CHAPTER VIII

SOME NEW MAPPINGS II

This Chapter, though continues the study of the type of the pairwise mappings discussed in Chapter VII, is a little diverted in its conception. In the previous chapter, the new notions introduced take into account $P_i$ (resp. $T_i$)-open sets or their stronger forms viz. $(i,i)$-regular open sets, $i,j=1,2$ and $i \neq j$. The concepts that we propose to introduce here, take into consideration $P_i$ (resp. $T_i$)-open sets and their weaker forms viz. $(i,j)$-semiopen sets and $P_i$ (resp. $T_i$)-semiopen sets.

In a bitopological space the concept of $(i,j)$-semiopen is due to Maheshwari and Prasad [54]. A subset $A$ of a space $(X, P_1, P_2)$ is termed $(i,j)$-semiopen if there exists a $P_i$-open set $O$ such that $O \subseteq A \subseteq P_j$-cl $O$, $i,j=1,2$ such that $i \neq j$. Every $P_i$-open set is $(i,j)$-semiopen but the converse may be false. The complement of an $(i,j)$-semiopen set is called $(i,j)$-semiclosed. The smallest $(i,j)$-semiclosed

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set which contains a set $A$ is denoted by $(i,j)$-scl $A$. In general, $A \subseteq (i,j)$-scl $A \subseteq P_1$-cl $A$ [54]. Moreover, every $P_1$-open set is $P_1$-semiopen but the converse may fail. It has been observed [54] that the notions of $(i,j)$-semiopen and $P_1$-semiopen are independent. The aim of this chapter is to make use of $(i,j)$-semiopen sets and $P_1$-semiopen sets in the study of pairwise mappings.

The three new concepts are conceived as follows:

**DEFINITION (8.1):** A mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is termed $\phi$-pairwise semiopen if the image of each $P_1$-open set is $(i,j)$-semiopen in $Y$, $i, j = 1, 2$ such that $i \neq j$.

**DEFINITION (8.2):** A mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is termed $\phi^e$-pairwise semiopen if the image of each $(i,j)$-semiopen set in $X$ is $(i,j)$-semiopen in $Y$, $i, j = 1, 2$ such that $i \neq j$.

**DEFINITION (8.3):** A mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is termed $\phi^e$-pairwise semiopen if the image of each $P_1$-semiopen set is $(i,j)$-semiopen in $Y$, $i, j = 1, 2$ such that $i \neq j$. 

THEOREM (8.1) : Every pairwise open mapping \( f : (X_1, P_1, P_2) \rightarrow (Y_1, T_1, T_2) \) is \( \phi \)-pairwise semiopen.

PROOF : It follows because every \( T_1 \)-open set is \((i,j)\)-semiopen in \( Y \). //

THEOREM (8.2) : Every \( \phi \)-pairwise semiopen mapping \( f : (X_1, P_1, P_2) \rightarrow (Y_1, T_1, T_2) \) is \( \phi \)-pairwise semiopen.

PROOF : It follows because every \( P_1 \)-open set is \((i,j)\)-semiopen in \( X \). //

THEOREM (8.3) : Every \( \phi \)-pairwise semiopen mapping \( f : (X_1, P_1, P_2) \rightarrow (Y_1, T_1, T_2) \) is \( \phi \)-pairwise semiopen.

PROOF : It follows because every \( P_1 \)-open set is \( P_1 \)-semiopen. //

Let us now consider few examples. Let \( X = \{a, b, c\} = Y = Z \), and \( f : (X_1, P_1, P_2) \rightarrow (Y_1, T_1, T_2) \) and \( g : (Y_1, T_1, T_2) \rightarrow (Z_1, Q_1, Q_2) \) be the identity mappings throughout the examples that follow in this chapter.

