CHAPTER IX

ON PAIRWISE SEMICONTINUITY

In 1976, Maheshwari and Prasad [53] have introduced the notion of pairwise irresolute mappings as follows:

**Definition (9.1):** A mapping \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau_1, \tau_2) \) is termed pairwise irresolute if the inverse image of each \((i,j)\)-semiopen set in \( Y \) is \((i,j)\)-semiopen in \( X \), \( i, j = 1, 2 \) such that \( i \neq j \).

**Remark (9.2):** The concepts of pairwise continuous mapping and pairwise irresolute mappings are independent [53].

Later in 1978, they have introduced the generalization of the concept of pairwise irresolute mappings in the form of pairwise semicontinuity (refer Definition (8.1.8)). It is well known [53] that pairwise semicontinuous mappings need not be pairwise irresolute. So far, the literature is silent on the question as to when pairwise semicontinuity may imply pairwise irresolute. This chapter investigates the conditions that ensures pairwise semicontinuity to imply pairwise irresolute.

Theorem (9.1): If \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) is pairwise semicontinuous and almost pairwise open then it is pairwise irresolute.

We shall need the following result:

Lemma (9.1): [34]. In the space \( (X, P_1, P_2) \) a set \( A \) is \((i,j)\)-semiopen iff \( A \subseteq P_j - \overline{\text{cl } P_i - \text{int } A} \), \( i, j = 1, 2 \), \( i \neq j \).

Proof of Theorem (9.1): Let \( V \) be \((i,j)\)-semiopen in \( Y \). Then, there exists a \( T_j \)-open set \( G \) such that \( G \subseteq V \subseteq T_j - \overline{\text{cl } G} \); hence, \( f^{-1}(G) \subseteq f^{-1}(V) \subseteq f^{-1}(T_j - \overline{\text{cl } G}) \). Since \( f \) is pairwise semicontinuous, \( f^{-1}(G) \) is \((i,j)\)-semiopen in \( X \) and hence by Lemma (9.1) \( f^{-1}(G) \subseteq P_j - \overline{\text{cl } P_i - \text{int } f^{-1}(G)} \). Now put, \( F = Y - f(X - P_j - \overline{\text{cl } P_i - \text{int } f^{-1}(G)}) \). Then, \( F \) is \( T_j \)-closed because \( f \) is almost pairwise open and \( P_j - \overline{\text{cl } P_i - \text{int } f^{-1}(G)} \) is \((j,1)\)-regular closed in \( X \). By a straightforward calculation we get, \( G \subseteq F \) and \( f^{-1}(F) \subseteq P_j - \overline{\text{cl } P_i - \text{int } f^{-1}(G)} \). Therefore, we have, \( f^{-1}(T_j - \overline{\text{cl } G}) \subseteq f^{-1}(F) \subseteq P_j - \overline{\text{cl } P_i - \text{int } f^{-1}(G)} \subseteq P_j - \overline{\text{cl } f^{-1}(G)} \). Thus, \( f^{-1}(G) \subseteq f^{-1}(V) \subseteq P_j - \overline{\text{cl } f^{-1}(G)} \). It follows by Lemma (8.2), that \( f^{-1}(V) \) is \((i,j)\)-semiopen in \( X \). Hence, \( f \) is pairwise irresolute. //
THEOREM (9.2): If \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) is pairwise semicontinuous, \( \emptyset \)-pairwise semiopen and the space \((Y, T_1, T_2)\) is \(p\)-extremely disconnected, then \( f \) is pairwise irresolute.

The proof requires the following lemma:

LEMMA (9.2): If a space \((X, P_1, P_2)\) is \(p\)-extremely disconnected then for every \((i,j)\)-semi open subset \(U \) of \(X\),

\[(j,1)\)-scl \( U = P_j\)-cl \( U \), \( i,j = 1,2, i \neq j \).

PROOF OF LEMMA (9.2): In general, \((j,1)\)-scl \( S \subseteq P_j\)-cl \( S \), for every subset \(S\) of \(X\). Therefore, it is sufficient to show that \( P_j\)-cl \( U \subseteq (j,1)\)-scl \( U \) for every \((i,j)\)-semiopen subset \(U\) of \(X\). Let \((j,1)\)-scl \( U \neq \emptyset \) and let \( x \notin (j,1)\)-scl \( U \).

