CHAPTER 4

COMMON FIXED POINT THEOREMS OF
SEQUENCE OF MAPPINGS
CHAPTER IV

COMMON FIXED POINT THEOREMS OF SEQUENCE OF MAPPINGS

4.1 In 1922, Banach[8] obtained the following:

**THEOREM A**: Let \( f \) be a mapping of a complete metric space \((X,d)\) into itself satisfying

\[
(4.1.1) \quad d(fx, fy) \leq k \, d(x, y)
\]

for all \( x, y \) in \( X \) and \( 0 \leq k < 1 \), then \( f \) has a unique fixed point.

Many authors have extended the Theorem of Banach[8] in different directions. Bonsall[10], Collins[28], Dubey and Singh[33], Singh and Russel[145], and Singh[143] have investigated the conditions under which the convergence of a sequence of contraction mapping to a mapping \( f \) of a metric space into itself implies the convergence of their fixed points to the fixed point of \( f \).

A partial solution of this problem was given by Bonsall[10] for the first time as follows:

**THEOREM B**: Let \((X,d)\) be a complete metric space. Let \( f_n \) \((n=1,2,\ldots)\) and \( f \) be contraction mapping of \( X \) into itself with the same Lipschitz's constant \( k < 1 \), and
with fixed points $z_n$ and $z$ respectively. Suppose that
\[ \lim_{n \to \infty} f_n x = f x \text{ for every } x \text{ in } X, \text{ then } \lim_{n \to \infty} z_n = z. \]

In 1974, Iseki[55] obtained the following result for sequence of self mappings.

**THEOREM C**: Let $(X,d)$ be a complete metric space, $f_n (n=1,2,\ldots)$ be a sequence of mappings of $X$ into itself. Suppose that there are constants $a_1, a_2, a_3 > 0$ such that for $x,y$ in $X$.

\[
(4.1.2) \quad d(f_i(x), f_j(y)) \leq a_1 [d(x,f_i(x))+d(y,f_j(y))] \\
+ a_2 [d(x,f_j(y))+d(y,f_i(x))]+a_3 d(x,y)
\]

where $2a_1 + 2a_2 + a_3 < 1$. Then the sequence of mappings $\{f_n\}$ has a unique fixed point.

The present chapter is devoted to a study of fixed points of sequence of contraction mappings. Here, we shall prove some fixed point theorems for sequence of mappings and then further extend the results so obtained to a family of mappings which maps $X$ into itself and satisfy a common condition.

4.2 In fact we prove:

**THEOREM (l)**: Let $X$ be a complete metric space, $\{f_n\}$ $(n=1,2,\ldots)$ a sequence of mappings of $X$ into itself.
Suppose that there are non-negative numbers $a_1, a_2, a_3, a_4, a_5$ such that for $x, y$ in $X$

\[(4.2.1) \quad d(f_i(x), f_j(y)) \leq a_1[d(x, f_1(x)) + d(y, f_j(y))] + a_2[d(x, f_j(y)) + d(y, f_i(x))] + a_3[d(x, y) + d(f_i(x), f_i^2(x))] + a_4[d(x, f_i^2(x)) + d(f_j(y), f_i^2(x))] + a_5[d(y, f_i^2(x))]\]

where $2a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 < 1$. Then the sequence of mappings \(\{f_n\}\) has a unique common fixed point.

**Proof**: Let \(x_0\) be any point in $X$, take $x_n = f_n(x_{n-1})$ \((n = 1, 2, \ldots)\), then we have,

\[d(x_1, x_2) = d(f_1(x_0), f_2(x_1)) \leq a_1[d(x_0, x_1) + d(x_1, x_2)] + a_2[d(x_0, x_2) + d(x_1, x_1)] + a_3[d(x_0, x_1) + d(x_1, x_2)] + a_4[d(x_0, x_2) + d(x_2, x_2)] + a_5 d(x_1, x_2)\]

or,

\[d(x_1, x_2) \leq (a_1 + a_3) d(x_0, x_1) + (a_1 + a_2 + a_3 + a_5) d(x_1, x_2) + (a_2 + a_4) d(x_0, x_2)\]
On simplifying we get,

\[ d(x_1, x_2) \leq p d(x_0, x_1) \text{ where } p = \frac{a_1 + a_2 + a_3 + a_4}{1 - a_1 - a_2 - a_3 - a_4 - a_5} \]

Similarly,

\[ d(x_2, x_3) \leq p d(x_1, x_2) \leq p^2 d(x_0, x_1) \]

inductively,

\[ d(x_n, x_{n+1}) \leq p^n d(x_0, x_1). \]

This means that sequence \( \{x_n\} \) is a Cauchy sequence.