EXAMPLE (8.1) : Let \( P_1 = \{\emptyset, \{a\}, X\} \), \( P_2 = \{\emptyset, \{b, c\}, X\} \), and \( T_1 = \{\emptyset, \{a\}, Y\} \), \( T_2 = \{\emptyset, \{b\}, Y\} \).
Then, the mapping $f$ is $\Phi^*$-pairwise semiopen (and hence, $\Phi$-pairwise semiopen) but it is neither $W^*$-almost pairwise open nor weakly pairwise open. Obviously, it is neither $W$-almost pairwise open, nor almost pairwise open nor pairwise open. Note also that, it is not $\Phi^*_e$-pairwise semiopen.

**Example (8.2):** Let $P_1 = \{\emptyset, \{a\}, X\}$, $P_2 = \{\emptyset, X\}$ and $T_1 = \{\emptyset, \{a\}, Y\}$, $T_2 = \{\emptyset, \{b\}, Y\}$.

Then, the mapping $f$ is pairwise open but it is not $\Phi^*$-pairwise semiopen. Obviously, it is almost pairwise open, weakly pairwise open, pairwise semiopen, $W$-almost pairwise open, $W^*$-almost pairwise open, $R$-almost pairwise open and weakly $R$-almost pairwise open. Note also that, it is not $\Phi^*_e$-pairwise semiopen.

**Example (8.3):** Let $P_1, P_2$ be as in Example (8.2) and $T_1 = \{\emptyset, \{a, b\}, Y\}$, $T_2 = \{\emptyset, \{c\}, Y\}$.

Then, the mapping $f$ is almost pairwise open, $W$-almost pairwise open and $R$-almost pairwise open but it is not $\Phi$-pairwise semiopen. Obviously, it is weakly pairwise open, $W^*$-almost pairwise open and weakly $R$-almost pairwise open.
**Example (8.4):** Let $T_1, T_2$ be as in Example (8.1). Let

$$p_1 = \{\emptyset, \{a, b\}, X\}, \quad p_2 = \{\emptyset, \{b, c\}, X\}.$$

Then, the mapping $f$ is pairwise semiopen but it is not $\phi$-pairwise semiopen. Obviously, it is not $\phi^*$-pairwise semiopen.

**Example (8.5):** Let

$$p_1 = \{\emptyset, \{a, c\}, X\}, \quad p_2 = \{\emptyset, \{b, c\}, X\}$$

and

$$T_1 = \{\emptyset, \{a\}, \{b, c\}, Y\}, \quad T_2 = \{\emptyset, \{b\}, Y\}.$$

Then, the mapping $f$ is $\phi^*$-pairwise semiopen (and hence, $\phi$-pairwise semiopen) but it is not pairwise semiopen. Note that, it is $\phi^*_g$-pairwise semiopen.

**Example (8.6):** Let

$$p_1 = \{\emptyset, \{b, c\}, X\}, \quad p_2 = \{\emptyset, \{a\}, X\}$$

and

$$T_1 = T_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}.$$

Then, the mapping $f$ is $\phi^*$-pairwise semiopen (and hence, $\phi$-pairwise semiopen) but it is not weakly $R$-almost pairwise open (and hence, it is not $R$-almost pairwise open). It is $\phi^*_g$-pairwise semiopen also.

**Example (8.7):** Let

$$p_1 = \{\emptyset, \{a\}, X\}, \quad p_2 = \{\emptyset, \{b, c\}, X\}$$
and
\[ T_1 = \{ \emptyset, \{ a \}, \{ a, b \}, Y \}, \quad T_2 = \{ \emptyset, \{ b \}, Y \} \]

Then, the mapping \( f \) is \( \Phi_0 \)-pairwise semiopen but it is neither \( W^* \)-almost pairwise open nor weakly pairwise open.

**Example (8.8):** Let
\[ p_1 = \{ \emptyset, \{ a \}, \{ b, c \}, X \}, \quad T_1 = \{ \emptyset, \{ a \}, \{ b, c \}, Y \} \]
and
\[ p_2 = \{ \emptyset, X \}, \quad T_2 = \{ \emptyset, \{ c \}, Y \} \]

Then, the mapping \( f \) is \( \Phi_0 \)-pairwise semiopen but it is not \( \Phi^* \)-pairwise semiopen.