This implies that there exists a \((j,1)\)-semiopen set \(V\) in \(X\) such that \( x \in V \) and \( U \cap V = \emptyset \). Hence, \( P_1\)-int \( U \cap P_j\)-int \( V = \emptyset \), showing that \( P_j\)-cl \( P_1\)-int \( U \cap P_j\)-int \( V = \emptyset \). This implies that \( P_j\)-cl \( P_1\)-int \( U \cup P_1\)-cl \( P_j\)-int \( V = \emptyset \), for \(X\) being \(p\)-extremely disconnected \( P_j\)-cl \( P_1\)-int \( U \) is \(P_1\)-open. Since \(V\) is \((j,1)\)-semiopen therefore by Lemma (9.1), \( V \subseteq P_1\)-cl \( P_j\)-int \( V \). Since \( x \in V \), it follows that \( x \notin P_j\)-cl \( P_1\)-int \( U \).
Again by Lemma (9.1), $U \subseteq P_j - cl \ P_1 - int \ U$, showing that $P_j - cl \ U \subseteq P_j - cl \ P_1 - int \ U \subseteq P_j - cl \ U$, and so, $P_j - cl \ U = P_j - cl \ P_1 - int \ U$. Therefore, $x \notin P_j - cl \ U$. //

**Proof of Theorem (9.2):** Let $V$ be $(i,j)$-semiopen in $Y$, there exists $T_i$-open set $G$ such that $G \subseteq V \subseteq T_j - cl \ G$, hence $f^{-1}(G) \subseteq f^{-1}(V) \subseteq f^{-1}(T_j - cl \ G)$. Since $Y$ is $p$-extremely disconnected and $G$ being $T_i$-open is $(i,j)$-semiopen in $Y$, therefore by Lemma (9.2), we get, $(j,i)$-scl $G = T_j - cl \ G$.

Now, if $x \in f^{-1}(j,i)$-scl $G$ then $f(x) \in (j,i)$-scl $G$. Let $O$ be any $P_j$-open set containing $x$. Then $f(x) \in f(O)$ which is $(j,i)$-semiopen since $f$ is $\emptyset$-pairwise semiopen. Therefore, $G \cap f(O) \neq \emptyset$, and hence $O \cap f^{-1}(G) \neq \emptyset$. Therefore, $x \in P_j - cl \ f^{-1}(G)$. Thus, $f^{-1}((i,j)$-scl $G) \subseteq P_j - cl \ f^{-1}(G)$. Therefore, $f^{-1}(G) \subseteq f^{-1}(V) \subseteq P_j - cl \ f^{-1}(G)$. Since $f$ is pairwise semicontinuous, $f^{-1}(G)$ is $(i,j)$-semiopen in $X$, and hence by Lemma (8.2), $f^{-1}(V)$ is $(i,j)$-semiopen in $X$. Consequently, $f$ is pairwise irresolute. //

**Remark (9.1):** The following example shows that a pairwise semicontinuous $\emptyset$-pairwise semiopen function is not necessarily
pairwise irresolute if the codomain is not p-extremely disconnected.

**Example (9.1):** Let \( X = \{a, b, c, d\} \) and \( Y = \{a, b, c\} \). Let,

\[
P_1 = \{\emptyset \cup \{a\}, X\}, \quad P_2 = \{\emptyset \cup \{b\}, \{b, c\}, X\}
\]

and

\[
T_1 = \{\emptyset \cup \{a\}, Y\}, \quad T_2 = \{\emptyset \cup \{b\}, Y\}.
\]

Then, the function \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) defined by \( f(a) = a, f(b) = b, f(c) = f(d) = c \), is pairwise semicontinuous and \( \emptyset \)-pairwise semiopen but it is not pairwise irresolute. Note that the space \( (Y, T_1, T_2) \) is not p-extremely disconnected.

**Remark (9.2):** The concepts of p-extremely disconnected (Definition (9.2)) and extremely disconnected (Definition (8.4)) are independent. For,

**Example (9.2):** Let \( X = \{a, b, c\} \), \( P_1 = \{\emptyset \cup \{a\}, \{a, c\}, X\} \), and \( P_2 = \{\emptyset \cup \{b\}, \{b, c\}, X\} \). Then, the space \( (X, P_1, P_2) \) is p-extremely disconnected but it is not extremely disconnected.

