] showed that it implies \( \Theta \)-continuity. On the other hand, strong continuity [44], strongly open mappings (that is mappings under which inverse image of every set is open).
perfect mappings [15], G-mapping [14], supercontinuity [65] and separate mappings (that is, continuous mappings under which any two distinct point with distinct images have disjoint neighbourhoods) etc. are certain stronger types of continuities. Several notions of functions in topology which are independent of the concept of continuity have also been obtained and studied. Among these may be cited open mappings [36, 106], closed mappings [21], irresolute functions [9a], α-irresolute mappings [57], σ-continuity [71], semiconnected mappings [43], strong semicontinuity [35], strong feebly continuous mappings [39], β-continuity [101], ασ-continuity [80], α-irresolute functions [98] and others.

The open mappings and closed mappings have their own distinct places in the study of topological spaces. Many useful and important properties of topological space are preserved under continuous mappings and in several cases they fail to do so. For example, though the continuous image of a compact space is compact and that of a connected space is connected, but this is not the case if the space is taken to be Hausdorff or normal. It is wellknown that the property of being Hausdorff is preserved under homeomorphism (i.e. one-one, onto, open continuous mapping)
and normality is preserved under closed continuous mappings. As such, in view of its own importance several weaker forms of open mappings such as three types of almost open defined by Arhangel'skiĭ [1], Singal and Singal [95] and Wilansky [105], inductively open [1], pseudo open [3], quasi compact [103], almost quasi compact [95], semiopen [6], weakly open [91], feebly open [35], almost feebly open [35], g-semiopen [101] etc. have been introduced. Similar to open mappings, the class of closed mappings also occupies central place in the study of topological spaces. A large part of the work done on closed mappings and its various forms may be found in Arhangel'skiĭ [2], Hafner [24], Hanai [25-to-29], Hyman [33], Mannings [59], McDouglo [61], Michael [62], Noiri [68,70], Ronnow [90], Whyburn [104], Maheshwari and Jain [50], Thakur [101] and others.

The concept of bitopological spaces originated from quasi metric spaces (that is, spaces equipped with a metric that lacks the property of symmetry), and quasi uniform spaces (that is, spaces equipped with a uniformity that lacks the
property of symmetry). With every quasi metric d on X there is associated another quasi metric p on X defined by \( p(x,y) = d(x,y) \). The d-open spheres and p-open spheres generate two topologies on X say \( P^* \) and \( Q^* \) respectively. Thus with every quasi metric space X with the quasi metric d there is an associated set X endowed with two topological structures \( P^* \) and \( Q^* \). Similarly, if \( U \) is a quasi-uniformity on a set X, then \( U^{-1} = \{ U^{-1} | u \in U \} \) is also a quasi uniformity on X. There is a unique topology \( J(U) \) on X such that \( \{ U(x) | u \in U \} \) is the filter of all the neighbourhoods of the point \( x \in X \). Similarly, there is another unique topology \( J(U^{-1}) \) on X. Thus with any quasi uniform space X with the quasi uniformity \( U \), there is associated a set X endowed with two topological structures \( J(U) \) and \( J(U^{-1}) \) respectively. Now if one studies X with two topologies P and Q some of the symmetry of the classical metric situation in regained and one can obtain symmetric generalization of the standard results such as Urysohn's lemma, Urysohn's metrization theorem, Tietze's extension theorem etc. Inspired by these observations in 1963, Kelly [37] introduced the concept of a bitopological space as follows:
DEFINITION (1.8) : A bitopological space \( (X, P, Q) \) is a nonempty set \( X \) equipped with two arbitrary topological structures \( P \) and \( Q \).

After the publication of Kelly's paper, a number of mathematicians such as Pervin [76], Kim [38], Lane [42], Patty [73], Fletcher [16], Gaatt[22], Murdeshwar and Naimpally [67], Thampuran [102], Reilly [82-to-87], Óirsan [5], Singal and his associates [93,94,96,97], Sunderjal [100], Popa [78], Cirić [9], Datta [11,12], Milena [63] Maheshwari and his associates [51-to-56] and others have contributed to the theory of bitopological spaces. In almost all the cases the aim has been to generalise results from the theory of general topological spaces to bitopological spaces in such a way that classical theorems in general topology become particular cases of the analogous theorems for bitopological spaces, provided one makes the convention that a topological space \( (X, J) \) is also a bitopological space \( (X, J, J) \).

In 1977, Sunderjal [100] differentiated among the bitopological notions. He observed that some bitopological notions are very much bitopological whereas some are not so
much bitopological. For example, the concept of pairwise $T_1$ in Reilly's sense (i.e. when both the topological spaces $(X, P)$ and $(X, Q)$ are $T_1$) is not so much bitopological. On the other hand, pairwise normality of $(X, P, Q)$ is independent of the normality of both $(X, P)$ and $(X, Q)$. Thus the concept of pairwise normality is very much bitopological. In 1967, Pervin [76] extended the concept of continuity to bitopological spaces. And thence, as in topological spaces several generalizations of pairwise continuity have been introduced and studied in bitopological spaces. For example, pairwise semi-continuous functions [54], quasi continuous functions [11], quasi semi-continuity [58a], semibicontinuity [63] and pairwise feebly continuous functions [58b] are some of the generalizations of the concept of pairwise continuity. On the other hand, functions which are independent of pairwise continuity have also been introduced and studied. Among them may be cited pairwise open functions [78], pairwise irresolute functions [53], birresolute mappings [63] and quasi irresolute mappings [58a].

The above observations led us to the following questions:
"Though the concept of continuity (respectively open mapping) in topological spaces has been extended to the concept of pairwise continuity (respectively pairwise open mapping) in bitopological spaces but several of its generalizations e.g. almost continuities, θ-continuity, weak continuity etc. (respectively e.g. almost open, weakly open mapping etc.) still await their analogues in bitopological spaces."

The aim of the DISsertATION is to present a study of mappings in this direction.