Thus, we arrive at the following diagram of implications:
where the new abbreviations stand as follows:

\( \phi\text{-p.s.o.} \) : \( \phi\)-pairwise semiopen.

\( \phi^*\text{-p.s.o.} \) : \( \phi^*\)-pairwise semiopen.

\( \phi_6\text{-p.s.o.} \) : \( \phi_6\)-pairwise semiopen.

**Remark (8.1)**: The composition of two \( \phi\)-pairwise semiopen mappings need not be \( \phi\)-pairwise semiopen. For,

**Example (8.9)**: Let,

\[
P_1 = \{ \emptyset, \{a,b\} \} \times X, \quad P_2 = \{ \emptyset, \{c\} \} \times X,
\]

\[
T_1 = \{ \emptyset, \{a\} \} \times Y, \quad T_2 = \{ \emptyset, \{c\} \} \times Y,
\]

and

\[
Q_1 = \{ \emptyset, \{a\} \} \times Z, \quad Q_2 = \{ \emptyset, \{b\} \times \{c\} \times \{b,c\} \times Z \}.
\]

Then, the mapping \( f \) is \( \phi\)-pairwise semiopen and the mapping \( g \) is pairwise open (and hence, \( \phi\)-pairwise semiopen) but \( g \circ f \) is not \( \phi\)-pairwise semiopen.

**Remark (8.2)**: The composition of two \( \phi_6\)-pairwise semiopen mappings may not be \( \phi_6\)-pairwise semiopen. For,

**Example (8.10)**: Let,

\[
P_1 = \{ \emptyset, \{a,b\} \} \times X, \quad P_2 = \{ \emptyset \} \times X,
\]

\[
T_1 = \{ \emptyset, \{a\} \} \times \{b,c\} \times Y, \quad T_2 = \{ \emptyset, \{c\} \} \times \{a,c\} \times Y
\]
and
\[ q_1 = \{\emptyset, \{a\}, \{b,c\}, Z\}, \]
\[ q_2 = \{\emptyset, \{c\}, \{b,c\}, \{a,c\}, Z\}. \]

Then, the mappings \( f \) and \( g \) are \( \phi^e \)-pairwise semiopen but \( g \circ f \) fails to be so.

However, we have the following results:

**Theorem (8.4):** The composition of a pairwise open mapping and a \( \phi \)-pairwise semiopen mapping is \( \phi \)-pairwise semiopen.

**Theorem (8.5):** The composition of a \( \phi \)-pairwise semiopen mapping and a \( \phi^e \)-pairwise semiopen mapping is \( \phi \)-pairwise semiopen.

**Theorem (8.6):** The composition of a \( \phi^e \)-pairwise semiopen mapping and a \( \phi^e \)-pairwise semiopen mapping is \( \phi^e \)-pairwise semiopen.

**Theorem (8.7):** The composition of two \( \phi^e \)-pairwise semiopen mappings is \( \phi^e \)-pairwise semiopen.

The concept of pairwise presemiopen mappings is due to Papp [78]. A mapping \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) is presemiopen if the image of each \( P_1 \)-semiopen set is \( T_1 \)-semiopen. We see that,
THEOREM (8.8) : The composition of a pairwise prosemiopen mapping and a \( \emptyset \)-pairwise semiopen mapping is \( \emptyset \)-pairwise semiopen.

THEOREM (8.9) : The composition of a pairwise semiopen mapping and a \( \emptyset \)-pairwise semiopen mapping is \( \emptyset \)-pairwise semiopen.

The proofs of the above theorems are straightforward and hence are omitted.

REMARK (8.3) : The mapping \( f \) considered in Example (8.9) is \( \emptyset \)-pairwise semiopen and if we let \( A = \{ a, c \} \) then \( f/A \) is not \( \emptyset \)-pairwise semiopen.

THEOREM (8.10) : If \( f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2) \) is \( \emptyset \)-pairwise semiopen and \( A \) is a biopen set in \( X \), then \( f/A \) is \( \emptyset \)-pairwise semiopen.