Hence, by the completeness of \( X \), \( \{x_n\} \) converges to some point \( z \) in \( X \), such that

\[
(4.2.2) \quad \lim_{n \to \infty} x_n = z.
\]

Now, we shall show that \( z \) is a fixed point. For each \( m \) in \( I^+ \), we have

\[
d(z, f_n z) \leq d(z, x_{m+1}) + d(x_{m+1}, f_n(z))
\]

\[
= d(z, x_{m+1}) + d(f_{m+1}(x_m), f_n(z))
\]

\[
\leq d(z, x_{m+1}) + a_1 [d(x_m, f_{m+1}(x_m)) + d(z, f_n(z))] + a_2 [d(x_m, f_n z) + d(z, f_{m+1} x_m)]
\]

\[
+ a_3 [d(x_m, z) + d(f_{m+1}(x_m), f_{m+1}^2(x_m))]
\]

(inequality continued on next page)
+ a_4 [d(x_m, f_{m+1}^2(x_m)) + d(f_n(z), f_{m+1}^2(x_m))]  \\
+ a_5 [d(z, f_{m+1}^2(x_m))]

Letting m → ∞, we have

\[ d(z, f_n(z)) ≤ (a_1 + a_2 + a_4) d(z, f_n(z)) < d(z, f_n(z)). \]

Therefore \( d(z, f_n(z)) = 0 \), that is the point \( z \) is a common fixed point of \( f_n \).

To show that \( z \) is a unique common fixed point of all \( f_n \), we consider a point \( u \) in \( X \) such that \( f_n(u) = u \) for every \( n \). We have

\[ d(z, u) = d(f_n(z), f_n(u)) \]
\[ ≤ a_1 [d(z, f_n(z)) + d(u, f_n(u))] + a_2 [d(z, f_n(u)) + d(u, f_n(z))] \]
\[ + a_3 [d(z, u) + d(f_n(z), f_n^2(z))] \]
\[ + a_4 [d(z, f_n^2(z)) + d(f_n(u), f_n^2(z))] + a_5 d(u, f_n^2(z)) \]
\[ = (2a_2 + a_3 + a_4 + a_5) d(z, u) < d(z, u) \]

Hence \( d(z, u) = 0 \) i.e. \( z = u \).

This completes the proof of the theorem.

4.3 Nadler[91], considered separately the uniform convergence and pointwise convergence of a sequence of contraction mapping, with this object in view we shall prove:
THEOREM (2): Let \( \{f_n\} \) be a sequence of mappings of a complete metric space \( X \) into itself. Let \( x_n \) be a fixed point of \( f_n \) \( (n=1,2,...) \), and suppose that \( f_n \) converges uniformly to \( f_0 \). If \( f_0 \) satisfies the condition:

\[
(4.3.1) \quad d(f_0(x), f_0(y)) \leq a_1 [d(x, f_0(x)) + d(y, f_0(y))] \\
+ a_2 [d(x, f_0(y)) + d(y, f_0(x))] \\
+ a_3 [d(x, y) + d(f_0(x), f_0^2(x))] \\
+ a_4 [d(x, f_0^2(x)) + d(f_0(y), f_0^2(x))] \\
+ a_5 d(y, f_0^2(x))
\]

where \( a_1, a_2, a_3, a_4, a_5 \) are non-negative constants and \( 2a_1 + 2a_2 + 2a_3 + 2a_4 + a_5 < 1 \), then \( \{x_n\} \) converges to the fixed point \( x_0 \) of \( f_0 \).

We need the following Lemma.

Lemma 1: Let \( f_0, f : X \rightarrow X \) be a pair of maps on a metric space \( (X,d) \). If for all \( x, y \) in \( X \)

\[
(4.3.2) \quad d(f_0(x), f(y)) \leq a_1 [d(x, f_0(x)) + d(y, f(y))] \\
+ a_2 [d(x, f_0(y)) + d(y, f_0(x))] \\
+ a_3 [d(x, y) + d(f_0(x), f_0^2(x))] \\
+ a_4 [d(x, f_0^2(x)) + d(f(y), f_0^2(x))] + a_5 d(y, f_0^2(x))
\]
where $a_1, a_2, a_3, a_4, a_5$ are non-negative constants and 
$(2a_1+2a_2+a_3+a_4) < 1$, and $F(f)$ is a non empty set, then
$F(f)$ is a singleton of $F(f) = F(f)$, $(F(f) = x \in S : x = fx)$.

**Proof of Lemma**: Let $x_o \in F(f) \subset X$ be any fixed point
and $x_o \neq fx_o$ then by (4.3.2) we have,

$$d(x_o, f(x_o)) = d(f_o(x_o), f(x_o))$$

$$\leq a_1[d(x_o, f_o(x_o)) + d(x_o, f(x_o))]$$

$$+ a_2[d(x_o, f(x_o)) + d(x_o, f_o(x))]$$

$$+ a_3[d(x_o, x_o) + d(f_o(x_o), f_o^2(x_o))]$$

$$+ a_4[d(x_o, f_o^2(x_o)) + d(f(x_o), f_o^2(x_o))]$$

$$+ a_5 \; d(x_o, f_o^2(x_o))$$

$$= (a_1 + a_2 + a_4) d(x_o, f(x_o))$$

or, $(1-a_1-a_2-a_4) \; d(x_o, f(x_o)) \leq 0$.

