CHAPTER 3

FIXED POINT THEOREMS FOR WEAKLY COMMUTING MAPPINGS
UNDER ASYMPTOTICALLY REGULARITY AT A POINT
CHAPTER III

FIXED POINT THEOREMS FOR WEAKLY COMMUTING MAPPINGS
UNDER ASYMPTOTICALLY REGULARITY AT A POINT.

3.1 This chapter is devoted to the study of fixed point for weakly commuting mappings under asymptotically regularity and we prove some theorems for three maps in metric spaces, complete metric spaces and in normed linear spaces.

(3.1.1) The notion of asymptotic regularity was first introduced by Browder and Petryshyn[14] for Banach space X. Asymptotically regularity is found to be very useful in proving the existence of fixed points of the mapping f:X→ X as well as in showing that in certain cases, the sequence of iterates at some point xo in X also converges to the fixed point of f.

Let X denotes a Banach space. A mapping f:X→ X is said to be asymptotically regular at a point xo in X if,

\[ \|f^n x_o - f^{n+1} x_o\| \to 0, \text{ as } n \to \infty, \]

where \( f^n x_o \) denotes the n\textsuperscript{th} iterates of f at xo in X.
The equivalent form of the definition of asymptotically regularity can be stated as follows:

A mapping \( f: X \to X \) of a metric space \((X,d)\) is said to be asymptotic regular at a point \( x \) in \( X \) if,

\[
\lim_{n \to \infty} d(f^n x, f^{n+1} x) \to 0,
\]

where \( f^n(x) \) denotes the \( n^{th} \) iterates of \( f \) at \( x \) in \( X \).

It is remarkable that asymptotic regularity at a point \( x \) in \( X \) is not only fruitful in proving the existence of fixed points of self mapping \( f \) of \( X \), but it is equally helpful in showing that in certain cases the sequence of iterates at a point \( x \) in \( X \) converges to a point of \( f \).

Before going to state our results we need the following definitions.

**DEFINITION 3.1.2**: A sequence \( \{x_n\} \) in \( X \) is said to be asymptotically \( h \)-regular with respect to \( g \) if

\[
\lim_{n \to \infty} d(hx_n, gx_n) = 0.
\]

Further due to symmetry, we may also say that \( \{x_n\} \) is asymptotically \( g \)-regular with respect to \( h \).

When \( h \) is the identity map the above definition reduced to that of Engl[38].
**DEFINITION (3.1.2)**: A pair \( (h,g) \) is said to be weakly commuting, if \( d(hgx,ghx) \leq d(hx,gx) \), for all \( x,y \) in \( X \).

Clearly a commuting pair is weakly commuting but the converse is not necessarily true, as is shown by the simple example.

**Example (3.1.3)**: Let \( X = [0,1] \) with the usual metric \( d \).

Define \( h(x) = x/r \) and \( g(x) = x/(r+x) \), where \( r \) is any non-zero constant then, for all \( x \) in \( X \),

\[
d(hgx, ghx) = \frac{x}{r^2 + x} - \frac{x}{r(r+x)} = x\left(\frac{1}{r^2 + x} - \frac{1}{r(r+x)}\right)
\]

\[
= \frac{(r-1)x^2}{r(r+x)(r^2+x)} \leq \frac{x^2}{r(r+x)} , \text{ as } \frac{r-1}{r^2+x} < 1
\]

\[
= \frac{x}{r} - \frac{x}{r+x} = h(x) - g(x)
\]

i.e., \( d(hgx, ghx) \leq d(hx, gx) \)

and \( h \) and \( g \) commute weakly. But for any non-zero \( x \) in \( X \), we have

\[
g(h(x)) = \frac{x}{r^2 + x} > \frac{r}{r(r+x)} = h(g(x))
\]

whence \( gh \neq hg \). Thus \( h \) and \( g \) are not commuting mappings.
DEFINITION (3.1.4) : A pair \((h,g)\) is said to be compatible if \(\lim_{n} d(hgx_n,ghx_n) = 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n} h(x_n) = \lim_{n} g(x_n) = t\) for some point \(t\) in \(X\).

A weakly commuting pair is a compatible pair but a compatible pair is not necessarily weakly commuting.

Hardy and Rogers[51] proved a fixed point theorem in complete metric space satisfying :

\[
d(fx,fy) \leq a_1 d(x,fx) + a_2 d(y,fy) + a_3 d(x, fy) + a_4 d(y,fx) + a_5 d(x, y),
\]

for \(a_i \geq 0\) and \(\sum_{i=1}^{5} a_i \leq 1\).

Recently Sastry and others[123] proved the following :

THEOREM A : Let \(f, g\) and \(h\) be self mappings on a metric space \((X,d)\) satisfying :

(i) either \(fh = hf\) or \(gh = hg\),

(ii) \(d(fx,gy) \leq k \max\{d(hx,hy),d(hx,fx),d(hy,gy),d(hx,gy),d(hy,fx)\}\)

for all \(x,y\) in \(X\), where \(k \in (0,1)\) and

(iii) \((f,g)\) is asymptotically regular with respect to \(h\) at \(x_0\) in \(X\).
Then f, g and h have a unique common fixed point in X.

