ON FIXED POINT THEOREMS IN HAUSDORFF SPACES
CHAPTER V

ON FIXED POINT THEOREMS IN HAUSDORFF SPACES

5.1 Some unique fixed point theorems in Hausdorff space have been established in the present chapter.

Let \((X,d)\) be a metric space. A mapping \(f: X \rightarrow X\) is called a contraction map if

\[(5.1.1) \quad d(fx, fy) \leq k \, d(x, y)\]

for all \(x, y\) in \(X\) and \(0 \leq k < 1\). The well known Banach contraction principle is as follows:

A contraction mapping of a complete metric space into itself, always has a unique fixed point.

A mapping \(f : X \rightarrow X\) is said to be non-expansive if

\[(5.1.2) \quad d(fx, fy) \leq d(x, y)\]

for all \(x, y\) in \(X\).

Let \((X,d)\) be a metric space. A mapping \(f\) of \(X\) into itself is said to be contractive if

\[(5.1.3) \quad d(fx, fy) < d(x, y), \quad x \neq y \in X.\]

Kannan[66] proved the following theorem.
THEOREM A: Let \((X,d)\) be a metric space and \(f: X \rightarrow X\) be a mapping, such that

\[(5.1.4)\quad d(fx, fy) \leq k[d(x, fx) + d(y, fy)]\]

for all \(x, y\) in \(X\), \(0 \leq k < 1/2\), then \(f\) has a unique fixed point.

In 1971 Singh and Zorzitto[146] obtained the following theorem in Hausdorff space.

THEOREM B: Let \((X,d)\) be a Hausdorff space and \(f:X \rightarrow X\) be a continuous mapping. Let \(F: XxX \rightarrow [0, \infty)\) be a continuous mapping such that

\[(5.1.5)\quad F(fx, fy) \leq F(x, y) \text{ for all } x, y \text{ in } X.\]

If there exists \(x \in X\) such that \((f^nx_0)\) has a convergent subsequence, then \(f\) has a fixed point.

Recently in 1985 Kiventidis[77] obtained the following theorem.

THEOREM C: Let \(X\) be a Hausdorff space, \(f: X \rightarrow X\) continuous function, and \(F: XxX \rightarrow R^+\) be a continuous function such that

\[(5.1.6)\quad F(x, y) \neq 0, x \neq y.\]

\[(5.1.7)\quad F(fx, fy) \leq F(x, y) - \omega(F(x, y)) \text{ for all } x, y \in X.\]

Where \(\omega: R^+ \rightarrow R^+\) is continuous function, with \(0 < \omega(r) < r\) for all \(r \in R^* - \{0\} \).
If for some $x_0$ in $X$ the sequence $x_n = \{f^nx_0\}$ has a convergent subsequence, then $f$ has a unique fixed point.

5.2 Now we combine the idea of Kannan[66] and Kiventidis[77] and have obtained, the following Theorem.

**THEOREM 1**: Let $f$ be a continuous function of a Hausdorff space $X$ into itself, and let $F : XX \rightarrow R^+$ be a continuous function satisfying (5.1.6) and

\[(5.2.1) F(fx, fy) \leq \frac{1}{2} [F(x, fx) + F(y, fy)] - \omega(F(x, y))\]

for all $x \neq y$ in $X$, where $\omega : R^+ \rightarrow R^+$ as a continuous function, with $0 < \omega(r) < r$ for all $r \in R^+ - \{0\}$.

If for some $x_0 \in X$, the sequence of iterates $\{f^nx_0\}$ has a convergent subsequence $\{f^kx_0\}$ converging to $z$ in $X$, then $z$ is a unique fixed point of $f$.

**Proof**: From (5.2.1) we have,

$$\omega[F(x, y)] \leq \frac{1}{2} [F(x, fx) + F(y, fy)] - F(fx, fy).$$

Then the series,
\[
\sum_{k=0}^{\infty} (F(f^kx, f^{k+1}x)) = \omega(F(f^0x, f^1x)) + \omega(F(f^1x, f^2x)) + \ldots
\]
\[
+ \omega(F(f^{n-1}x, f^nx)) + \omega(F(f^nx, f^{n+1}x))
\]
\[
\leq \frac{1}{2}[F(x, fx) + F(fx, f^2x)] - F(fx, f^2x)
\]
\[
+ \frac{1}{2}[F(fx, f^2x) + F(f^2x, f^3x)] - F(f^2x, f^3x) + \ldots
\]