Thus we arrive at a contradiction which implies 
$d(x_o, fx_o) = 0$. Therefore, $x_o \in F(f)$. Now let $z$ in $F(x_o)$
be arbitrary such that $z \neq x_o$. Then $z \in F(f)$ and using
(4.3.2) we have

$$d(x_o, z) = d(f_o(x_o), fz)$$
or,
\[
\begin{align*}
d(x_0, z) & \leq a_1[d(x_0, f_0(x_0)) + d(z, f_2)] \\
& \quad + a_2[d(x_0, f_2) + d(z, f_0(x_0))] \\
& \quad + a_3[d(x, z) + d(f_0(x_0), f_0^2(x_0))] \\
& \quad + a_4[d(x_0, f_0^2(x_0)) + d(f_2, f_0^2(x_0))] + a_5d(z, f_0^2(x_0)) \\
& \leq p \cdot d(x_0, z), \quad (p = 1 - a_1 - a_2 - a_4 < 1)
\end{align*}
\]
again a contradiction, hence \( x_0 = z \). Therefore
\[F(f_0) = x_0 = F(f)\]
This completes the proof of Lemma.

**Proof of the Theorem**: Let \( \epsilon > 0 \), then there exists a natural number \( N \) such that:

\[(4.3.3) \quad d(f_n(x), f_0(x)) < \epsilon\]

for all \( x \) in \( X \) and \( N \leq n \), hence
\[
d(x_n, x_0) = d(f_n(x_n), f_0(x_0))
\]
or
\[
d(f_n(x_n), f_0(x_0)) < d(f_n(x_n), f_0(x_n)) + d(f_0(x_n), f_0(x_n))
\]
\[
\leq d(f_n(x_n), f_0(x_n)) + a_1[d(x_n, f_0(x_n)) + d(x_n, f_0(x_n))]
\]
\[
+ a_2[d(x_0, f_0(x_n)) + d(x_n, f_0(x_n))]
\]
\[ + a_3 [d(x_o, x_n) + d(f_o(x_o), f_o^2(x_o))] \\
+ a_4 [d(x_o, f_o^2(x_o)) + d(f_o(x_n), f_o^2(x_o))] \\
+ a_5 d(x_n, f_o^2(x_o)) \leq d(f_n(x_n), f_o(x_n)) + a_1 [d(f_o(x_o), f_o(x_n)) + d(f_n(x_n), f_o(x_n))]} \\
+ a_2 [d(f_o(x_o), f_n(x_n)) + d(f_n(x_n), f_o(x_n)) + d(f_n(x_n), f_o(x_o))]} \\
+ a_3 [d(f_o(x_o), f_n(x_n)) + d(f_o(x_n), f_o(x_n))]} \\
+ a_4 [d(f_o(x_o), f_o(x_n)) + d(f_o(x_n), f_n(x_n)) + d(f_n(x_n), f_o(x_o))]} \\
+ a_5 d(f_n(x_n), f_o(x_o)) \\
= (1 + a_1 + a_2 + a_4) d(f_n(x_n), f_o(x_n)) \\
+ (2a_2 + a_3 + a_4 + a_5) d(f_n(x_n), f_o(x_o)) \\
[1 - (2a_2 + a_3 + a_4 + a_5)] d(f_n(x_n), f_o(x_o)) \leq (1 + a_1 + a_2 + a_4) d(f_n(x_n), f_o(x_n)).} \\

From the hypothesis, \( 2(a_1 + a_2 + a_3 + a_4 + a_5) < 1 \). Hence for \( n \geq N \), we have \\
\[ d(x_n, x_o) \leq \frac{1 + a_1 + a_2 + a_4}{2a_2 + a_3 + a_4 + a_5} \varepsilon. \]

Which shows that \( \{x_n\} \) converges to \( x_o \). Under the condition (4.3.1) and hence \( f_o \) has a unique fixed point by the Lemma 1.
THEOREM 3: Let \( \{f_n\} \) (n=1,2,...) be a sequence of mappings with fixed point \( x_n \) of a metric space \( X \) into itself. Suppose that

\[
(4.3.4) \quad d(f_n(x), f_n(y)) \leq a_1[d(x, f_n(x)) + d(y, f_n(y))] \\
+ a_2[d(x, f_n(y)) + d(y, f_n(x))] \\
+ a_3[d(x, y) + d(f_n(x), f_n^2(x))] \\
+ a_4[d(x, f_n^2(x)) + d(f_n(y), f_n^2(x))] \\
+ a_5d(y, f_n^2(x)).
\]

Where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are non-negative constants and \((2a_1 + 2a_2 + 2a_3 + 2a_4 + a_5) < 1\). If \( \{f_n\} \) converges to a mapping \( f_0 \), and \( x_0 \) is an accumulation point of \( \{x_n\} \), then \( x_0 \) is fixed point of \( f_0 \).