Khan and Swaleh[73] obtained the following:

**THEOREM B**: Let \((X,d)\) be a complete metric space, f, g
and h are three self mappings on X satisfying:

\[
d(fx,gy) \leq a_1d(fx,hx) + a_2d(gy,hy) + a_3d(fx,hy) \\
+ a_4d(gy,hx) + a_5d(hx,hy)
\]

for all \(x, y\) in \(X\), where \(a_i\) (i = 1, 2, ..., 5) are non-negative reals and \(\max\{(a_3+a_4), (a_3+a_4+a_5)\} < 1\), h is continuous, {h, f} and {h, g} are weakly commuting pairs and there exists a sequence which is asymptotically f-regular as well as g-regular with respect to h then f, g and h have a unique common fixed point.

Further Fisher[41] obtained the following

**Theorem for a pair of mappings.**

**THEOREM C**: Let f and g be a pair of self mappings satisfying the inequality

\[
[d(fx,gy)]^2 \leq b \cdot d(x,fx)d(x,gy) + c \cdot d(y,fx)d(y,gy)
\]

for all \(x, y\) in \(X\), b, c \(\geq 0\) and \([b+(b^2+4b)^{\frac{1}{2}}][c+(c^2+4c)^{\frac{1}{2}}] < 4\).

Then f and g have a unique common fixed point.

In 1985, Devi prasad[31] proved a common fixed point theorem for three self maps of a complete metric space \((X,d)\), as follows,
THEOREM D: Let \( f, g \) and \( h \) be three self mappings of a complete metric space \((X, d)\), which satisfy:

\[
(i) \quad [d(fx, gy)]^2 \leq \phi[d(hx, fx) d(hy, gy), d(hx, gy) d(hy, gx), d(hx, fx) d(hx, gy), d(hy, fx) d(hy, gy)],
\]

\[(ii) \quad fh = hf, gh = hg, f(X) \subseteq h(X) \text{ and } g(X) \subseteq h(X),\]

where \( \phi \) is an upper semi-continuous function. Further, let \( h \) be continuous, then \( f, g \) and \( h \) have a unique common fixed point in \( X \).

3.2 We shall now prove some fixed point theorems for more general class of mappings which generalize the above Theorem A and Theorem D. In our results we have weakened the commutativity condition of mappings by considering weakly commuting mappings:

In fact we prove:

THEOREM (i): Let \( f, g, h \) be three self mappings of a complete metric space \((X, d)\) satisfying:

\[
(3.2.1) \quad [d(hx, hy)]^2 \leq \phi[\max(d(fx, hx) d(gy, hy),
\quad d(fy, hy) d(gx, hx), d(fx, gy) d(fy, gy),
\quad d(fx, hy) d(gx, hy), d(fy, hx) d(gy, hx))]\]

for all \( x, y \) in \( X \), where \( \phi \) is non-decreasing continuous
function from \( R^+ \) to \( R^+ \), satisfying \( \phi(t) < t \) for each \( t > 0 \) and \( \phi(0) = 0 \),

(3.2.2) \( h \) is continuous,

(3.2.3) \( h \) weakly commutes with \( f \) and \( g \) and

(3.2.4) there exists a sequence which is asymptotically \( f \)-regular and \( g \)-regular with respect to \( h \),
then \( f, g \) and \( h \) have a unique common fixed point.

Proof: Let \( \{x_n\} \) be a sequence satisfying (3.2.4).

Using (3.2.1), we have

\[
[d(hx_n, hx_m)]^2 \leq \phi(\max(d(fx_n, hx_n)d(gx_m, hx_m), d(fx_m, hx_m)d(gx_n, hx_n), d(fx_n, gx_m)d(fx_m, gx_n), d(fx_n, hx_m)d(gx_n, hx_m), d(fx_m, hx_n)d(gx_m, hx_n)))
\]

or,

\[
[d(hx_n, hx_m)]^2 \leq \phi(\max(d(fx_n, hx_n)d(gx_m, hx_m), d(fx_m, hx_m)d(gx_n, hx_n), [d(fx_n, hx_n)+d(hx_n, hx_m)+d(hx_m, gx_m)])
\]

\[
[d(fx_m, hx_n)+d(hx_m, hx_n)+d(hx_n, gx_n)])
\]

\[
[d(fx_n, hx_n)+d(hx_n, fx_m)][d(hx_n, gx_n) + d(hx_n, hx_m) + d(hx_m, hx_n)]
\]

\[
[d(gx_m, hx_m)+d(hx_m, hx_n)])
\]

using (3.2.4) we get,
\[ [d(hx_n, hx_m)]^2 \leq \phi(\max\{d(hx_n, hx_m)^2, 0, 0, 0, d(hx_n, hx_m)^2\}) \]

or \[ [d(hx_n, hx_m)]^2 \leq \phi((d(hx_n, hx_m))^2 < (d(hx_n, hx_m))^2 \]

which implies that \( \{hx_n\} \) is a Cauchy sequence.