PROOF : If \( U \) is \( P_1A \)-open in \( A \) then \( U \) is \( P_1 \)-open for \( A \) is \( P_1 \)-open. Since \( (f/A)(U) = f(U) \), which is by hypothesis \( (1, j) \)-semiopen in \( Y \). //

REMARK (8.4) : The mapping \( f \) considered in Example (8.1) is \( \emptyset \)-pairwise semiopen and if we let \( A = \{ a, c \} \), then \( f/A \)
is not $\emptyset^*$-pairwise semiopen. However, we have,

**Theorem (8.11):** If $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is $\emptyset^*$-pairwise semiopen and $A$ is biopen in $X$, then $f/A$ is $\emptyset^*$-pairwise semiopen.

It's proof requires the following result:

**Lemma (8.1):** If $A$ is a $P_1$-open subset of $(X, P_1, P_2)$ and $S$ is $(i,j)$-semiopen in $A$ then $S$ is $(i,j)$-semiopen in $X$.

**Proof of Theorem (8.11):** Let $U$ be any $(i,j)$-semiopen set in $A$ then by Lemma (8.1), it is $(i,j)$-semiopen in $X$ for $A$ is $P_1$-open. And so, $(f/A)(U) = f(U)$ which is $(i,j)$-semiopen in $Y$ since $f$ is $\emptyset^*$-pairwise semiopen. Hence, $f/A$ is $\emptyset^*$-pairwise semiopen. //

**Remark (8.3):** The mapping $f$ considered in Example (8.5) is $\emptyset_6$-pairwise semiopen and if we let $A = \{a, c\}$ then $f/A$ fails to be $\emptyset_6$-pairwise semiopen. However, we get,

**Theorem (8.12):** If $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is $\emptyset_6$-pairwise semiopen and $A$ is bisemiopen in $X$ then $f/A$ is $\emptyset_6$-pairwise semiopen.
PROOF: It results in view of the fact that if \( U \) is any \( p_{1A} \)-semiopen set in \( A \) then it is \( p_{1} \)-semiopen in \( X \) for \( A \) is \( p_{1} \)-semiopen \([51] \).  

The following concept is due to Popa \([78]\):

**DEFINITION (8.3):** A mapping \( f : (X, p_{1}, p_{2}) \rightarrow (Y, T_{1}, T_{2}) \) is said to be pairwise semicontinuous (Popa's sense) if the inverse image of each \( T_{1} \)-open set is \( p_{1} \)-semiopen, \( i = 1, 2 \).

We introduce the following concept:

**DEFINITION (8.4):** A space \( (X, p_{1}, p_{2}) \) is termed extremely disconnected if for each \( p_{1} \)-open set \( U \), \( p_{j} = 1 \) \( U \) is \( p_{j} \)-open.

The following theorem investigates the situation under which the concept of \( \emptyset \)-pairwise semiopen implies \( \emptyset^{*} \)-pairwise semiopen.

**THEOREM (8.13):** Let \( f : (X, p_{1}, p_{2}) \rightarrow (Y, T_{1}, T_{2}) \) be pairwise semicontinuous (Popa's sense) and \( \emptyset \)-pairwise semiopen. If the space \( X \) is extremely disconnected, then \( f \) is \( \emptyset^{*} \)-pairwise semiopen.

We shall need the following lemma:

**Lemma (8.2)\([54]\).** If \( A \) is \((i, j)\)-semiopen in a space \( (X, p_{1}, p_{2}) \) and \( A \subseteq B \subseteq p_{j} = 1 A \), then \( B \) is \((i, j)\)-semiopen in \( X \).
PROOF OF THEOREM (6.13): Let \( U \) be \((i,j)\)-semiopen in \( X \).