**Proof**: Since \( x_0 \) is an accumulation point of the sequence \( \{x_n\} \), there is a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) which converges to \( x_0 \).

\[
d(x_0, f_0(x_0)) \leq d(x_0, f_{n_i}(x_{n_i})) + d(f_{n_i}(x_{n_i}), f_{n_i}(x_0)) \\
+ d(f_{n_i}(x_0), f_0(x_0)).
\]

Let \( \varepsilon > 0 \), then there is a natural number \( N \) such that,

\[
d(x_0, x_{n_i}) < \varepsilon \\
d(f_{n_i}(x_0), f_0(x_0)) < \varepsilon \quad \text{for} \quad N \leq n_i.
\]
Hence for $N \leq n_i$, we have

\[(4.3.5) \quad d(x_0, f_n(x_0)) < 2 \varepsilon + d(f_{n_i}(x_{n_i}), f_{n_i}(x_0))\]

To estimate $d(f_{n_i}(x_{n_i}), f_{n_i}(x_0))$, we use condition (4.3.4), then,

\[d(f_{n_i}(x_{n_i}), f_{n_i}(x_0)) \leq a_1 [d(x_{n_i}, f_{n_i}(x_{n_i}))+d(x_0, f_{n_i}(x_0))]\]

\[+ a_2 [d(x_{n_i}, f_{n_i}(x_0))+d(x_0, f_{n_i}(x_0))]\]

\[+ a_3 [d(x_{n_i}, x_0)+d(f_{n_i}(x_{n_i}), f_{n_i}(x_0))]\]

\[+ a_4 [d(x_{n_i}, f_{n_i}(x_{n_i}))+d(f_{n_i}(x_0), f_{n_i}(x_0))]\]

\[+ a_5 d(x_0, f_{n_i}^2(x_{n_i}))\]

for $N \leq n_i$, we have

\[d(f_{n_i}(x_{n_i}), f_{n_i}(x_0)) \leq a_1 d(x_0, f_{n_i}(x_0)) + (a_2 + a_3 + a_5) \varepsilon\]

\[+ (a_2 + a_4) d(x_{n_i}, f_{n_i}(x_0)),\]

hence,

\[(4.3.6) \quad (1-a_2-a_4) d(x_{n_i}, f_{n_i}(x_0)) \leq a_1 d(x_0, f_{n_i}(x_0)) + (a_2 + a_3 + a_5) \varepsilon.\]

Now consider, $d(x_0, f_{n_i}(x_0))$, then

\[d(x_0, f_{n_i}(x_0)) \leq d(x_0, x_{n_i}) + d(x_0, f_{n_i}(x_0)).\]

For $N \leq n_i$, we have
\[(4.3.7)\ d(x_0, f_{n_i}(x_0)) \leq \varepsilon + d(x_{n_i}, f_{n_i}(x_0)).\]

from (4.3.6) and (4.3.7) we have

\[(1-a_2-a_4) d(x_0, f_{n_i}(x_0)) \leq (1-a_2-a_4) \varepsilon + a_1 d(x_0, f_{n_i}(x_0)) + (a_2 + a_3 + a_5) \varepsilon ,\]

hence,

\[(4.3.8)\ d(x_0, f_{n_i}(x_0)) \leq \frac{1+a_2-a_4+a_5}{1-a_1-a_2-a_4} \varepsilon\]

from (4.3.5), (4.3.6) and (4.4.8) we have

\[d(x_0, f_0(x_0)) \leq 2\varepsilon + \frac{1}{(1-a_2-a_4)} \left[ a_1 d(x_0, f_{n_i}(x_0)) + (a_2 + a_3 + a_5) \varepsilon \right]\]

\[\leq [2+ \frac{1}{(1-a_2-a_4)} \left\{ \frac{1+a_2-a_4-a_5}{1-a_1-a_2-a_4} + (a_2 + a_3 + a_5) \right\}] \varepsilon\]

Which shows that $x_0$ is a fixed point of $f_0$.

Hence the proof is complete.