Now,

\[ d(fx_n, z) \leq d(fx_n, hx_n) + d(hx_n, z) \]

implies, \( \{fx_n\} \to z \), similarly \( \{gx_n\} \to z \). Also using continuity of \( h, \{h^2x_n\} \to hz, \{hfx_n\} \to hz \) and \( \{hgx_n\} \to hz \).

From (3.2.3) we have,

\[ d(fh_n, hz) \leq d(fh_n, hfx_n) + d(hfx_n, hz) \]

\[ \leq d(hx_n, fx_n) + d(hfx_n, hz) \]

whence \( \{fh_n\} \to hz \). Similarly \( \{ghx_n\} \to hz \).

Further, from (3.2.1) we have

\[ [d(hhx_n, hx_n)]^2 \leq \phi(\max\{d(fhx_n, hhx_n), d(gx_n, hx_n), d(fh_n, hx_n) d(gx_n, hx_n), d(fhx_n, fx_n) d(gx_n, hx_n), d(fh_n, hx_n) d(gx_n, hx_n), d(fh_n, hx_n) d(gx_n, hhx_n), d(fh_n, hx_n) d(gx_n, hhx_n), d(fh_n, hx_n) d(gx_n, hhx_n), d(fh_n, hx_n) d(gx_n, hhx_n)\}) \]

From the right continuity of \( h \) and taking limit as \( n \) tends to infinity, yields,
\[(d(hz,z))^{2} \leq \phi(d(hz,z))^{2} < (d(hz,z))^{2}\]

Therefore \(hz = z\).

Now by virtue of (3.2.3) we have

\[d(hgx_{n},ghz) \leq d(gx_{n},hz)\]

Letting \(n\) tends to infinity we have

\[d(hz,ghz) \leq d(z,hz) = d(hz,hz) = 0\]

implies \(hz = gz\). Similarly \(fz = gz\).

Thus we obtain \(hz = gz = z\).

For uniqueness, let \(z\) and \(u\) are two distinct common fixed points of \(f, g\) and \(h\). Then,

\[(d(z,u))^{2} = [d(hz,hu)]^{2}\]

\[\leq \phi(\max\{d(fz,hz)d(gu,hu),d(fu,hu)d(gz,hz),
\]

\[d(fz,gu)d(fu,gz),d(fz,hu)d(gz,hu),
\]

\[d(fu,hu)d(gu,hz)\})\]

\[\leq \phi(\max\{0,0,d(z,u)d(z,u),d(z,u)d(z,u),d(u,z)d(u,z)\})\]

or, \([d(z,u)]^{2} \leq \phi(\{(d(z,u))^{2}\}) < (d(z,u))^{2}\) (as \(\phi(t) < t\))

which is a contradiction, therefore \(z = u\).

This competes the proof of the Theorem.
Now we shall remove weakly commutativity to more general case compatibility. In fact we prove:

**Theorem (2):** Let \( f, g, h \) be three self mappings of a complete metric space \((X,d)\) and satisfying (3.2.1), (3.2.2) where \( \phi : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\} \) is nondecreasing continuous from the right and satisfying \( \phi(t) < t \) for each \( t > 0 \) and \( \phi(0) = 0 \),

(3.2.5) \((h,f)\) and \((h,g)\) are compatible pairs and (3.2.4) also holds, then mapping \( h \) has a fixed point.

**Proof:** First we shall show that \( \{hx_n\} \) is a Cauchy sequence, substituting \( x=x_n \) and \( y=y_m \) in (2.2.1), we have

\[
[d(hx_n,hx_m)]^2 \leq \phi(\max(d(fx_n,hx_n)d(gx_m,hx_m),
\quad d(fx_m,hx_m)(gx_n,hx_n),d(fx_n,gx_m)d(fx_m,gx_n),
\quad d(fx_n,hx_m)d(gx_n,hx_m),d(fx_m,hx_n)d(gx_m,hx_n)))
\]

Using (3.2.4), we get

\[
[d(hx_n,hx_m)]^2 \leq \phi(\max(0,0,(d(hx_n,hx_m))^2,(d(hx_n,hx_m))^2),
\quad (d(hx_n,hx_m))^2))
\]

\[
\leq \phi(d(hx_n,hx_m))^2 < (d(hx_n,hx_m))^2,
\]

from definition of \( \phi \), we deduce that \( \{hx_n\} \) is a Cauchy sequence. Since \( X \) is complete, \( \{hx_n\} \) converges to a
point \( z \), since \( h \) is continuous, the sequences \( \{ h^2 x_n \} \), \( \{ h x_n \} \) and \( \{ h g x_n \} \) also converges to \( h z \).

\[
d(f x_n, h z) \leq d(f x_n, h x_n) + d(h x_n, h z)
\]

Using (3.2.5), we get \( \{ f x_n \} \) also converges to \( h z \).

