COMMON FIXED POINT THEOREMS FOR
THREE MAPPINGS
CHAPTER I

COMMON FIXED POINT THEOREMS FOR THREE MAPPINGS

1.1 Let \((X,d)\) be a metric space. A mapping \(f:X \rightarrow X\) is called a contraction, if there exists some \(k, 0 \leq k < 1\) such that for all \(x, y \in X\)

\[
(1.1.1) \quad d(f(x), f(y)) \leq k \ d(x, y)
\]

A well known Banach contraction principle states that if \(X\) is complete and \(f\) is continuous mapping, then \(f\) has a unique fixed point.

Further Sehgal[127] generalized the above result given as follows:

**THEOREM A**: Let \(f : X \rightarrow X\) be a continuous mapping of a complete metric space \((X,d)\). If for each \(x\) in \(X\) there exists a positive integer \(n(x)\) such that for all \(x, y \in X\)

\[
(1.1.2) \quad d(f^{n(x)}(x), f^{n(x)}(y)) \leq k \ d(x, y)
\]

for some constant \(0 \leq k < 1\). Then \(f\) has a unique fixed point \(z \in X\) and \(f^{n(x_0)}(x_0) \rightarrow z\) for each \(x_0 \in X\).

The following generalization of the well known Banach contraction principle for the first time is due to Jungck[63].
THEOREM B: Let $f$ and $g$ be two continuous and commuting self mappings of a complete metric space $(X,d)$ satisfying:

\[(1.1.3) \quad (a)f(x) \equiv g(x) \quad (b) \quad d(f(x),f(y)) \leq k \quad d(g(x),g(y))\]

for all $x, y \in X$, where $k$ is a non-negative real number, $k < 1$. Then $f$ and $g$ have a unique common fixed point.

While, Yeh[154] proved an interesting extension of common fixed point theorem due to Jungck[63], for three continuous self mappings in the following way:

THEOREM C: Let $f$, $g$ and $h$ be three continuous self mappings of a complete metric space $(X,d)$ satisfying:

\[(1.1.4) \quad fh=hf, \quad gh=hg; \quad f(X) \subset h(X) \quad \text{and} \quad g(X) \subset h(X)\]

and

\[(1.1.5) \quad d(f(x),g(y)) \leq k(d(h(x),h(y)),d(h(x),f(x)))\]

\[d(h(x),g(y)), \quad d(h(y),f(x)), \quad d(h(y),g(y))\]

where $k$ is upper semi-continuous and nondecreasing with respect to each variable and $h(t,t,at,at,t) < t$, for each $t$ in $R^+-\{0\}$, where $a + b = 2$. Then $f, g$ and $h$ have a unique common fixed point in $X$.

THEOREM D: Let $f$ be a mapping of metric space $X$ into itself such that for all $x, y$ in $X$,

\[ d(f(x), f(y)) \leq a_1 d(y, f(x)) \frac{1 + d(x, f(x))}{1 + d(x, y)} + a_2 d(x, y) \]

and for some $a_1, a_2 \in (0, 1)$ with $a_1 + a_2 < 1$, then $f$ has a unique fixed point in $X$.

Jaggi[60] has proved that

THEOREM E: Let $f$ be a continuous mapping of a complete metric space $X$ into itself satisfying:

\[ d(f(x), f(y)) \leq a_1 \frac{d(x, f(x)) d(y, f(y))}{d(x, y)} + a_2 d(x, y) \]

for all $x, y$ in $X$, $x \neq y$, and for some $a_1, a_2 \in [0, 1)$, with $a_1 + a_2 < 1$, then $f$ has a unique fixed point in $X$.