4.4 In 1979, Pachpatte\[93\] gave the following result.

**THEOREM D**: Let $f$ and $g$ be self mappings of a nonempty complete metric space satisfying the inequality

\[\max\{[d(x, y)]^2, [d(x, fx)]^2, [d(y, gy)]^2, \frac{1}{2}[d(x, gy)]^2, \frac{1}{2}[d(y, fx)]^2\}
\]

\[(4.4.1)\ d(fx, gy) \leq q \frac{\min\{d(x, fx), d(y, gy)\}}{[d(x, fx) + d(y, gy)]}\]

for all $x, y$ in $X$ for which $d(x, fx) + d(y, gy) \neq 0$, $q \in (0, 1)$,
then $f$ and $g$ have a common fixed point $z$. Further, if $d(x,fx) + d(y,gy) = 0$ implies that $d(fx,gy)=0$, then $z$ is the unique common fixed point of $f$ and $g$.

Now we prove the following theorem[104] which is the generalization of Theorem D, for sequence of mappings.

**Theorem (4):** Let $f_0$ and $\{f_n : n \in I^+\}$, ($I^+$ denotes the set of positive integers) be mappings of a non-empty complete metric space $X$ into itself satisfying the inequality

$$
(4.4.2) \quad d(f_0(x), f_n(y)) \leq q \frac{\max\{[d(x,y)]^2, [d(x,f_0(x))]^2, [d(y,f_0(y))]^2, \frac{1}{2}[d(x,f_n(y))]^2, \frac{1}{2}[d(y,f_n(x))]^2\}}{[d(x,f_0(x)) + d(y,f_n(y))]}
$$

for all $x,y$ in $X$. For each $n = 1, 2, \ldots$ for which $d(x,f_0(x)) + d(y,f_n(y)) \neq 0$, $q \in (0,1)$ then there exists a point $z$ in $X$ such that $f_n z = z$ for each $n=0, 1, 2, \ldots$ and for arbitrary $x_0$ in $X$, the sequence,

$$
(4.4.3) \quad x_0, x_1 = f_0 x_0, x_2 = f_1 x_0, x_3 = f_0 x_2, \ldots
$$

$$
x_{2n-1} = f_0 x_{2n-2}, x_{2n} = f_n x_{2n-2}, x_{2n+1} = f_0 x_{2n}
$$

converging to $z$. Further, if $d(x,f_0(x)) + d(y,f_n(y)) = 0$ then $z$ is the unique fixed point $f_n$ for $n = 1, 2, \ldots$.
Proof: First, we prove that the sequence \( \{x_n\} \) defined by (4.4.3) is Cauchy sequence. By (4.4.2) for \( x=x_{2n-2} \) and \( y=x_{2n-1} \), we have when \( x_{2n} \neq x_{2n-1} \),

\[
d(x_{2n-1}, x_{2n}) = d(f_{x_{2n-2}}, f_{x_{2n-1}})^2 \leq q \frac{[d(x_{2n-2}, x_{2n-1})]^2}{[d(x_{2n-2}, x_{2n-1})+d(x_{2n-1}, x_{2n})]}
\]

\[
\max\{[d(x_{2n-2}, x_{2n-1})]^2, [d(x_{2n-2}, x_{2n})]^2, [d(x_{2n-1}, x_{2n})]^2, \frac{1}{2}[d(x_{2n-2}, x_{2n})]^2 \}
\]

\[
\leq q \frac{[d(x_{2n-2}, x_{2n-1})+d(x_{2n-1}, x_{2n})]}{[d(x_{2n-2}, x_{2n-1})+d(x_{2n-1}, x_{2n})]}
\]

(4.4.4)

\[
d(x_{2n-1}, x_{2n}) \leq q \frac{[d(x_{2n-2}, x_{2n})]^2}{[d(x_{2n-2}, x_{2n-1})+d(x_{2n-1}, x_{2n})]}
\]

If possible, let maximum of numerator in right hand side of (4.4.4) be \([d(x_{2n-1}, x_{2n})]^2\), then we get,

\[
d(x_{2n-1}, x_{2n}) \leq q \frac{[d(x_{n-1}, x_{2n})]^2}{d(x_{2n-1}, x_{2n-2})+d(x_{2n-1}, x_{2n})}
\]

or,

\[
d(x_{2n}, x_{2n-2})+d(x_{2n-1}, x_{2n}) \leq q d(x_{2n-1}, x_{2n})
\]
i.e. \( d(x_{2n-2}, x_{2n-1}) \leq (q-1)d(x_{2n-1}, x_{2n}) < 0 \)

which is impossible,

\[ d(x_{2n-2}, x_{2n-1}) \neq d(x_{2n-1}, x_{2n}) \]

Again if possible, let \( [d(x_{2n-2}, x_{2n})]^2 \) is the maximum in numerator of right hand side of (4.4.4), then we have

\[
d(x_{2n-1}, x_{2n}) \leq q \frac{\frac{1}{2}(d(x_{2n-2}, x_{2n}))^2}{[d(x_{2n-1}, x_{2n-2})+d(x_{2n-1}, x_{2n})]}
\]

\[
\leq q \frac{\frac{1}{2}(d(x_{2n-2}, x_{2n-1})+d(x_{2n-1}, x_{2n}))}{[d(x_{2n-1}, x_{2n-2})+d(x_{2n-1}, x_{2n})]}
\]

or,

\[
d(x_{2n-1}, x_{2n}) \leq \left(\frac{q}{2-q}\right) d(x_{2n-2}, x_{2n-1})
\]

\[
< q d(x_{2n-2}, x_{2n-1})
\]