Further (3.2.1) yields,

\[
[d(h x_n, h z)]^2 \leq \phi(\max\{d(f x_n, h^2 x_n) + d(gh x_n, h x_n), d(f x_n, g x_n) + d(f x_n, g h x_n), d(f x_n, h x_n) + d(gh x_n, h x_n), d(f x_n, h^2 x_n) + d(g x_n, h^2 x_n)\})
\]

letting \( n \to \infty \) we obtain,

\[
[d(h z, z)]^2 \leq \phi(\max\{d(h z, h z) + O, 0, 0, d(h z, h z)\})
\]

\[
(d(h z, z) + O), (0 + d(z, h z)) \quad d(h z, z) d(h z, z),
\]

\[
d(z, h z)(O + d(z, h z)))
\]

or,

\[
[d(h z, z)]^2 \leq \phi([d(h z, z)]^2) \leq [d(h z, z)]^2
\]

which is a contradiction. Therefore \( h z = z \) and thus \( z \) is a fixed point of \( h \).

This completes the proof.

We shall now prove some more general Theorem.

We prove:

**THEOREM (3)** : Let \( f, g \) and \( h \) be three self mappings of a complete metric space \( (X, d) \) satisfying (3.2.1) for all
\(x, y \in X\). If (3.2.4) holds and \(f\) is continuous then \(f\) has a fixed point provided the pairs \(\{h, f\}\) and \(\{h, g\}\) are compatible.

**Proof:** As in Theorem 2, one shows that the sequences \(\{h_n\}, \{f_n\}\) and \(\{g_n\}\) converges to a point \(z\). Since \(f\) is continuous, the sequences \(\{f_n h_n\}, \{f^2_n x_n\}\) and \(\{f g_n x_n\}\) converge to the point \(fz\). Using the compatibility of the pairs \(\{h, g\}\) and \(\{f, g\}\) it is immediately seen that the sequence \(\{h_n f x_n\}\) and \(\{g h x_n\}\) also converge to \(fz\).

Now applying the condition (3.2.1) we obtain.

\[
[d(h_n f x_n, h x_n)]^2 \leq \phi(\max(d(f^2_n x_n, h f x_n) d(g x_n, h x_n),
\quad d(f x_n, h x_n) d(g f x_n, h f x_n), d(f x_n, g x_n) d(f x_n, g f x_n),
\quad d(f^2_n x_n h x_n) d(f x_n, g f x_n), d(f^2_n x_n, h x_n) d(g f x_n, h x_n)
\quad d(f x_n, h f x_n) d(g x_n, h f x_n)]
\]

letting \(n \to \infty\), we have

\[
[d(fz, z)]^2 \leq \phi(\max(d(fz, fz) d(z, z), d(z, z) d(fz, fz),
\quad d(fz, z) d(z, fz), d(fz, z) d(fz, z) d(z, fz))
\]

\[= \phi((d(fz, z))^2) < (d(fz, z))^2\]

which is a contradiction. Therefore \(fz = z\), implies \(z\) is a fixed point of \(f\).
This completes the proof of the theorem.

**Remark 1:** A result analogous to Theorem 3 can be obtained by using the continuity of \( g \) instead of \( f \) and the compatibility of the pairs \( \{ h, g \} \) and \( \{ f, g \} \).

### 3.3

In this section we prove some fixed point theorems for more general class of mappings which are further generalizations of Theorem (B) and Theorem (D).

We prove:

**Theorem (4):** Let \( f, g \) and \( h \) be self mappings of a complete metric space \((X,d)\) satisfying

\[
(3.3.1) \quad [d(fx,gy)]^2 \leq a_1 [d(hx,fx)d(hy,gy)] \\
+ a_2 [d(fy,hy)d(gx,hx)] + a_3 [d(fx,hy)d(gx,hy)] \\
+ a_4 [d(fy,hx)d(gy,hx)] + a_5 [d(hx,hy)]^2
\]

for all \( x, y \) in \( X \) where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are non-negative real numbers,

\[
(3.3.2) \quad \max\{a_2+a_3+a_4,(a_2+a_4+a_5)\} < 1,
\]

\[
(3.3.3) \quad h \text{ is continuous},
\]

\[
(3.3.4) \quad \{h,f\} \text{ and } \{h,g\} \text{ are weakly commuting pairs &}
\]

\[
(3.3.5) \quad \text{there exists a sequence which is asymptotically } f\text{-regular and } g\text{-regular with respect to } h. \text{ Then } f, g \text{ and } h \text{ have a unique common fixed point.}
\]
Proof: Let \( \{x_n\} \) be a sequence satisfying (3.3.4).