1.2 The object of this chapter is to prove further generalization of Theorem D and Theorem E for three mappings:

THEOREM (1): Let $f, g$ and $h$ be three self-mappings of a complete metric space $(X, d)$ such that the following holds:

(1.2.1) $fh = hf$, $gh = hg$; $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$,
(1.2.2) for all \( x, y \) in \( X \)

\[
d(f(x), g(y)) \leq a_1 \frac{d(h(x), f(x))d(h(y), g(y))}{d(h(x), h(y))} + a_2 \frac{d(h(y), g(y))}{[1+d(h(x), f(x))]} + a_3 \frac{d(h(x), h(y))}{[1+d(h(x), h(y))]}\]

where \( h(x) \neq h(y) \). If \( h \) is continuous, then \( f, g \) and \( h \) have a unique common fixed point.

**Proof**: Let \( x_0 \) be any point in \( X \). Construct a sequence \( \{x_n\} \) in the following way

(1.2.3) \( h x_{2n+1} = f x_{2n} \); \( h x_{2n+2} = g x_{2n+1} \)

for \( n = 0, 1, 2, \ldots \), we can do this since \( f(X) \) and \( g(X) \) are subsets of \( h(X) \).

Now consider,

\[
d(h x_{2n+1}, h x_{2n+2}) = d(f x_{2n}, g x_{2n+1})
\]

\[
\leq a_1 \frac{d(h x_{2n}, f x_{2n})d(h x_{2n+1}, g x_{2n+1})}{d(h x_{2n}, h x_{2n+1})} + a_2 \frac{d(h x_{2n+1}, g x_{2n+1})[1+d(h x_{2n}, f x_{2n})]}{[1+d(h x_{2n}, h x_{2n+1})]}\]
\[+a_3 \frac{d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+1})} + \frac{d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} \leq a_1 \frac{d(x_{2n+1}, x_{2n+2})[1+d(x_{2n}, x_{2n+1})]}{[1+d(x_{2n}, x_{2n+1})]}\]

Or,

\[+a_2 \frac{d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} \leq a_3 \frac{d(x_{2n}, x_{2n+1})}{1-a_1-a_2}\]

Since \(a_1 + a_2 + a_3 < 1\), it follows that \(c < 1\).

Proceeding in this way, we get

\[d(x_{2n+1}, x_{2n+2}) \leq c^2d(x_{2n-1}, x_{2n}) \leq c^3d(x_{2n-2}, x_{2n-1}) \leq \ldots \leq c^{2n+1}d(x_0, x_n)\]

By routine calculation the following inequality holds for \(k > n\),

\[d(x_n, x_{n+k}) \leq \sum_{i=1}^{k} d(x_{n+i-1}, x_{n+i})\]
\[ \leq \sum_{i=1}^{k} c^{n+i-1} d(hx_0, hx_1) \]
\[ \leq \left( \frac{c^n}{1-c} \right) d(hx_0, hx_1) \]

\[ \rightarrow 0 \text{ as } n \rightarrow \infty \quad (\text{Since } c < 1) \]

Hence \( \{hx_n\} \) is a Cauchy sequence. Since \( X \) is complete, sequence \( \{hx_n\} \) will converge to a point \( z \) in \( X \). Since \( \{fx_{2n}\} \) and \( \{gx_{2x+1}\} \) are subsequences of \( \{hx_n\} \), they will also converge to the same point \( z \).

Using continuity of \( h \) and \((1.2.1)\), we see that
\( h(hx_{2n}) \rightarrow hz \) and \( f(hx_{2n}) = h(fx_{2n}) \rightarrow hz \). Then from
\((1.2.2)\), we have

\[
\begin{align*}
\text{d}(fhx_{2n}, g_z) & \leq a_1 \frac{d(hx_{2n}, fhx_{2n})}{d(hhx_{2n}, h(z))} \frac{d(h(z), g(z))}{d(hhx_{2n}, h(z))} \\
& \quad + a_2 \frac{d(h(z), g(z))[1+d(hhx_{2n}, fhx_{2n})]}{[1+d(hhx_{2n}, h(z))]} \\
& \quad + a_3 d(hhx_{2n}, hz).
\end{align*}
\]