In the same manner, if we let \( [d(x_{2n-2}, x_{2n-1})]^2 \) as the maximum in numerator of right hand side of (4.4.4), then we have

\[
d(x_{2n-1}, x_{2n}) \leq q \frac{[d(x_{2n-2}, x_{2n-1})]^2}{[d(x_{2n-1}, x_{2n-2})+d(x_{2n-1}, x_{2n})]}
\]

\[
\leq q \frac{[d(x_{2n-2}, x_{2n-1})]^2}{d(x_{2n-1}, x_{2n-2})}
\]

\[= q d(x_{2n-2}, x_{2n-1})
\]
Thus, in both the cases, we have

\[ d(x_{2n-1}, x_{2n}) \leq q \ d(x_{2n-2}, x_{2n-1}) \]

Similarly we can show that,

\[ d(x_{2n-2}, x_{2n-1}) = d(f_n x_{2n-3}, f_o x_{2n-2}) \]
\[ \leq q \ d(x_{2n-3}, x_{2n-2}) \]

Proceeding in the similar manner, we get

\[ d(x_{2n-1}, x_{2n}) \leq q \ d(x_{2n-2}, x_{2n-1}) \]
\[ \leq q^2 \ d(x_{2n-3}, x_{2n-2}) \]
\[ \ldots \ldots \ldots \]
\[ \leq q^{2n-1} \ d(x_0, x_1) \text{ for } n = 1, 2, \ldots \]

Then by routine calculation, we can easily show that the following inequality hold for \( k > n \).

\[ d(x_n, x_{n+k}) \leq \sum_{i=1}^{k} d(x_{n+i-1}, x_{n+i}) \]
\[ \leq \sum_{i=1}^{k} q^{n+i-1} d(x_0, x_1) \]
\[ \leq \frac{q^n}{1-q} d(x_0, x_1) \]

Since \( q < 1 \), the right hand side of above inequality tends to zero as \( n \) tends to infinity. Therefore the sequence \( \{x_n\} \) is a Cauchy sequence.
As \( X \) is complete there exists a point \( z \) in \( X \) such that

\[
\lim_{n \to \infty} x_n = z.
\]

Using (4.4.2) and if \( z \neq f_0 z \),

\[
d(z, f_0 z) \leq d(z, x_{2n}) + d(x_{2n}, f_0(z)) + d(f_0(z), x_{2n-1})
\]

\[
\leq d(z, x_{2n}) + d(f_0 z, x_{2n-1})
\]

\[
\leq \max\{d(z, x_{2n-1})^2, [d(z, f_0 z)]^2, [d(x_{2n-1}, f_0 z)]^2, [d(x_{2n-1}, f_0 z)]^2, [d(x_{2n}, f_0 z)]^2, [d(x_{2n}, f_0 z)]^2\}
\]

\[
\leq \frac{1}{2} [d(z, x_{2n}) + d(f_0 z, x_{2n-1})] + q \frac{1}{2} [d(z, x_{2n}) + d(f_0 z, x_{2n-1})] + q [d(z, f_0 z) + d(x_{2n-1}, x_{2n})]
\]

On letting \( n \to \infty \), we have \( d(z, f_0 z) = 0 \), which implies that \( f_0 z = z \).

Now let us consider that \( z \neq f_n z \), then

\[
d(z, f_n z) = d(f_0 z, f_n z)
\]
\[
\max\{[d(z,z)]^2, [d(z,f_\sigma z)]^2, [d(z,f_n z)]^2, \\
\frac{\beta [d(z,f_{n+1} z)]^2, \gamma [d(z,f_\sigma z)]^2}{[d(z,f_\sigma z)+d(z,f_n z)]}
\]

or, \(d(z,f_n z) \leq q \frac{\beta [d(z,f_{n+1} z)]^2, \gamma [d(z,f_\sigma z)]^2}{d(z,f_n z)}\)

\[
= q \frac{\max\{[d(z,f_{n+1} z)]^2, \gamma [d(z,f_n z)]^2\}}{d(z,f_n z)}
\]

= \(q \cdot d(z,f_n z)\)

a contradiction. Hence it follows that \(z = f_n z\).

Further in order to prove the uniqueness of \(z\). Suppose an additional condition \(d(x,f_\sigma x)+d(y,f_n y) = 0\), implies \(d(f_\sigma x,f_n y) = 0\) and let \(f_n\) has another fixed point \(u(\neq z)\) in \(X\). Then \(d(z,f_\sigma z)+d(u,f_n u) = 0\) implies \(d(f_\sigma z,f_n u) = 0\), therefore we have

\[
d(z,u) = d(f_\sigma z,f_n u) = 0
\]

which implies \(z = u\).