Using (3.3.1),

\[
d(hx_n, hx_m) \leq d(hx_n, fx_n) + d(fx_n, gx_m) + d(gx_m, hx_m)
\]

Now,

\[
[d(fx_n, gx_m)]^2 \leq a_1 [d(hx_n, fx_n)d(hx_m, gx_m)] + a_2 [d(fx_n, hx_m)d(gx_n, hx_n)] + a_3 [d(fx_n, hx_m)d(gx_n, hx_m)] + a_4 [d(fx_m, hx_n)d(gx_m, hx_n)] + a_5 [d(hx_n, hx_m)]^2
\]

\[
\leq a_1 [d(hx_n, fx_n)d(hx_m, gx_m)] + a_2 [d(fx_m, hx_n)d(gx_m, hx_m)] + a_3 [d(fx_n, hx_n)d(hx_n, hx_m)] + a_4 [(d(fx_m, hx_n)d(hx_m, hx_n)] + (d(gx_m, hx_m)d(hx_m, hx_n))] + a_5 [d(hx_n, hx_m)]^2
\]

\[
[d(fx_n, gx_m)]^2 \leq (a_3 + a_4 + a_5) [d(hx_n, hx_m)]^2
\]

i.e. \( [d(fx_n, gx_m)] \leq (a_3 + a_4 + a_5)^{1/2}[d(hx_n, hx_m)] \)

So that,

\[
d(hx_n, hx_m) \leq d(hx_n, fx_n) + (a_3 + a_4 + a_5)^{1/2}d(hx_n, hx_m)
\]

\[
[1-(a_3 + a_4 + a_5)^{1/2}]d(hx_n, hx_m) \leq d(hx_n, fx_n)
\]

from (3.3.5), this implies that \( \{hx_n\} \) is a Cauchy sequence. Put \( \lim_{n \to \infty} hx_n = z \), then it follows that \( \lim_{n \to \infty} fx_n = z \).
and \( \lim_{n \to \infty} g_n x_n = z \). Also by virtue of the continuity of \( h \) we find that \( h^2 x_n \to h z, \ h f x_n \to h z \) and \( h g x_n \to h z \). To show that \( f x_n \to h z \) and \( g x_n \to h z \), consider the inequality

\[
  d(f x_n, h z) \leq d(f x_n, f x_n) + d(h f x_n, h z) \leq d(h x_n, f x_n) + d(h f x_n, h z)
\]

whence \( f x_n \to h z \). Similarly we have \( g x_n \to h z \).

Now from (3.3.1),

\[
  [d(f x_n, g z)]^2 \leq a_1 [d(h x_n, f x_n) d(h z, g z)] + a_2 [d(f z, h z) d(g h x_n, h h x_n)] + a_3 [d(f h x_n, h z) d(g h x_n, h h z)] + a_4 [d(f z, h h x_n) d(g z, h h x_n)] + a_5 [d(h h x_n, h z)]^2
\]

or,

\[
  [d(h z, g z)]^2 \leq a_1 [d(h z, h z) d(h z, g z)] + a_2 [d(f z, h z) d(g z, h z)] + a_3 [d(f z, h z) d(g z, h z)] + a_4 [d(f z, h z) d(g z, h z)] + a_5 [d(h z, h z)]^2
\]

or,

\[
  [d(h z, g z)]^2 \leq (a_2 + a_3 + a_4) d(f z, h z)^2
\]

Similarly we have

\[
  [d(f z, h z)]^2 \leq (a_2 + a_3 + a_4) [d(h z, g z)]^2,
\]

then from (3.3.2) if follows that

\( h z = f z = g z \).
From (3.3.1), consider
\[
[d(fx_n, gz)]^2 \leq a_1 [d(hx_n, fx_n) d(hz, gz)] + a_2 [d(fz, hz) d(gx_n, hx_n)] + a_3 [d(fx_n, hz) d(fx_n, hz)] + a_4 [d(fz, hx_n) d(gz, hx_n)] + a_5 [d(hx_n, hz)]^2
\]

letting \( n \to \infty \),
\[
[d(z, gz)]^2 \leq (a_3 + a_5) [d(z, gz)]^2
\]

which from (3.3.2), implies \( z = gz \), and hence \( z \) is a common fixed point of \( f, g \) and \( h \).

To prove the uniqueness of \( z \), suppose \( z \) and \( u \) are common fixed points of \( f, g \) and \( h \). Then from (3.3.1)
\[
[d(z, u)]^2 = [d(fz, gu)]^2 \leq a_1 [d(hz, fz) d(hu, gu)] + a_2 [d(fu, hu) d(gz, hz)] + a_3 [d(z, u) d(z, u)] + a_4 [d(u, z) d(u, z)] + a_5 [d(z, u)]^2
\]
or,
\[
[d(z, u)]^2 \leq (a_3 + a_4 + a_5) [d(z, u)]^2
\]

which shows that \( z \) is a unique common fixed point of \( f, g \) and \( h \).

This completes the proof.