as \( n \rightarrow \infty \), we get

\[
\text{d}(h(z), g(z)) \leq a_1 \text{d}(h(z), g(z)) + a_2 \text{d}(h(z), g(z))
\]
or,

\[
\text{d}(h(z), g(z)) \leq a_1 \text{d}(h(z), g(z)) + a_2 \text{d}(h(z), g(z))
\]
\[ d(h(z), g(z)) \leq (a_1 + a_2) \ d(h(z), g(z)) \]

\[ \leq d(h(z), g(z)) \quad \text{as} \quad (a_1 + a_2 < 1) \]

which is a contradiction, hence \( hz = gz \). Similarly \( hz = fz \). Again from (1.2.2) we have,

\[
d(fx_{2n}, gz) \leq a_1 \frac{d(hx_{2n}, fz_{2n}) d(hz, gz)}{d(hx_{2n}, hz)} + a_2 \frac{d(hz, gz)[1 + d(hx_{2n}, fx_{2n})]}{[1 + d(hx_{2n}, hz)]} + a_3 d(hx_{2n}, hz)\]

As \( n \to \infty \), we get \( d(z, g(z)) \leq a_3 d(z, gz) \)

So that \( z = gz \).

Thus \( z \) is a common fixed point of \( f, g \) and \( h \).

Now to prove the uniqueness of the fixed point, let \( u \) and \( v \) (\( u \neq v \)) be two points of \( X \) such that \( fu = gu = hu = u \) and \( fv = gv = hv = v \). Then

\[
d(u, v) = d(fu, gv) \]

\[ \leq a_1 \frac{d(hu, fu) \ d(hv, gv)}{d(hu, hv)} + a_2 \frac{d(hv, gv)[1 + d(hu, fu)]}{[1 + d(hu, hv)]} + a_3 d(hu, hv)\]

\[ \leq a_3 d(u, v) < d(u, v) \]

which is a contradiction. Hence \( u = v \). This completes our proof.
Remark 1: If we set $f=g$, $h=I_x$ (Identity map on $X$) and $a_2=0$ in Theorem 1, we get result identical with theorem D.

Remark 2: If $f=g$ and $a_1=a_2=0$, then Theorem C appears as a special case of Theorem 1.

Consider the following example which shows the generality of the Theorem 1.

Example 1: Let $X = [0, 1]$ with usual metric $d$. If $f, g$ and $h$ maps $X$ into itself and defined as follows:

$$
fx = \begin{cases} 
x/4 + 3/8, & x \neq 0 \\
1/2 & x = 0,
\end{cases}
$$

and $hx = 1-x$, $x \in X$.

Clearly $h$ is continuous in $[0, 1]$, $h$ commutes with each of the maps $g$ and $h$. Also

$$
fX = [3/8, 5/8] \quad hX = X,
$$

$$
gX = [2/5, 3/5] \quad hX = X,
$$

$$
hX = X,
$$

Finally $f$, $g$ and $h$ have a unique common fixed point $1/2$.

Now we shall prove:

THEOREM (2): Let, $f, g$ and $h$ be three self mappings of a complete metric space $(X, d)$ such that following holds:
(1.2.4) \( fh = hf, \ gh = hg, \ f(X) \subseteq h^r(X) \) and \( g(X) \subseteq h^r(X) \), where \( r \) is some positive integer;

(1.2.5) there exist positive integers \( m, n \) such that for all \( x, y \) in \( X \),

\[
d(f^m x, g^n y) \leq a_1 \frac{d(h^r x, f^m x) \cdot d(h^r y, g^n y)}{d(h^r x, h^r y)} + a_2 \frac{d(h^r y, g^n y)[1 + d(h^r x, f^m x)]}{[1 + d(h^r x, h^r y)]} + a_3 \ d(h^r x, h^r y),
\]

where \( h x \neq h y \), \( a_1, a_2, a_3 \geq 0 \), \( a_1 + a_2 + a_3 < 1 \). If \( h^r \) is continuous, then \( f, g \) and \( h \) have a unique common fixed point.