Hence it follows that \(z\) is a unique fixed point of \(f_n\).

4.5  Before going to state our next result for family of mappings, we need the following lemma:

**Lemma 2**: Let \(f_\sigma, f : X \to X\) be a pair of mappings on a metric space \((X,d)\). If for all \(x,y \) in \(X\),
\[ \max\{[d(x,y)]^2, [d(x,f_0x)]^2, [d(y,fy)]^2, \]

\[ \frac{\frac{1}{2}[d(x,fy)]^2, \frac{1}{2}[d(y,f_0x)]^2}{[d(x,f_0x)+d(y,fy)]} \]

(4.5.1) \( d(f_0x, fy) \leq q \cdot \frac{\frac{1}{2}[d(x,fx_o)]^2, \frac{1}{2}[d(x,f_0x)]^2}{[d(x,f_0x)+d(x,fx_o)]} \)

holds for some \( q, 0 \leq q < 1 \) and \( F(f_0), F(f) \) are non-empty sets, then \( F(f_0) \) is singleton and \( F(f)=F(f_0). \) Further if \( d(x,f_0x)+d(y,fy) = 0 \) implies \( d(f_0x, fy)=0, \) then common fixed point of \( f_0 \) and \( f \) is unique.

**Proof:** Let \( x_o \in F(f_0) \subset X \) be any fixed point and \( x_o \neq fx_o \) then by (4.5.1) we have,

\[ d(x_o, fx_o) = d(f_0x_o, fx_o) \]

\[ \max\{[d(x_o,x_o)]^2, [d(x_o,f_0x_o)]^2, [d(x_o,fx_o)]^2, \]

\[ \frac{\frac{1}{2}[d(x_o,fx_o)]^2, \frac{1}{2}[d(x_o,f_0x_o)]^2}{[d(x_o,f_0x_o)+d(x_o,fx_o)]} \]

\[ = (q/2)d(x_o,fx_o) < d(x_o,fx_o) \]

which is a contradiction and implies that \( d(x_o,fx_o)=0. \) Therefore \( x_o \in F(f). \) Now let \( z \in F(f_0) \) and using (4.5.1) we have,

\[ d(x_o, z) = d(f_0x_o, f(z)) \]

\[ \max\{[d(x_o,z)]^2, [d(x_o,f_0x_o)]^2, [d(z,fz)]^2, \]

\[ \frac{\frac{1}{2}[d(x_o,fz)]^2, \frac{1}{2}[d(z,f_0x_o)]^2}{[d(x_o,f_0x_o)+d(z,fz)]} \]
or, \[ d(x_0, z) \leq q \frac{\max\{[d(z, fz)]^2, \frac{1}{4}[d(x_0, fz)]^2\}}{d(z, fz)} \]

\[ \leq q \ d(x_0, fz) \leq d(x_0, z) \]

again a contradiction. Hence \( x_0 = z \). Therefore,

\[ F(f_0) = \{x_0\} = F(f). \]

This completes the proof of Lemma.

We use the above Lemma to prove the following:

**THEOREM (5):** Let \( F = \{f_\lambda : \lambda \in (\lambda)\} \) be a family of mappings, which maps a complete metric space \((X, d)\) into itself and let \( q \in (0, 1) \). If there exists some \( f_{\lambda_0} \in F \) such that for each \( f_\lambda \in F(\lambda \neq \lambda_0) \) there are positive integers \( i_\lambda \) and \( j_\lambda \) such that

\[
\begin{align*}
(4.5.2) & \quad \max\{[d(x, y)]^2, [d(x, f_{\lambda_0}^{i_\lambda} (x))]^2, \\
& \quad [d(y, f_{\lambda_0}^{j_\lambda} (y))]^2, \frac{1}{4}[d(x, f_{\lambda_0}^{i_\lambda} (y))]^2, \\
& \quad \frac{1}{4}[d(y, f_{\lambda_0}^{j_\lambda} (x))]^{\frac{1}{2}} \} \\
\end{align*}
\]

\[ d(f_{\lambda_0}^{i_\lambda} (x), f_{\lambda_0}^{j_\lambda} (y)) \leq q \frac{\max\{[d(x, y)]^2, [d(x, f_{\lambda_0}^{i_\lambda} (x))]^2, \\
[\] [d(y, f_{\lambda_0}^{j_\lambda} (y))]^2, \frac{1}{4}[d(x, f_{\lambda_0}^{i_\lambda} (y))]^2, \\
\frac{1}{4}[d(y, f_{\lambda_0}^{j_\lambda} (x))]^{\frac{1}{2}} \} \]

\[ \leq \frac{d(x, f_{\lambda_0}^{i_\lambda} (x)) + d(y, f_{\lambda_0}^{j_\lambda} (y))}{2} \]

holds for all \( x, y \) in \( X \). Then every \( f_\lambda \in F \) has a unique fixed point in \( X \), which is a unique common fixed point \( f \) or \( F \).