Then, there exists a \( P_i \)-open set \( G \) in \( X \) such that \( G \subseteq U \subseteq P_j \text{cl} \ G \). Now, by Lemma (9.3) \( P_j \text{-scl} \ G = P_j \text{cl} \ G \), for the space \( X \) is extremely disconnected and \( G \) being \( P_i \)-open is \((i,j)\)-semiopen in \( X \). Since \( f \) is pairwise semicontinuous (Papa's), \( f^{-1}(T_j \text{cl} f(G)) \) is \( P_j \)-semiclosed and contains \( G \). Therefore, \( f(P_j \text{-scl} G) \subseteq T_j \text{cl} f(G) \). Thus, \( f(G) \subseteq f(U) \subseteq f(P_j \text{cl} G) = f(P_j \text{-scl} G) \subseteq T_j \text{cl} f(G) \). Since \( f \) is \( \emptyset \)-pairwise semiopen, \( f(G) \) is \((i,j)\)-semiopen in \( Y \). Consequently, by Lemma (8.2), \( f(U) \) is \((i,j)\)-semiopen in \( Y \), showing that \( f \) is \( \emptyset \)-pairwise semiopen. ///

REMARK (8.6): Theorem (6.13) may fail if the domain space is not extremely disconnected. For,

EXAMPLE (6.11): Let

\[
P_1 = \{ \emptyset, \{a\}, X \}, \quad P_2 = \{ \emptyset, \{b\}, X \},
\]

and

\[
T_1 = \{ \emptyset, \{a\}, \{a, c\}, Y \}, \quad T_2 = \{ \emptyset, \{b\}, Y \}.
\]

Then, the mapping \( f \) is pairwise semicontinuous (Papa's) and \( \emptyset \)-pairwise semiopen but it is not \( \emptyset \)-pairwise semiopen. Note that the domain space \((X, P_1, P_2)\) is not extremely disconnected.
Maheshwari and Prasad [54] have also introduced the term pairwise semicontinuous as follows:

**Definition (8.1)**: A mapping $f : (X_1, P_1, P_2) \rightarrow (Y_1, T_1, T_2)$ is termed pairwise semicontinuous if the inverse image of each $T_{i}$-open set is $(I_j, J)$-semiopen in $X$, $i, j = 1, 2$ such that $i \neq j$.

**Remark (8.7)**: The concepts of pairwise semicontinuity and pairwise semicontinuity (Papa's sense) are independent. For,

**Example (8.12)**: Let,

$$P_1 = \{\emptyset, \{a\}, \{a, c\}, X\}, \quad P_2 = \{\emptyset, \{b\}, X\}$$

and

$$T_1 = \{\emptyset, \{a, b\}, Y\}, \quad T_2 = \{\emptyset, \{b, c\}, Y\}.$$

Then, the mapping $f$ is pairwise semicontinuous (Papa's sense) but not pairwise semicontinuous.

**Example (8.13)**: Let,

$$P_1 = \{\emptyset, \{a\}, \{b, c\}, X\}, \quad P_2 = \{\emptyset, \{b\}, X\}$$

and

$$T_1 = \{\emptyset, \{a, c\}, Y\}, \quad T_2 = \{\emptyset, Y\}.$$

Then, the mapping $f$ is pairwise semicontinuous but not pairwise semicontinuous (Papa's sense).
We have:

**Theorem (8.14):** Let \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) be pairwise semicontinuous and \( \emptyset \)-pairwise semiopen. If the space \( X \) is \( p \)-extremely disconnected then \( f \) is \( \emptyset^* \)-pairwise semiopen.