Proof: From (1.2.4), we have

\[
f^m h^r = h^r f^m; \ g^n h^r = h^r g^n; \ f^m(X) \subseteq f(X) \subseteq h^r(X)
\]

and \( g^n(X) \subseteq g(X) \subseteq h^r(X) \). Therefore by Theorem 1, there exists a unique fixed point \( z \) in \( X \) such that

(1.2.6) \( z = h^r z = f^m z = g^n z \).

This gives, \( h z = h^r(h z) = f^m(h z) = g^n(h z) \).

Therefore \( h z \) is a common fixed point of \( f^m, g^n \) and \( h^r \), also,

\[
f z = h^r(f z) = f^m(f z).
\]
So $fz$ is a common fixed point of $h^r$ and $f^m$, similarly $gz$ is a common fixed point of $h^r$ and $g^n$. Using (1.2.5), for $x = fz$ and $y = gz$, we get $gz = fz$. Thus $fz = gz = hz$ are common fixed points of $f^m$, $g^n$ and $h^r$. The unicity of $z$ then implies that $z$ is unique common fixed point of $f$, $g$ and $h$.

This completes the proof.

**Theorem (3)**: Let $f, g$ and $h$ are three self mappings of a complete metric space $(X,d)$ and satisfying the following conditions:

1. $fgh = hfg$; $gfh = hgf$; $fg(X) \subseteq h(X)$ and $gf(X) \subseteq h(X)$ for all $x, y$ in $X$.
2. $d(fgx, gfy) \leq a_1 \frac{d(hx, fgx)d(hy, gfy)}{d(hx, hy)} + a_2 \frac{d(hy, gfy)[1+d(hx, fgx)]}{[1+d(hx, hy)]} + a_3 d(hx, hy)$

where $hx \neq hy$, $a_1, a_2, a_3 \geq 0$, $a_1 + a_2 + a_3 < 1$. If $h$ is continuous then $f, g$ and $h$ have a unique common fixed point in $X$.

**Proof**: Let $fg = S_1$ and $gf = S_2$, then by (1.2.8), we have

$$d(S_1x, S_2y) \leq a_1 \frac{d(hx, S_1x)}{d(hx, hy)} \frac{d(hy, S_2y)}{d(hx, hy)}$$

$$+ a_2 \frac{d(hy, S_2y)[1+d(hx, S_1x)]}{[1+d(hx, hy)]} + a_3 d(hx, hy)$$
holds for all $x, y$ in $X$, with $hx \neq hy$, $a_1 + a_2 + a_3 \geq 0$, $a_1 + a_2 + a_3 < 1$ and conditions $S_2 h = hS_2$, $S_1 h = hS_1$, $S_1(X) \subseteq h(X)$, $S_2(X) \subseteq h(X)$ are satisfied. Further $h$ is continuous self-mapping of $X$, therefore by Theorem 1, there exists a unique fixed point $z$ such that

$$z = S_1 z = S_2 z = hz.$$ 

Also $fz = f(S_2 z) = fg(fz) = S_1(fz)$

and $gz = g(S_1 z) = gf(fz) = S_2(fz)$.

This means that $fz$ is a fixed point of $S_1$ and $gz$ is a fixed point of $S_2$. The uniqueness of $z$ implies,

$$z = fz = gz = hz.$$ 

This completes the proof.

**Remark 3**: If we put $f = g$ and $h = Ix$ (Identity map) and $a_2 = 0$, then theorem E appears as a special case of our Theorems 3.

**Remark 4**: If we put $f = g$, $h = Ix$ and $a_1 = 0$, then theorem D appears as particular case of our Theorem 3.

1.3 In this section we obtain some fixed point theorems for three self maps satisfying a rational expression.

In 1978, Fishor[41] proved the following theorem:
THEOREM F: Let $f$ and $g$ be self mappings of a complete metric space $(X,d)$ such that

$$(1.3.1) \quad d(fx,gy) \leq a \frac{[d(x,fx)]^2 + [d(y,gy)]^2}{[d(x,fx) + d(y,gy)]}$$

for all $x,y$ in $X$, for which $d(x,fx) + d(y,gy) \neq 0$, where $0 \leq a < 1$. Then $f$ and $g$ have a common fixed point $z$. Further if $d(x,fx) + d(y,gy) = 0$ implies $d(fx,gy) = 0$, then $z$ is the unique common fixed point of $f$ and $g$.