(inequality continued on next page)
\[ + \frac{1}{2}[F(f^{n-1}x, f^n x) + F(f^n x, f^{n+1} x)] - F(f^n x, f^{n+1} x) \]
\[ + \frac{1}{2}[F(f^n x, f^{n+1} x) + F(f^{n+1} x, f^{n+2} x)] - F(f^{n+1} x, f^{n+2} x) \]
\[ = \frac{1}{2}[F(x, fx)] - \frac{1}{2}[F(f^{n+1} x, f^{n+2} x)] \]
\[ \leq \frac{1}{2} F(x, fx), \text{ for all } n \geq 1, x \in X. \]

Thus the series  \( \sum_{k=0}^{\infty} \Omega[F(f^k x, f^{k+1} x)] \) is a monotone non-increasing sequence of non-negative real numbers, which must converge along with all its subsequences. Thus, we have

\[ \lim_{k \to \infty} \omega(F(f^k x, f^{k+1} x)) = 0. \]

Since  \( \omega(0) = 0 \), from the continuity of  \( \omega \), we have

\[ \lim_{k \to \infty} (F(f^k x, f^{k+1} x)) = \omega(\lim_{k \to \infty} F(f^k x, f^{k+1} x)) = 0 \]

which implies  \( \lim_{k \to \infty} F(f^k x, f^{k+1} x) = 0. \)

Now from the continuity of  \( F \) and  \( f \), we arrive at the relation,

\[ F(u, fu) = F(\lim_{k \to \infty} f^k x_0, \lim_{k \to \infty} f^{k+1} x_0) \]
\[ = \lim_{k \to \infty} F(f^k x_0, f^{k+1} x_0) = 0. \]

From (5.1.6), we get  \( z = fz \).
If \( z \neq u \) be another fixed point, then

\[
F(z,u) = F(fz,fu) \leq \frac{1}{2}[F(z,z) + F(u,u)] - \omega(f(z,u))
\]

\[
= - \frac{1}{2} \omega(F(z,u))
\]

which is a contradiction. Hence \( f \) has a unique fixed point.

This completes the proof.

**COROLLARY (1):** Let \((X,d)\) be a complete metric space. If \( f \) be a continuous mapping of \( X \) into itself such that

\[(5.2.2) \quad d(fx, fy) \leq \frac{1}{2}[d(x, fx) + d(y, fy)] - \omega(d(x, y))\]

for all \( x, y \) in \( X \), where \( \omega: \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous function, with \( 0 < \omega(r) < r \) for all \( r \in \mathbb{R}^+ - \{0\} \), then \( f \) has a unique fixed point.

**Proof:** From Theorem 1,

\[
\lim_{k \to \infty} d(f^k x, f^{k+1} x) = 0, \text{ for all } x \text{ in } X,
\]

follows that the sequence \( \{f^k x\} \) is a Cauchy sequence in \( X \) and therefore is convergent.

We complete the proof as in Theorem 1.

**Remark 1:** The result of Kannan[66] follows by taking \( \omega = 0 \), in Corollary 1.
Remark 2: If $X$ is a compact space then the condition (5.2.1) in Theorem 1 can be weakened as follows [39]

$$F(fx, f^2x) < \frac{1}{2}[F(x, fx) + F(fx, f^2x)]$$

(5.2.3) $F(fx, fy) < \frac{1}{2}[F(x, fx) + F(y, fy)]$

where $F : X \times X \to \mathbb{R}^+$ is a lower semi-continuous function.

THEOREM (2): Let $f$ and $g$ be continuous functions of a Hausdorff space $X$ into itself and let $F : X \times X \to \mathbb{R}^+$ be a continuous function satisfying (5.1.6) and

(5.2.4) $F(fx, g^2y) \leq \frac{1}{2}[F(x, fx) + F(fx, f^2y)] - \omega[F(x, y)], x \neq y$ in $X$.

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function, with $0 < \omega(r) < r$ for all $r \in \mathbb{R}^+ - \{0\}$.

If for some $x_0 \in X$ the sequence $\{x_n\}$, where $fx_{2n} = x_{2n+1}$ and $gx_{2n+1} = x_{2n+2}$, for $n = 0, 1, 2, \ldots$ has a convergent subsequence of the type $\{x_{(2p+1)n}\}$, where $p$ in $N$ is fixed and $n$ in $N$, then $f$ and $g$ have a unique common fixed point.

Proof: For simplicity we suppose that $\{x_{3n}\}$ is convergent to $z$, we consider for sub-sequence $\{x_{3(2k)}\}$ which also converges to $z$.