**Proof:** Let \( f_\lambda \in F \) be arbitrary. For arbitrary \( x \) in \( X \), let us consider a sequence \( \{x_n\} \) defined as follows:
(4.5.3) \( x_0 = x, x_1 = f^{i\lambda}_{\lambda_0} (x_0), x_2 = f^{j\lambda}_{\lambda} (x_1), \ldots; \)

\( x_{2n-1} = f^{i\lambda}_{\lambda_0} (x_{2n-2}), x_{2n} = f^{j\lambda}_{\lambda} (x_{2n-1}), \ldots \)

By (4.5.2) for \( x = x_{2n-2} \) and \( y = x_{2n-1} \) we have

\[
\begin{align*}
&d(x_{2n-1}, x_{2n}) = d(f^{i\lambda}_{\lambda_0} (x_{2n-2}), f^{j\lambda}_{\lambda} (x_{2n-1})) \\
&\max\{d(x_{2n-2}, x_{2n-1})^2, [d(x_{2n-2}, f^{i\lambda}_{\lambda_0} (x_{2n-2}))]^2, \\
&[d(x_{2n-1}, f^{j\lambda}_{\lambda} (x_{2n-1}))]^2, \frac{1}{2} d(x_{2n-1}, f^{j\lambda}_{\lambda} (x_{2n-1}))^2\} \\
&\leq q \frac{\left[ d(x_{2n-2}, f^{i\lambda}_{\lambda_0} (x_{2n-2})) + d(x_{2n-1}, f^{j\lambda}_{\lambda} (x_{2n-1})) \right]^2}{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}
\end{align*}
\]

\[
\begin{align*}
&\max\{d(x_{2n-2}, x_{2n-1})^2, [d(x_{2n-2}, x_{2n-1})]^2, \\
&[d(x_{2n-1}, x_{2n})]^2, \frac{1}{2} d(x_{2n-1}, x_{2n})^2, \frac{1}{2} d(x_{2n-1}, x_{2n-1})^2\} \\
&= q \frac{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}{d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})}
\end{align*}
\]

Then from Theorem 4, we have

\[
d(x_{2n-1}, x_{2n}) \leq q d(x_{2n-2}, x_{2n-1}).
\]

Similarly,

\[
d(x_{2n-2}, x_{2n-1}) = d(f^{j\lambda}_{\lambda} (x_{2n-3}), f^{i\lambda}_{\lambda_0} (x_{2n-2})) = q d(x_{2n-3}, x_{2n-2}).
\]
Then by routine calculations, as followed in proof of Theorem 4 we can show that the sequence \( \{x_n\} \) defined by (4.5.3) is a Cauchy sequence. Using the completeness of \( X \), we have,

\[
(4.5.4) \quad \lim_{n \to \infty} x_n = z.
\]

Further for some \( z \) in \( X \), by (4.5.2), we have \( \delta(z, f^{i\lambda}_x(z)) = 0 \) hence \( z = f^{i\lambda}_x(z) \) i.e. \( z \) is a fixed point of \( f^{i\lambda}_x(z) \).

By Lemma 2, \( z \) is a unique fixed point of \( f^{i\lambda}_{\lambda_0} \) and \( f^{i\lambda}_x \), as (4.5.2) implies (4.5.1). Since

\[
f^{i\lambda}_{\lambda_0}(f_{\lambda_0}(z)) = f_{\lambda_0}(f^{i\lambda}_{\lambda_0}(z)) = f_{\lambda_0}(z).
\]

Hence \( f_{\lambda_0}(z) \) is also a fixed point of \( f^{i\lambda}_{\lambda_0} \) and therefore by uniqueness of the fixed point, it follows \( f_{\lambda_0}(x_0) = x_0 \). Similarly it follows that \( f_{\lambda}(z) = z \). So, we have proved that \( z \) is a unique fixed point of \( f_{\lambda_0} \) and \( f_{\lambda} \).

Now we shall show that \( z \) is a unique common fixed point for \( F \). Let \( f^{i\lambda'}_{\lambda} \in F, \ \lambda_0 \neq \lambda' \neq \lambda \), be arbitrary. Since \( z = f_{\lambda_0}(z) \) implies \( z = f^{i\lambda'}_{\lambda_0}(z) \), by (4.5.2) and Lemma 2, \( z \) is a unique fixed point \( f^{i\lambda'}_{\lambda_0} \). This implies that \( z \) is a unique fixed point of \( f^{i\lambda'}_{\lambda} \), and so we have the desired result.

This completes the proof of the Theorem.

* * * *