We generalize Theorem F, by adding an additional factor in numerator and denominator both. In fact we give our Theorem 1 of [105].

THEOREM(4): Let $f$, $g$ and $h$ be three continuous self mappings of a complete metric space $(X,d)$ which satisfy

$$(1.3.2) \quad fh = hf, \quad gh = hg; \quad f(X) \subseteq h(X) \text{ and } g(X) \subseteq h(X),$$

$$(1.3.3) \quad d(fx,gy) \leq a \frac{d(hx,fx) + [d(hx,hy)]^2 + d(hx,gy)d(hy,fx)}{d(hy,fx) + d(hx,hy) + d(hy,gy)}$$

for all $x,y$ in $X$ with $x \neq y$, $0 < k < 1$, and $d(hx,fx) + d(hx,hy) + d(hy,gy) \neq 0$ and let $h$ is continuous, then $f,g$ and $h$ have a unique common fixed point.

Proof: Let $x_0$ be any point in $X$, let $x_1$ in $X$ be such that $hx_1 = fx_0$ and $x_2$ in $X$ be such that $hx_2 = gx_1$. In general, we can choose $x_{2n+1}$ and $x_{2n+2}$ such that
(1.3.4) \( h_{x_{2n+1}} = f_{x_{2n}} ; h_{x_{2n+2}} = g_{x_{2n+1}} \) for \( n = 0,1,2 \ldots \),
we can do this since (1.3.2) holds.

Now from (1.3.3), we have

\[
d(h_{x_{2n+1}}, h_{x_{2n+2}}) = d(f_{x_{2n}}, g_{x_{2n+1}})
\]

\[
d(h_{x_{2n}}, h_{x_{2n+1}})d(h_{x_{2n+1}}, h_{x_{2n+2}}) + \left[ d(h_{x_{2n}}, h_{x_{2n+1}}) \right]^2 
+ \frac{d(h_{x_{2n}}, h_{x_{2n+2}})d(h_{x_{2n+1}}, h_{x_{2n+2}})}{d(h_{x_{2n+1}}, h_{x_{2n+1}}) + d(h_{x_{2n}}, h_{x_{2n+1}}) + d(h_{x_{2n+1}}, h_{x_{2n+2}})}
\]

\[
= a \ d(h_{x_{2n}}, h_{x_{2n+1}}).
\]

This yields that,

\[
d(h_{x_{2n+1}}, h_{x_{2n+2}}) \leq a \ d(h_{x_{2n}}, h_{x_{2n+1}}).
\]

Proceeding in this way, we get

\[
d(h_{x_{2n+1}}, h_{x_{2n+2}}) \leq a \ d(h_{x_{2n}}, h_{x_{2n+1}})
\leq a^2 \ d(h_{x_{2n-1}}, h_{x_{2n}}) \leq \ldots \leq a^{2n+1} d(h_{x_0}, h_{x_1}).
\]

Consider for \( k > n \), we have,

\[
d(h_{x_n}, h_{x_{n+k}}) \leq \sum_{i=1}^{k} d(h_{x_{n+i-1}}, h_{x_{n+i}})
\]

\[
\leq \sum_{i=1}^{k} a^{n+i-1} d(h_{x_0}, h_{x_1}) \leq \ldots \leq \frac{a^n}{1-a} d(h_{x_0}, h_{x_1}), (a < 1),
\]

which tends to zero as \( n \to \infty \). It follows that \( \{h_{x_n}\} \) is
a Cauchy sequence. By the completeness of $X$, there exists $z$ in $X$ such that $\{hx_n\}$ tends to $z$. From (1.3.2), $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ also converges to $z$.