3.4 Now we prove the following theorem for a pair of compatible mappings:
THEOREM (6) : Let \( f, g \) and \( h \) be three self mappings of a complete metric space \((X,d)\) satisfying (3.3.1), where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are non-negative reals with \( a_3+a_4+a_5 < 1 \),

(3.4.1) \( \{h,f\} \) and \( \{h,g\} \) are compatible pairs,

and if (3.3.3) and (3.3.5) holds, then \( h \) has a fixed point.

Proof : To prove \( \{hx_n\} \) is a Cauchy sequence, let \( \{x_n\} \) be a sequence satisfying (3.4.1) we have

\[
[d(fx_n,gx_m)]^2 \leq a_1[d(hx_n,fx_n)d(hx_m,hx_m)] \\
+ a_2[d(fx_m,hx_m)d(gx_n,hx_n)] + a_3[d(fx_n,hx_m)d(gx_n,hx_n)] \\
+ a_4[d(fx_m,hx_m)d(gx_m,hx_n)] + a_5[d(hx_n,hx_m)]^2
\]

\[
[d(fx_n,gx_m)]^2 \leq (a_3+a_4+a_5)[d(hx_n,hx_m)]^2
\]

i.e. \( d(fx_n,gx_m) \leq (a_3+a_4+a_5)^\frac{1}{2}d(hx_n,hx_m) \)

and so,

\[ d(hx_n,hx_m) \leq d(hx_n,fx_n)+d(fx_n,gx_m)+d(gx_m,hx_m), \]

implies

\[
[1-(a_3+a_4+a_5)^\frac{1}{2}]d(hx_n,hx_m) \leq d(hx_n,fx_n)+d(gx_m,hx_m)
\]

and from (3.3.3) and (3.4.1), \( \{hx_n\} \) is a Cauchy sequence, since \( X \) is complete \( \{hx_n\} \) converges to a point \( z \), also \( h \) is continuous, the sequence \( \{h^2x_n\}, \{hfx_n\} \) and \( \{hx_n\} \) converges to \( hz \).
Using (3.4.1), we obtain
\[ d(f h x_n, h z) \leq d(f h x_n, h f x_n) + d(h f x_n, h z), \]
which implies that \( \{ f h x_n \} \) converges to \( h z \). Similarly we can show that \( \{ g h x_n \} \) converges to \( h z \).

Being \[ d(f x_n, z) \leq d(f x_n, h x_n) + d(h x_n, z), \]
we have \( \{ f x_n \} \to z \), similarly \( \{ g x_n \} \to z \) also.

Further from (3.3.1), we have
\[
[d(f h x_n, g x_n)]^2 \leq a_1 [d(h h x_n, f h x_n) d(h x_n, g x_n)] \\
+ a_2 [d(f x_n, h x_n) d(g h x_n, h h x_n)] + a_3 [d(h h x_n, h h x_n)] \\
+ a_4 [d(f x_n, h h x_n) d(g x_n, h h x_n)] + a_5 [d(h h h x_n, h x_n)]^2
\]

Letting \( n \to \infty \) we have,
\[
[d(h z, z)]^2 \leq a_1 [d(h z, h z) d(z, z)] + a_2 [d(z, z) d(h z, z)] \\
+ a_3 [d(h z, z) d(h z, z)] + a_4 [d(z, h z) d(z, h z)] \\
+ a_5 [d(h z, z)]^2
\]
or
\[
[d(h z, z)]^2 \leq (a_3 + a_4 + a_5) [d(z, h z)]^2
\]
which means that \( h z = z \) and thus \( z \) is a fixed point of \( h \).

The next Theorem assures the existence of a fixed point of \( f \).
THEOREM (7) : Let f, g, and h be three self mappings of a complete metric space \((X,d)\) satisfying (3.3.1), for all \(x,y \in X\). If (3.3.5) hold with \(a_j + a_4 + a_5 < 1\) and \(f\) is continuous then \(f\) has a fixed point with the condition that \((h,f)\) and \((f,g)\) are compatible.

Proof : As in Theorem 6, we shows that the sequences \(\{hx_n\}, \{fx_n\}\) and \(\{gx_n\}\) converge to a point \(z\). Since \(f\) is continuous the sequences \(\{fx_n\}, \{f^2x_n\}\) and \(\{gx_n\}\) converges to the point \(fz\). Using the compatibility of the pairs \((h,f)\) and \((f,g)\) it is immediately seen that the sequences \(\{fx_n\}\) and \(\{gx_n\}\) converge also to \(fz\). Now, applying the condition (3.3.1), we obtain that

\[
[d(f^2x_n,gx_n)]^2 \leq a_1[d(hfx_n,ffx_n)d(hx_n,gx_n)] \\
+ a_2[d(fx_n,hx_n)d(gx_n,hfx_n)] + a_3[d(ffx_n,hx_n)d(gfx_n,hx_n)] \\
+ a_4[d(fx_n,hfx_n)d(gx_n,hfx_n)] + a_5[d(hfx_n,hx_n)]^2
\]

letting \(n\) tending to infinity we have,

\[
[d(fz,z)]^2 \leq a_1[d(fz,fz)d(z,z)] + a_2[d(z,z)d(fz,fz)] \\
+ a_3[d(fz,z)d(fz,z)] + a_4[d(z,fz)d(z,fz)] + a_5[d(fz,z)]^2
\]

or,

\[
[d(fz,z)]^2 \leq (a_3 + a_4 + a_5)[d(fz,z)]^2
\]

and this gives, \(fz \equiv z\), i.e. \(z\) is a fixed point of \(f\).