We shall need the following Lemma:

**Lemma (8.3):** A mapping \( f : (X, P_1, P_2) \to (Y, T_1, T_2) \) is pairwise semicontinuous iff for each subset \( A \) of \( X \),
\[
f[ (i, j) - \text{scl} A ] \subseteq T_j - \text{cl} f(A).
\]

**Proof of Theorem (8.14):** Let \( U \) be \((i, j)\)-semiopen in \( X \). There exists a \( P_j \)-open set \( G \) such that \( G \subseteq U \subseteq P_j - \text{cl} G \). Since \( G \) is \( P_j \)-open it is \((i, j)\)-semiopen in \( X \). Now the space \( X \) being \( p \)-extremely disconnected we have, by Lemma (9.2) that \((j, i)\)-\text{scl} \( G = P_j - \text{cl} G \). Since \( f \) is pairwise semicontinuous by Lemma (8.3), we obtain:
\[
f[(j, i) - \text{scl} G] \subseteq T_j - \text{cl} f(G).
\]
Therefore,
\[
f(G) \subseteq f(U) \subseteq f(P_j - \text{cl} G) = f[(j, i) - \text{scl} G] \subseteq T_j - \text{cl} f(G).
\]
Now \( f \) being \( \emptyset \)-pairwise semiopen, \( f(G) \) is \((i, j)\)-semiopen in \( Y \). And so, \( f(U) \) is \((i, j)\)-semiopen in \( Y \) by Lemma (8.2). Hence, \( f \) is \( \emptyset^* \)-pairwise semiopen. //
**Remark (8.8):** For the validity of Theorem (8.14), it is essential that the domain space $X$ must be $p$-extremely disconnected. This also is asserted by Example (8.11), where $f$ is pairwise semicontinuous and $\emptyset$-pairwise semiopen but it is not $\emptyset^*$-pairwise semiopen. Note that $(X, P_1, P_2)$ is not $p$-extremely disconnected.

**Theorem (8.15):** If a mapping $f : (X, P_1, P_2) \rightarrow (Y, T_1, T_2)$ is almost $H^*$-pairwise continuous and $\emptyset$-pairwise semiopen then it is $\emptyset^*$-pairwise semiopen.

**Proof:** Let $U$ be $(i,j)$-semiopen in $X$. There exists a $P_1$-open set $G$ such that $G \subseteq U \subseteq P_1$-$\text{cl} \ G$. If $x \in P_1$-$\text{cl} \ G$ and $M$ be any $T_j$-open neighbourhood of $f(x)$ then $P_1$-$\text{cl} \ f^{-1}(M)$ is a $P_1$-neighbourhood of $x$ since $f$ is almost $H^*$-pairwise continuous. Therefore, $G \cap P_1$-$\text{cl} \ f^{-1}(M) \neq \emptyset$. Since $G$ is $P_1$-open it follows that, $P_1$-$\text{cl} \ (G \cap f^{-1}(M)) \neq \emptyset$, and so $G \cap f^{-1}(M) \neq \emptyset$, showing that, $f(G) \cap M \neq \emptyset$. Thus, $f(x) \in T_j$-$\text{cl} \ f(G)$. Consequently, $f(G) \subseteq f(U) \subseteq f(P_1$-$\text{cl} \ G) \subseteq T_j$-$\text{cl} \ f(G)$. By hypothesis $f$ is $\emptyset$-pairwise semiopen and so $f(G)$ is $(i,j)$-semiopen in $Y$. And so by Lemma (8.2), $f(U)$ is $(i,j)$-semiopen in $Y$. Hence, $f$ is $\emptyset^*$-pairwise semiopen. //
Since every \( \emptyset \)-pairwise semiopen mapping is \( \emptyset \)-pairwise semiopen, we obtain,

**Corollary (8.1):** Every almost \( H^* \)-pairwise continuous \( \emptyset \)-pairwise semiopen mapping is \( H^* \)-pairwise semiopen.

**Theorem (8.16):** Every almost \( H \)-pairwise continuous pairwise semiopen mapping is pairwise presemiopen.

The proof is similar to Theorem (8.15) and uses the following result:

**Lemma (8.4):** [46]. In a space \((X, P_1, P_2)\) if \( A \) is \( P_1 \)-semiopen and \( A \subseteq B \subseteq P_1 \)-cl \( A \), then \( B \) is \( P_1 \)-semiopen.

Since every pairwise open mapping is pairwise semiopen we have,

**Corollary (8.2):** Every almost \( H \)-pairwise continuous pairwise open mapping is pairwise presemiopen.

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