Using continuity of $h$ and (1.3.2), we have

(1.3.5) $f(hx_{2n}) = h(fx_{2n}) \to hz$,

(1.3.6) $g(hx_{2n+1}) = h(gx_{2n+1}) \to hz$,

(1.3.7) $h(hx_{2n}) \to hz$.

Now, we show that $f(hx_{2n}) = fz$. If we assume that, $f(hx_{2n}) \neq fz$, then $d(fhx_{2n}, fz) \geq 0$. Consider the inequality

$$d(fhx_{2n}, fz) \leq d(fhx_{2n}, ghx_{2n+1}) + d(ghx_{2n+1}, fz).$$

Using (1.3.3), we have

$$d(f(hx_{2n}), fz)$$

$$d(hhx_{2n}, fhx_{2n})d(hhx_{2n+1}, gx_{2n+1}) + [d(hhx_{2n}, hhx_{2n+1})^2]$$

$$+d(hhx_{2n}, gx_{2n+1})d(hhx_{2n+1}, fhx_{2n})$$

$$\leq a \frac{d(hhx_{2n+1}, fhx_{2n}) + d(hhx_{2n}, hhx_{2n+1}) + d(hhx_{2n+1}, ghx_{2n})}{d(hhx_{2n+1}, fhx_{2n}) + d(hhx_{2n+1}, ghx_{2n})}.$$  

On letting $n \to \infty$ and applying (1.3.5), (1.3.6) and (1.3.7), we get

$$d(hz, fz) \leq 0.$$
leading to a contradiction. Hence it follows that, 
\[ d(hz, fz) = 0, \] which implies \( hz = fz \).

Similarly we can prove that, \( gz = hz \).

Thus we have \( fz = hz = gz \).

Also from (1.3.2), we have

\[ (1.3.8) \ h(hz) = h(fz) = f(hz) = f(fz) = h(gz) = g(hz) \]

\[ = g(fz) = g(gz) \]

By (1.3.3) and (1.3.8), if \( fz = g(fz) \), we have

\[ d(hz, hz) + d(hfzg(fz)) + [d(hz, h(fz))]^2 \]

\[ + d(hz, g(fz))d(h(fz), fz) \]

\[ d(fz, g(fz)) \leq a \frac{d(h(fz), fz) + d(hz, h(fz)) + d(h(fz), g(fz))}{d(h(fz), fz) + d(hz, h(fz)) + d(hz, g(fz))} \]

\[ = a \ d(fz, g(fz)), \]

which is a contradiction, therefore \( d(fz, g(fz)) = 0 \).

Hence, \( fz = g(fz) \). Now (1.3.8) gives,

\[ fz = g(fz) = f(fz) = h(fz), \]

which shows that \( fz \) is a common fixed point of \( f, g \) and \( h \).

Now to prove the uniqueness of fixed point, let \( u \) and \( v \) be two distinct points in \( X \) such that \( fu = gu = hu = u \) and \( fv = gv = hv = v \). Then (1.3.3) we have,
\[ \frac{d(u,v)d(u,v)+[d(u,v)]^2+d(u,v)d(u,v)}{d(u,v)+d(u,v)+d(u,v)} \leq (a/2) d(u,v) < d(u,v), \]

a contradiction. Hence \( u = v \).

This implies the uniqueness of common fixed point of \( f, g \) and \( h \). This completes the proof.

Remark 5: Yeh[154] has considered, three mappings \( f, g \) and \( h \) as continuous, but here we consider only one mapping as continuous.

**Corollary (1):** Let \( f \) be a self mapping of a complete metric space \( (X, d) \) and satisfying the inequality

\[ d(fx, fy) \leq a \frac{d(x, fx)d(y, fy) + [d(x, y)]^2+d(x, fy)d(y, fx)}{d(x, fx) + d(x, y) + d(x, fy)} \]

for all \( x, y \) in \( X \) with \( x \neq y \), \( 0 < a < 1 \) and \( d(x, fx) + d(x, y) + d(x, fy) \neq 0 \), then \( f \) has unique fixed point.