**Example (9.3):** Let \( X = \{a, b, c\} \), \( P_1 = \{\emptyset \cup \{a\}, \{b, c\}, X\} \) and \( P_2 = \{\emptyset \cup \{b\}, \{a, c\}, X\} \).
Then, the space \((X, P_1, P_2)\) is extremely disconnected but it is not \(p\)-extremely disconnected.

**Theorem (9.3)**: If \(f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)\) is pairwise semicontinuous, pairwise semiopen and the space \((Y, T_1, T_2)\) is extremely disconnected, then \(f\) is pairwise irresolute.

We shall need the following lemma:

**Lemma (9.3)**: If a space \((X, P_1, P_2)\) is extremely disconnected then for every \((i,j)\)-semiopen subset \(U\) of \(X\), \(P_j\)-cl \(U = P_j\)-scl \(U\), where \(i,j = 1, 2\), \(i \neq j\).

**Proof of Lemma (9.3)**: It is sufficient to show that \(P_j\)-scl \(U \supseteq P_j\)-cl \(U\) for each \((i,j)\)-semiopen subset \(U\) of \(X\). Let \(x \notin P_j\)-scl \(U\). Then, there exists a \(P_j\)-semiopen set \(V\) in \(X\) such that \(x \in V\) and \(U \cap V = \emptyset\). Hence, \(P_i\)-int \(U \cap P_j\)-int \(V = \emptyset\).

This implies that \(P_j\)-cl \(P_i\)-int \(U \cap P_j\)-int \(V = \emptyset\), showing that \(P_j\)-cl \(P_i\)-int \(U \cap P_j\)-cl \(P_j\)-int \(V = \emptyset\) for \(X\) being extremely disconnected, \(P_j\)-cl \(P_i\)-int \(U\) is \(P_j\)-open. Since \(V\) is \(P_j\)-semiopen it follows that \(V \subseteq P_j\)-cl \(P_j\)-int \(V\). Since \(x \in V\), we get, \(x \notin P_j\)-cl \(P_i\)-int \(U\). Now, \(U\) being \((i,j)\)-semiopen therefore, \(U \subseteq P_j\)-cl \(P_i\)-int \(U\) implying that \(P_j\)-cl \(U \subseteq P_j\)-cl \(P_i\)-int \(U\)
proof of theorem (9.3): Let $V$ be $(1,j)$-semiopen in $Y$. There exists a $T_1$-open set $G$ such that $G \subseteq V \subseteq T_j$-cl $G$. And so, $f^{-1}(G) \subseteq f^{-1}(V) \subseteq f^{-1}(T_j$-cl $G)$. Since $Y$ is extremely disconnected we have by Lemma (9.3), $T_j$-cl $G = T_j$-cl $G$.

Now, if $x \in f^{-1}(T_j$-cl $G)$ then $f(x) \in T_j$-cl $G$. Let $O$ be any $P_j$-open set containing $x$. Then, $f(x) \in f(O)$ and $f(O)$ is $T_j$-semiopen since $f$ is pairwise semiopen. Therefore, $G \cap f(O) \neq \emptyset$, and hence, $O \cap f^{-1}(G) \neq \emptyset$, so that, $x \in P_j$-cl $f^{-1}(G)$.

Therefore, $f^{-1}(T_j$-cl $G) \subseteq P_j$-cl $f^{-1}(G)$. Therefore, $f^{-1}(T_j$-cl $G) \subseteq P_j$-cl $f^{-1}(G)$. Thus, $f^{-1}(G) \subseteq f^{-1}(V) \subseteq P_j$-cl $f^{-1}(G)$.

Since $f$ is pairwise semicontinuous, $f^{-1}(G)$ is $(1,j)$-semiopen in $X$. Thus, by Lemma (8.2), $f^{-1}(V)$ is $(1,j)$-semiopen in $X$. Consequently, $f$ is pairwise irresolute. //

remark (9.3): The mapping $f$ considered in Example (9.1) is pairwise semicontinuous and pairwise semiopen but it is not pairwise irresolute. We note that the space $(Y, T_1 T_2)$ is not extremely disconnected.