From the continuity of $f$ and $g$, we have

$$fz = f \lim x_{3(2k)} = \lim f(x_{3(2k)}) = \lim x_{3(2k+1)}$$

$$g(fz) = g \lim x_{3(2k+1)} = \lim gx_{3(2k+1)} = \lim x_{3(2k+2)}$$
Thus as \( k \to \infty \), we have,

\[
F(z, fz) = F(\lim x_3(2k), \lim x_3(2k+1)) \\
= \lim F(x_3(2k), x_3(2k+1)) = z \\
= \lim F(x_3(2k+1), x_3(2k+2)) \\
= F(fz, gfz)
\]

where \( z = \lim \frac{F(x_n, x_{n+1})}{n \to \infty} \).

As in theorem 1, we can prove that \( z = 0 \).

From (5.1.6) implies \( z = fz \) and \( gz = z \).

If \( u \neq z \) be another fixed point of \( f \) and \( g \), then

\[
F(z, u) = F(fz, gu) \leq \frac{1}{\alpha} [F(z, fz) + F(u, gu)] - \omega(F(z, u)) \\
\leq \frac{1}{\alpha} [F(z, z) + F(u, u)] - \omega(F(z, u))
\]

which is impossible. Therefore \( z = u \).

Hence \( f \) and \( g \) have a unique fixed point.

We completes the proof.

**COROLLARY (2)**: Let \( (X, d) \) be a metric space and \( f : X \to X \), \( g : X \to X \) be continuous functions such that

\[
d(fx, gy) \leq \frac{1}{\alpha} [d(x, fx) + d(y, fy)] - \omega(d(x, y)),
\]

for all \( x \neq y \) in \( X \), where \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous
function, with \(0 < \omega(r) < r\), for all \(r \in R^+ - \{0\}\). If for some \(x_0\) in \(X\) the sequence \(\{x_n\}\), where \(f_{2n}=x_{2n+1}\) and \(g_{2n+1}=x_{2n+2}\), \(n = 0,1,2,\ldots\) has a convergent subsequence of the type \(\{x_{3n}\}\), then \(f\) and \(g\) have a unique common fixed point.

Remark 3: If in Corollary 2, \((X,d)\) is a complete metric space then the sequence \(\{x_n\}\) is convergent for all \(x_0\) in \(X\), because

\[
\lim_{n \to \infty} d(x_n,x_{n+1}) = 0.
\]

Now we give Theorem for sequence of mappings.

**THEOREM (3):** Let \(f_1, f_2, \ldots, f_k\) be continuous functions of a Hausdorff space \(X\) into itself and let \(F : XX \to R^+\) be a continuous function, which satisfy (5.1.6) and

\[
(5.2.5) \quad F(f_{i+1}x,f_{i+1}y) \leq \frac{1}{k}[F(f_ix,f_ix) + F(y,f_{i+1}y)] - \omega(F(x,y))
\]

\(\forall x \neq y\) in \(X\), where \(\omega : R^+ \to R^+\) is a continuous function, with \(0 < \omega(r) < r\), for all \(r \in R^+ - \{0\}\). If for some \(x_0\) in \(X\), the sequence \(\{x_n\}\), where \(x_1 = f x_0\), \(x_2 = g x_1\), \(x_k = f x_{k-1}, x_{k+1} = f x_k, x_{k+2} = g x_{k+1}, \ldots x_{2k} = f_k x_{2k-1}\)

\(x_{nk+1} = f x_{nk}, x_{nx+2} = g x_{nk+1}, \ldots x_{(n+1)k} = f_k x_{(2n+1)k-1}\)

for \(n = 0,1,2,\ldots\) has a convergent subsequence of the type \(x_{(mk+1)n}\), where \(m\) in \(N\) is fixed and \(n\) in \(N\) then \(f_1, f_2, \ldots, f_k\) have a unique fixed point.
Proof: It is enough to observe that
\[ F(x_n, x_{n+1}) \leq \frac{1}{2}[F(x_{n-1}, x_n) + F(x_n, x_{n+1})] \]
for \( n = 0, 1, 2, \ldots \), and \( \lim_{n \to \infty} F(x_n, x_{n+1}) = 0 \) and the proof follows as in Theorem 2.

Since every sequence in a compact Hausdorff space have a convergent subsequence, we have the following theorems.