Consider the following example, which shows the generality of above Corollary 1.

**Example 2:** Let \( X = [0, 1] \) be a usual metric space and \( f : [0, 1/2] \to [0, 1/2] \) be defined such that

\[ fx = \begin{cases} \frac{x}{3} & \text{when } 0 \leq x \leq 1/2, \\ 1/4 & \text{when } x = 1/2. \end{cases} \]
This example satisfies all the conditions and inequality of the above Corollary for $1/2 \leq a \leq 1$. Although the function is discontinuous at $x = 1/2$, also $f(0) = 0$ i.e. zero is the unique fixed point of $f$.

**COROLLARY (2)**: Let $A$ be a family of continuous self mappings of a complete metric space $(X,d)$. Suppose there is a map $h$ in $A$ such that to each pair $f$ and $g$, the conditions (1.3.2) and (1.3.3) holds, for all $x, y$ in $X$. Then $f$ has a unique fixed point, which is a unique common fixed point for the family of $A$.

1.4 Following Sehgal[127], we prove a Theorem which is a generalization of Theorem 4.

**THEOREM (5)**: Let $f$, $g$ and $h$ be three self mappings of a complete metric space $(X,d)$ such that $h$ is continuous and $f$, $g$ and $h$ satisfy the condition (1.3.2) and there exists two positive integers $m$ and $n$ such that

$$d(hx,f^m x) = d(hy,g^n y) + d[(hx,hy)]^2$$

$$d(f^m x, g^n y) \leq a + \frac{d(hx,g^n y) d(hy,f^m x)}{d(hy,f^m x) + d(hx,hy) + d(hy,g^n y)}$$

for all $x, y$ in $X$, with $x \neq y$. $0 < a < 1$ and $d(hy,f^m x) + d(hx,hy) + d(hy,g^n y) \neq 0$, then $f, g$ and $h$ have unique common fixed point.

**Proof**: It follows from (1.3.2), that $f^m h = hf^m$; $g^n h = hg^n$;
\[ f^m(X) \subseteq f(X) \subseteq h(X) \text{ and } g^n(X) \subseteq g(X) \subseteq h(X), \] thus by Theorem 4 there is a unique fixed point \( z \) in \( X \) such that

\[ z = hz = f^mz = g^nz. \]

Also, \( h(fz) = f(hz) = fz = f(f^mz) = f^m(fz) \).

This means that \( fz \) is a common fixed point of \( h \) and \( f^m \).
Similarly \( g \) is a common fixed point of \( h \) and \( g^n \). The uniqueness of \( z \) implies \( fz = gz = hz = z \).

This completes the proof.

Finally, we prove

**THEOREM 6**: Let \( h \) and \( h_i \) (\( i=1,2,...,k \)) be self mappings of a complete metric space \((X,d)\), such that \( h \) is continuous and \( h, h_i \) satisfy the following conditions:

(1.4.2) \( h_i h_j = h_j h_i \); \( hh_i = h_i h \) for \( i, j = 1,2,...,k \).

(1.4.3) \( f(X) \subseteq h(X) \), where \( f = h_1, h_2, h_3, ..., h_k \).

(1.4.4) for \( f=g \), the condition (1.4.2) holds, then \( h \)

\( h_i \) (\( i = 1,2,...,k \)) have a unique common fixed point.

**Proof**: By Theorem 5, \( f \) and \( g \) have a unique common fixed point \( z \) in \( X \). Thus \( fz = hz = z \). Then

\[ h_i(fz) = h_i(hz) = h_i z. \]

Hence \( h_i z \) is a common fixed point of \( f \) and \( h \). By the uniqueness of the common fixed point of \( f \) and \( h \), we have \( h_i z = z \). This completes the proof of the theorem.

\[ * * * * \]