This completes the proof.
Remark 2: A result analogous to Theorem 7 can be obtained, using the continuity of \( g \) instead of \( f \) and the compatibility of the pairs \( \{h, g\} \) and \( \{f, g\} \).

3.5 Now we shall prove a theorem for Banach space. It is known (Goebel et al. [45]) that in a uniformly convex Banach space an asymptotically regular sequence always exists therefore from Theorem 4, some fixed point Theorems in certain Banach spaces can be derived. We first prove the following Lemma.

Lemma 1: Let \( X \) be a strictly convex, Banach space, and \( K \) a closed convex subset of \( X \) and \( f, g \) and \( h \) are three self mappings of \( K \) satisfying (3.3.1) for \( x, y \) in \( X \) where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are non-negative reals satisfying \( a_3 + a_4 + a_5 < 1 \). If \( h \) is continuous then the set \( F \) of common fixed points of \( f, g \) and \( h \) is closed.

Proof: Let \( \{x_n\} \) be a Cauchy sequence in \( F \) with limit \( x \) in \( K \) then,

\[
\|x - hx\| \leq \|x - x_n\| + \|x_n - hx\|
\]

\[
= \|x - x_n\| + \|hx_n - hx\| \to 0,
\]

since \( h \) is continuous. Thus \( hx = x \).

Now,
\[
\begin{align*}
\left( \| f x_n - g x \| \right)^2 & \leq a_1 \| h x_n - f x_n \| \| h x_n - g x \| + a_2 \| f x - h x \| \| g x_n - h x_n \| \\
& + a_3 \| f x_n - h x \| \| g x_n - h x \| + a_4 \| f x - h x \| \| g x - h x_n \| \\
& + a_5 \| h x_n - h x \|^2 \\
& = a_1 \| x_n - x_n \| \| x - x \| + a_2 \| x - x \| \| x_n - x_n \| \\
& + a_3 \| x_n - x \| \| x_n - x \| + a_4 \| f x_n - x_n \| \| g x - x_n \| \\
& + a_5 \| x_n - x \|^2
\end{align*}
\]

or, \[\left( \| x_n - g x \| \right)^2 \leq (a_3 + a_5) \| x_n - x \| \| x_n - x \| + a_4 \| f x_n - x_n \| \| g x - x_n \|\]

Letting \( n \rightarrow \infty \), we get
\[
\| x - g x \|^2 \leq a_4 \| f x - x \| \| g x - x \|
\]

or,
\[
\| x - g x \| \leq a_4 \| f x - x \| < \| f x - x \| , (as \ a_4 < 1)
\]

Similarly it can be shown that
\[
\| f x - x \| < \| x - g x \|
\]

It follows that \( f x = g x = x = h x \). Thus \( x \) is in \( F \) and \( F \) is closed.

Now we are in a position to prove:
THEOREM (8): Let $X$ be a strictly convex Banach space, and $K$ a closed convex subset of $X$. Let $f, g$ and $h$ be three self mappings on $K$ such that the following hold:

$$
(3.5.1) \left(\|fx-gy\|\right)^2 \leq a_1 \|hx-fx\| \|hy-gy\| + a_2 \|fy-hy\| \|gx-hx\| + a_3 \|fx-hy\| \|gx-hy\| + a_4 \|fy-hx\| \|gy-hx\| + a_5 \|hx-hy\|^2
$$

for all $x, y$ in $X$ where $a_1, a_2, a_3, a_4$ and $a_5$ are non-negative reals satisfying

$$
(3.5.2) a_3 + a_4 + a_5 < 1,
$$

$$
(3.5.3) \text{if } h \text{ is continuous and affine,}
$$

then the set $F$ of common fixed points of $f, g$ and $h$ is closed and convex.

**Proof**: From Lemma 1, $F$ is closed. To show convexity, let $x_1, x_2$ in $F$ and put $x = (x_1 + x_2)/2$.

Since $K$ is convex, $x$ is in $K$ and $hx = x$, since $h$ is affine. Now we consider the following cases:

**Case 1**: Suppose $\|x-fx\| \leq \|x-gx\|$, then

$$
\|x-gx\| \leq \left(\frac{1}{2}\right) \left[ \|x_1-gx\| + \|x_2-gx\| \right]
$$
With out loss of generality we may assume,

$$\| x_2 - gx \| \leq \| x_1 - gx \|$$

Then from (3.5.1)

$$\| x - gx \| \leq \| x_1 - gx \|^2 = \| f x_1 - gx \|^2$$

$$\leq a_1 \| h x_1 - f x_1 \| \| h x_1 - gx \| + a_2 \| f x - h x \| \| g x_1 - h x_1 \|$$

$$+ a_3 \| f x_1 - h x \| \| g x_1 - h x \| + a_4 \| f x - h x_1 \| \| g x - h x_1 \|$$

$$+ a_5 \| h x_1 - h x \|^2$$

or,

$$\| x - gx \|^2 \leq (a_3 + a_5) \| x_1 - x \| \| x_1 - x \| + a_4 \| x_1 - f x \| \| x_1 - g x \|$$

i.e.

$$\| x - g x \|^2 \leq (a_3 + a_5) \| x_1 - x \|^2 + a_4 \| x_1 - f x \| \| x_1 - g x \|.$$

**Case 1 (a):** Assume $$\| x_1 - f x \| \leq \| x_1 - g x \|.$$ Then

$$\| x - g x \|^2 \leq (a_3 + a_5) \| x_1 - x \|^2 + a_4 \| x_1 - g x \|^2.$$

Now,

$$\| x_1 - g x \|^2 = \| f x_1 - g x \|^2$$

$$\leq a_1 \| h x_1 - f x_1 \| \| h x - g x \| + a_2 \| f x - h x \| \| g x_1 - h x_1 \|$$

$$+ a_3 \| f x_1 - h x \| \| g x_1 - h x \| + a_4 \| f x - h x_1 \| \| g x - h x_1 \|$$

$$+ a_5 \| h x_1 - h x \|^2$$
or,

\[(3.5.4) \| x_1 - gx \|^2 \leq (a_3 + a_5) \| x_1 - x \|^2 + a_4 \| x_1 - gx \|^2 \]

Thus, we get

\[\| x - gx \|^2 \leq \frac{a_3 + a_5}{1 - a_4} \| x_1 - x \|^2.\]

From (3.5.2), we have \(\| x - gx \| \leq \| x_1 - x_2 \| / 2\) and substituting in (3.5.4), we have

\[\| x_1 - gx \|^2 \leq (a_3 + a_4 + a_5) \| x_1 - x \|^2\]

or, \(\| x_1 - gx \| \leq \| x_1 - x \| \leq \| x_1 - x_2 \| / 2\).

Thus, \(\| x_1 - x_2 \| \leq \| x_1 - gx \| + \| x_2 - gx \| \leq 2 \| x_1 - gx \| \leq \| x_1 - x_2 \|\)

and since \(X\) is strictly convex \(gx = x\) and since

\[\| fx - x \| \leq \| gx - x \| ,\]

we have \(fx = x\) too. Thus \(F\) is convex.

**Case 1 (b):** Assume \(\| x_1 - gx \| \leq \| x_1 - fx \|\). Then

\[\| x - gx \|^2 \leq (a_3 + a_5) \| x_1 - x \|^2 + a_4 \| fx - x_1 \|^2\]

Now,

\[\| fx - x_1 \|^2 = \| fx - gx_1 \|^2 \leq a_1 \| hx - fx \| \| hx_1 - gx_1 \| + a_2 \| fx_1 - hx_1 \| \| gx - hx \| + a_3 \| fx - hx \| \| gx - hx_1 \| + a_4 \| fx_1 - hx \| \| gx - hx \| + a_5 \| hx - hx_1 \|^2.\]
\[ \| f(x) - x_1 \|^2 \leq a_3 \| x_1 - f(x) \|^2 + a_4 \| x - x_1 \|^2 \]

Or,
\[ (3.3.5) \| f(x) - x_1 \|^2 \leq \frac{(a_4 + a_5)}{(1-a_3)} \| x_1 - x \|^2 \]

and so,
\[ \| x - g(x) \|^2 \leq (a_3 + a_5 + \frac{a_4(a_5 + a_4)}{1-a_3}) \| x_1 - x \|^2 \]

Thus
\[ \| x - g(x) \|^2 \leq b \| x_1 - x \|^2 \text{ (where } b = a_3 + a_5 + \frac{a_4(a_5 + a_4)}{1-a_3} < a_3 + a_5 + a_4 < 1 \) \]

So that
\[ \| x - g(x) \| \leq \frac{\| x_1 - x_2 \|}{2} \]

Substituting this in (3.5.5) yields, \( \| x_1 - f(x) \| \leq \| x_1 - x_2 \|/2 \).

Thus,
\[ \| x_1 - x_2 \| \leq \| x_1 - g(x) \| + \| x_2 - g(x) \| \leq 2 \| x_1 - g(x) \| \leq 2 \| x_1 - f(x) \| \]

\[ \leq \| x_1 - x_2 \| \]

and since \( X \) is strictly convex, \( g(x) = x \). As in case I(a), \( f(x) = x \) too and \( F \) is convex.

**Case II**: Assume \( \| x - g(x) \| \leq \| x - f(x) \| \). This proof is similar to case I and will therefore be omitted.

This completes the proof.