**THEOREM (4):** Let \( f \) and \( g \) are continuous functions of a compact Hausdorff space \( X \) into itself. Let \( F \) be a continuous function of \( X \times X \) into the non-negative reals satisfying (5.1.6) and

\[ (5.2.6) \quad F(fx, gy) \leq \frac{1}{2} [F(x, fx) + F(y, gy)] - \omega(F(x, y)) \]

for all \( x \neq y \) in \( X \), where \( \omega: \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function with \( 0 < \omega(r) < r \) for all \( r \in \mathbb{R}^+ - \{0\} \), then \( f \) and \( g \) have a unique common fixed point.

**THEOREM (5):** Let \( f_1, f_2, \ldots, f_k \) be continuous functions of a compact Hausdorff space \( X \) into itself and let \( F: X \times X \to \mathbb{R}^+ \) be continuous function such that condition (5.1.6) holds and

\[ (5.2.6) \quad F(f_i x, f_{i+1} y) \leq \frac{1}{2} [F(x, f_i x) + F(y, f_{i+1} y)] - \omega(F(x, y)) \]

for all distinct \( x, y \) in \( X \), where \( \omega: \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous.
function with $0 < \omega(r) < r$, for all $r \in \mathbb{R}^+ - \{0\}$, then $f_1, f_2, \ldots, f_k$ have a unique fixed point.

5.3 In this section we obtain some results for a new class of mappings. In fact we prove:

**THEOREM (6):** Let $f$ be a continuous function of a Hausdorff space $X$ into itself and $F$ be a continuous function of $X \times X \times X$ into the non-negative reals satisfying (5.1.6) and

$$(5.3.1) \quad F(fx,fy) \leq \frac{1}{4}[F(x,fx)+F(y,fy)] + \frac{1}{4}[F(x,fy)+F(y,fx)] - \omega(F(x,y))$$

for all $x \neq y$ in $X$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function with $0 < \omega(r) < r$ for all $r \in \mathbb{R}^+ - \{0\}$. If for some $x_0$ in $X$, the sequence of iterates $(f^n x_0)$ has a convergent subsequence, then $f$ has a unique fixed point.

**Proof:** By (5.3.1), we have

$$\omega(F(x,y)) \leq \frac{1}{4}[F(x,fx)+F(y,fy)] + \frac{1}{4}[F(x,fy)+F(y,fx)] - F(fx,fy)$$

Then

$$\sum_{k=0}^{n} \omega(F(f^k x, f^{k+1} x)) \leq \frac{1}{4}[F(x,fx)+F(fx,f^2 x)] + \frac{1}{4}[F(x,f^2 x)+F(fx,fx)]$$

$$- F(fx,f^2 x) + \frac{1}{4}[F(fx,f^3 x)+F(f^2 x,f^2 x)] - F(f^2 x,f^3 x) + \cdots$$

$$+ \frac{1}{4}[F(f^n x,f^{n+1} x) + F(f^{n+1} x,f^{n+2} x)]$$

(inequality continued on next page)
\[ + \frac{1}{4}[F(f^{n-1}x, f^{n+1}x) + F(f^{n}x, f^{n}x)] - F(f^{n}x, f^{n+1}x) \]
\[ + \frac{1}{4}[F(f^{n}x, f^{n+1}x) + F(f^{n+1}x, f^{n+2}x)] + \frac{1}{4}[F(f^{n}x, f^{n+2}x) \]
\[ + F(f^{n+1}x, f^{n+1}x)] - F(f^{n+1}x, f^{n+2}x) \]
\[ \leq \frac{1}{2}F(x, fx) - \frac{1}{2}F(f^{n+1}x, f^{n+2}x) \]
\[ \leq F(x, fx) \quad \text{for all } n \geq 1, \ x \text{ in } X. \]

Thus the series \[ \sum_{k=0}^{\infty} \omega(F(f^kx, f^{k+1}x)) \] is a monotone non-increasing sequence of non-negative reals, converges along with all its subsequences. Thus, we have
\[ \lim_{k \to \infty} \omega(F(f^kx, f^{k+1}x)) = \omega(\lim_{k \to \infty} F(f^kx, f^{k+1}x)) = 0 \]
which implies,
\[ \lim_{k \to \infty} F(f^kx, f^{k+1}x) = 0 \]

For the point \( x_0 \) in \( X \), we have \( \lim_{k \to \infty} f^nx_0 = z \), where\( \{f^kx_0\} \)
is a convergent subsequence of \( \{x_n\} \), so we get the relation
\[ F(z, fz) = F(\lim_{k \to \infty} f^kx_0, \lim_{k \to \infty} f^{k+1}x_0) \]
\[ = \lim_{k \to \infty} F(f^kx_0, f^{k+1}x_0) = 0 \]
from (5.1.6), we get \( z = fz \).

Therefore \( z \) is a fixed point of \( f \). To prove uniqueness of \( z \) suppose that \( u \) is another fixed point, then
\[ F(z,u) = F(fz, fu) \]
\[ \leq \frac{1}{4}[F(z, z) + F(u, u)] + \frac{1}{4}[F(z, u) + F(z, u)] - \omega(F(z, u)) \]

or,
\[ F(z, u) \leq -2 \omega(F(z, u)) \]

which is impossible. Hence \( f \) has a unique fixed point.

**COROLLARY (3)**: Let \((X, d)\) be a complete metric space. If \( f : X \to X \) is a function such that

\[(5.3.2) \ d(fx, fy) \leq \frac{1}{4}[d(x, fx) + d(y, fy)] + \frac{1}{4}[d(x, fy) + d(y, fx)] - \omega(d(x, y))\]

for all \( x = y \) in \( X \), where \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous function, with \( 0 < \omega(r) < r \) for all \( r \in \mathbb{R}^+ - \{0\} \), then \( f \) has a unique fixed point.

**Proof**: From theorem 6, we have

\[ \lim_{k \to \infty} d(f^k x, f^{k+1} x) = 0 \quad \text{for all } x \text{ in } X, \]

it follows that the sequence \( \{f^k x\} \) is Cauchy sequence in \( X \) and therefore is convergent.

We complete the proof as the Theorem 6.

**Remark 4**: If \( X \) is compact space, then the condition \((5.3.1)\) of Theorem 6 can be weekend as follows:

\[(5.3.3) \ \text{for all } x \in X, \ x \neq fx \text{ holds} \]
\[ F(fx, f^2x) \leq \frac{1}{4} [F(fx, f^2x) + F(fx, f^2x)] + \frac{1}{4} [F(fx, f^2x) + F(fx, fx)] \]

or,

\[ F(fx, fy) \leq \frac{1}{4} [F(x, fx) + d(y, fy)] + \frac{1}{4} [F(x, f^2y) + F(y, fx)] \]

where \( F : X \times X \rightarrow \mathbb{R}^+ \) is a lower semi continuous function.

**THEOREM (7):** Let \( f \) and \( g \) be continuous function of a Hausdorff space \( X \) into itself and let \( F : X \times X \rightarrow \mathbb{R}^+ \) be a continuous function satisfying (5.1.6) and

\[ (5.3.4) \ F(fx, gy) \leq \frac{1}{4}[F(x, fx)+F(y, gy)]+\frac{1}{4}[F(x, gy)+F(y, fx)] - \omega(F(x, y)) \]

for all \( x \neq y \) in \( X \), where \( \omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a continuous function with \( 0 < \omega(r) < r \) for all \( r \in \mathbb{R}^+ - \{0\} \).

If for some \( x_0 \) in \( X \) the sequence \( \{x_n\} \), where \( fx_{2n} = x_{2n+1} \) and \( gx_{2n+1} = x_{2n+2} \), \( n = 0, 1, 2, \ldots \), has a convergent subsequence of the type \( \{x_{(2p+1)n}\} \), \( p \) in \( N \) is fixed, then \( f \) and \( g \) have a unique common fixed point.

**Proof:** For simplicity we suppose \( \{x_{3n}\} \) is convergent to \( z \). We consider the subsequence \( \{x_{3(2k)}\} \), which converges to \( z \).

From the continuity of \( f \) and \( g \) we have

\[ F(u, fu) = F(\lim x_{3(2k)}, \lim x_{3(2k+1)}) \]
or, \[ F(u, fu) = \lim_{n \to \infty} F(x_{3(2k)}, x_{3(2k+1)}) = z \]

\[ = \lim_{n \to \infty} F(x_{3(2k)+1}, x_{3(2k+2)}) \]

\[ = F(fz, gfz) \]

where \( z = \lim_{n \to \infty} F(x_n, x_{n+1}) \).

As in Theorem 1 we can prove that \( z = 0 \), from (5.1.6) implies \( z = fz \) and \( gz = z \). If \( u \neq z \) be another fixed point of \( f \) and \( g \), then

\[ F(z, u) = F(fz, gu) \]

\[ \leq \frac{1}{2}[F(z, z) + F(u, u)] + \frac{1}{4}[F(z, u) + F(u, z)] \]

\[ - \omega(F(z, u)) \]

\[ = \frac{1}{2}F(z, u) - \omega(F(z, u)) \leq F(z, u) \]

which is a contradiction.

Hence \( f \) and \( g \) have a unique common fixed point.

Thus the proof of the Theorem completes.

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